EXTERIOR BALLISTICS

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Preface

As a rule, the authors of a scientific book presumably hope that it will prove useful. The authors of this book wish devoutly that it will turn out to be quite useless, and that the application of exterior ballistics, with lethal intent, may cease. Nevertheless it is clear that armies will mass and nations stand in readiness for war until "Homo sapiens" succeeds in better deserving his self-bestowed name. While this endures, there must be many who know something about the flight of projectiles, and a few who know much about it. For these, this book is written.

To the optimist who feels that this book is pointless because there will never be another war we can say only that we hope he is right. To the pessimist who feels that this book is pointless because the next war will be fought with weapons of such wide destructive power that it matters little where they are delivered, we would say that few weapons indeed, having once been useful, have been entirely discarded. The battle-axe survived in the hatchet of the commandos; the spear survived in the bayonet; and even sticks and stones killed many in the first days of the independence of India. If there is another war, we may feel reasonably sure that guns, bombs and rockets will not be useless.

This book is not meant to be a compendium of ballistic knowledge. Many of the older techniques have been omitted, and the historical references are few. This is due, in part at least, to the fact that a considerable amount of second-rate material has been published on problems of ballistics, and even the search for the origin of a useful idea would entail much winnowing.

The material here incorporated was chosen chiefly on a basis of utility. Those subjects are treated which proved important in the work done by the authors during or near the time of the Second World War; and to these are added related topics in exterior ballistics. Not all the work of the authors is included, and still less is the work of their associates, of whom we shall mention J. W. Green, A. P. Morse, A. S. Peters, H. Federer, H. L. Meyer and M. K. Fort. To select almost at random nothing appears about the work of Professor Green on a method for setting a certain British bombsight for use with U. S. bombs, nor on the work of Professor Morse on separating the dispersion due to an aircraft-mounted gun turret from other intermingled quantities in the experimental data. The first of these could not be described without violation of security regulations, and is in any case a problem of mechanism rather than of projectile flight; the second is a problem in statistics rather than in ballistics. Their omission from this volume, along with a number of other subjects which exercised one or more of us during the war years, is in no way an indication that they lacked interest or importance; rather, they were either "classified" or else not entirely within the province of exterior ballistics.

Exterior ballistics may be regarded as a fairly complicated exercise in dynamics, and thus requires some knowledge both of mathematics and of physics. Experience indicated that not all physicists had as much mathematics as was needed, and still less could the mathematicians be depended upon to know the requisite physics. Hence the first chapter in this book. Presumably the mathematics in it will appear quite trivial to mathematicians, and the physics equally trivial to physicists. Let each be tolerant of the ignorance of the other. The authors have attempted to make the book intelligible to anyone who has had a reasonably good undergraduate course either in mathematics or in physics. Furthermore, the elementary physics in the first chapter is not invariably discussed rigorously in textbooks. In particular, the authors do not happen to have encountered any published proof of the Buckingham Π-theorem which is above reproach.

When the manuscript was first completed to the approximate satisfaction of the authors, it was submitted to the Ballistic Research Laboratories at Aberdeen Proving Ground, Maryland. It was read with great care by Mr. R. H. Kent, who is an associate director of the Laboratories and is also head of the Exterior Ballistics Laboratory. Mr. Kent favored us with some comments which have been incorporated in the text. Later each chapter was also read either by Dr. L. S. Dederick, who is the other associate director of the Laboratories, or by Dr. T. E. Sterne, who is the head of the Terminal Ballistics Laboratory.

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The preceding paragraphs were written on July 12, 1949. We have left them unchanged, because the intervening years have done nothing to require their revision, save for the abandonment of the wisp of optimism in the second paragraph. But we now have the pleasant duty of acknowledging our gratitude to several persons besides those already named. Mr. C. H. Murphy of the Laboratories has examined several chapters and has greatly improved Chapter XI by correcting a number of errors. Finally, the authors are grateful to several persons who prepared the manuscript for lithoprinting, including Mrs. Jeannette Sheehan, who typed a substantial part of the text proper; Mr. George E. Proust, who typed and corrected much of the draft for master sheets and verified the trajectory computation in Chapter VI; and, especially, to Miss Lida Libby who typed about one-half the text proper, and entered machine symbolism throughout the book, and to Mrs. Pauline Weaver and Mrs. Hazel Spare, who entered many proof symbols by hand and made final corrections.

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March 1952.

CONCERNING THE NUMBERING OF REFERENCES:

All theorems, equations, formulas, etc., which had enough importance to deserve designation have been numbered together in one single scheme. References to preceding numbered statements ordinarily give three numbers in parentheses; the first is a Roman number, and indicates the chapter referred to; the second is Arabic, and indicates the section; the third is also Arabic, and indicates the number of the statement in that section. To facilitate reference, each spread shows the numbers of the chapter and section to which it belongs. For brevity, references to statements in the same chapter dispense with the chapter number, and references to other statements in the same section dispense with both chapter and section numbers. Thus in Section 4 of Chapter VIII we find a reference to (VII.1.8), which is equation 8 of Section 1 of Chapter VII; a reference to (3.11) which is equation 11 of Section 3 of the same chapter (VIII); and a reference to (11), which is in the same section (1) of the same chapter (VIII).

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Chapter I

MECHANICS,

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1. Definition of a vector.

Many quantities encountered in physics are not fully specified by a single real number but require also the specification of a direction. For example, if all we know is that some one has moved thirty-nine miles from the center of Washington, his location is not determined, but, if he has moved thirty-nine miles in a direction forty degrees east of north, he is in Baltimore. In order to be able to discuss such quantities conveniently it is desirable to use mathematical entities called vectors. These entities can best be regarded as purely mathematical; although they have uses in physics as well as in mathematics, it is preferable for the sake of clarity of thought to define them in a geometrical way and to study their properties mathematically, and later to use these established properties to help us in studying problems of physics.

The geometry we shall use is the familiar "Solid Geometry"; that is, we assume the axioms for Euclidean three-dimensional space. A "translation" is a motion, or mapping, of the space onto itself such that, if A goes to A' and B to B', the distance A'B' is equal to the distance AB, and the line through A' and B' is parallel to the line through A and B.

To reach a definition of a vector we start off with the idea of an ordered pair of points. Let A and B be any two points of three-dimensional space. They form an ordered pair of points if we designate one of them as first, or beginning, point and the other as last, or end, point. If we wish to represent such an ordered point pair graphically, it is permissible and in fact convenient to draw in the line segment from A to B, and to put an arrowhead at the point B to mark it as end point. But this is only a pictorial convenience; if we know where A and B are and which is designated first, we know everything we need to know about the line segment from A to B. It is not required that A and B be different; ordered pairs of points AB with A and B coincident are very important. Graphically such a pair would be represented by a single point.

At this stage one is tempted to define a vector as an ordered pair of points, or (what amounts to the same thing) a line segment AB with a specified beginning and a specified end. However, this would create a certain logical difficulty. It is usually desirable to regard line segments with the same length and direction as representing the same vector. Thus, with rectangular coordinates in three-dimensional space, the line segment from (0, 0, 0) to (1, - 3, 2) has the same length and direction as the line segment from (5, 4, 2) to (6, 1, 4). But they are distinct line segments with no points in common. So, if we would choose to define the vector as being the line segment, these (being different segments) would be different vectors.

This difficulty can easily be avoided by use of a classical device. We say that two ordered pairs of points (A, B) and (C, D) are <u>equivalent</u> if by means of a translation of the space it is possible to bring A and B to the position formerly occupied by C and D respectively. Thus the two point pairs mentioned in the preceding paragraph are equivalent. Then the following statements are almost self-evident. Each ordered pair of points is equivalent to itself. If (A, B) is equivalent to (C, D), then (C, D) is equivalent to (A, B). If (A, B) and (C, D) are both equivalent to (E, F), they are equivalent to each other. From these statements it follows (rather obviously) that if with each ordered pair of points we associate all the pairs equivalent to it, and call the result an "equivalence class," every ordered pair of points belongs to exactly one such class. Each such class, as a whole, will be called a vector. If (A, B) is an ordered point pair, it represents the vector to which it belongs; and every ordered pair of points equivalent to (A, B) also represents that same vector. In particular, all ordered pairs (A, A) with the same beginning and end are equivalent to each other; the vector of which they all are representations is the zero vector.

Vectors will be represented by letters in boldface type, such as $\mathbf{x}$, $\mathbf{y}$, $\boldsymbol{\omega}$. The zero vector will be denoted by $\mathbf{O}$.

All the pairs of points belonging to any one vector $\mathbf{x}$ have the same distance between beginning point and end point. This distance, being common to all pairs which represent the vector, may be regarded as a property of the vector. It is named the length of the vector $\mathbf{x}$, and will be denoted by the symbol $|\mathbf{x}|$. In a certain hazy sense, which will later be made more precise, all the representations of any vector $\mathbf{x}$ other than $\mathbf{O}$ have the same direction, so this direction too is a property of the vector $\mathbf{x}$. But the representations of $\mathbf{x}$ do not have any position in common, so the vector $\mathbf{x}$ may not be regarded as having any specific position. Nevertheless, for the sake of brevity one often says "we construct the vector $\mathbf{x}$ with beginning at P" as an abbreviation for "we construct the line segment PQ such that the ordered pair of points (P, Q) is a representation of the vector $\mathbf{x}$."

2. Addition of vectors; multiplication of vectors by real numbers.

The simplest operations which can be performed on vectors are the addition of two vectors and the multiplication of a vector by a real number. Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors, and let A be any point. The vector $\mathbf{x}$ has exactly one representation (A, B) beginning at A. The vector $\mathbf{y}$ has exactly one representation (B, C) beginning at B. The vector represented by (A, C) is called the sum of $\mathbf{x}$ and $\mathbf{y}$, and is denoted by $\mathbf{x} + \mathbf{y}$. It is evident that this is independent of the choice of A; for if any other beginning point A' is chosen, and representations (A', B') and (B', C') of $\mathbf{x}$ and $\mathbf{y}$ respectively are constructed, the translation which carries A into A' will also carry B into B' (since (A, B) and (A', B') both represent $\mathbf{x}$) and will carry C into C'. So (A, C) will be equivalent to (A', C').

Let $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ be any three vectors and A a point. If (A, B) represents $\mathbf{x}$, and (B, C) represents $\mathbf{y}$, and (C, D) represents $\mathbf{z}$, then (A, C) represents $\mathbf{x} + \mathbf{y}$ so that (A, D) represents $(\mathbf{x} + \mathbf{y}) + \mathbf{z}$. Also (B, D) represents $\mathbf{y} + \mathbf{z}$ so (A, D) represents $\mathbf{x} + (\mathbf{y} + \mathbf{z})$. Hence

(1) $$(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z}).$$

This, the associative law of addition, permits us to omit parentheses in successive sums such as

$$\mathbf{x} + \mathbf{y} + \mathbf{z} + \mathbf{w}$$

without causing any ambiguity. It is easy to see that

(2) $$\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$$

for every vector $\mathbf{x}$. If $\mathbf{x}$ is a vector and (A, B) is a representation of it, then (B, A) is a representation of a vector, different from $\mathbf{x}$ unless $\mathbf{x} = \mathbf{0}$. This new vector we denote by $-\mathbf{x}$. Clearly

(3) $$(-\mathbf{x}) + \mathbf{x} = \mathbf{x} + (-\mathbf{x}) = \mathbf{0}.$$

If $\mathbf{x}$ and $\mathbf{y}$ are two non-zero vectors, having representations (A, B) and (A, C) respectively which do not lie along the same line, we complete the parallelogram ABDC. Since by a translation we bring A to C and B to D, (C, D) is equivalent to (A, B) and represents $\mathbf{x}$. Similarly (A, C) and (B, D) both represent $\mathbf{y}$. Hence by definition (A, D) represents both $\mathbf{x} + \mathbf{y}$ and $\mathbf{y} + \mathbf{x}$, and we have established the commutative law

$$(4) \qquad \mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$$

if $\mathbf{x}$ and $\mathbf{y}$ are non-zero and their representations are not parallel.

If $\mathbf{x}$ or $\mathbf{y}$ is zero the equation is also true, by (2). There still remains the case of non-zero, parallel vectors $\mathbf{x}$, $\mathbf{y}$ to consider. This can be reduced to the case already considered by using an auxiliary vector $\mathbf{z}$ not parallel to $\mathbf{x}$ or $\mathbf{y}$. Then the order of addition of $\mathbf{x}$ and $\mathbf{z}$, of $\mathbf{y}$ and $\mathbf{z}$, and of $(\mathbf{x} + \mathbf{z})$ and $(-\mathbf{z} + \mathbf{y})$ is immaterial. So, using (1), (2) and (3), we find

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= \mathbf{x} + (\mathbf{z} - \mathbf{z}) + \mathbf{y} \\
&= (\mathbf{x} + \mathbf{z}) + (-\mathbf{z} + \mathbf{y}) \\
&= (\mathbf{y} - \mathbf{z}) + (\mathbf{z} + \mathbf{x}) \\
&= \mathbf{y} + (-\mathbf{z} + \mathbf{z}) + \mathbf{x} \\
&= \mathbf{y} + \mathbf{z}.
\end{aligned}$$

So (4) holds in all cases.

We now define the operation of multiplying a vector by a real number (for which the alternative name <u>scalar</u> is often used). Let $\mathbf{x}$ be a vector and c a real number. If $\mathbf{x} = \mathbf{O}$ or if $c = 0$, we define

$$(5) \qquad c\mathbf{x} = \mathbf{x}c = \mathbf{O}.$$

Otherwise, let AB be a representation of $\mathbf{x}$. On the line through A and B we find the point C such that the distance from A to C is $|c| \cdot |\mathbf{x}|$ and which is on the same side of A as B is if $c > 0$, on the opposite side if

$c < 0$. Then $c\mathbf{x}$ and $\mathbf{x}c$ are both defined to be the vector represented by AC. It is easily seen that this does not depend on the representation chosen for $\mathbf{x}$. The expression $\mathbf{x}/c$ shall mean $(1/c)\mathbf{x}$.

By methods familiar from elementary coordinate geometry we can show that

$$(6) \qquad (a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x},$$

and

$$(7) \qquad a(b\mathbf{x}) = (ab)\mathbf{x}.$$

It is also clear from the construction that if $\mathbf{x}$ is a non-zero vector, and $\mathbf{y}$ is a vector parallel to $\mathbf{x}$ (that is, a vector whose representations are parallel to those of $\mathbf{x}$), then there is a real number c such that

$$\mathbf{y} = c\mathbf{x}.$$

Suppose that $\mathbf{x}$ and $\mathbf{y}$ are non-zero vectors which are not parallel. Let AB represent $\mathbf{x}$ and BC represent $\mathbf{y}$; then AC represents $\mathbf{x} + \mathbf{y}$. Let AB' represent $c\mathbf{x}$ and B'C' represent $c\mathbf{y}$; then AC' represents $c\mathbf{x} + c\mathbf{y}$. But the triangles ABC and AB'C' are similar, so C' is on the line AC, and AC' represents $c(\mathbf{x} + \mathbf{y})$. That is,

$$(8) \qquad c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}.$$

The restriction that $\mathbf{x}$ and $\mathbf{y}$ are non-parallel is easily removed; if they are parallel, then by the preceding paragraph we have, say, $\mathbf{y} = k\mathbf{x}$, so

$$\begin{aligned} c(\mathbf{x} + \mathbf{y}) &= c(\mathbf{x} + k\mathbf{x}) = c(1 + k)\mathbf{x} \\ &= (c + ck)\mathbf{x} = c\mathbf{x} + ck\mathbf{x} \\ &= c\mathbf{x} + c\mathbf{y}. \end{aligned}$$

In short, the operations of addition of vectors and of multiplication of vectors by real numbers obey the same laws as the corresponding computations with real numbers. However, the multiplication of a vector by another vector has not been assigned any meaning.

We can now give a precise meaning to the word "direction." If $\mathbf{x}$ is any non-zero vector, the vector $\mathbf{x}\,(1/|\mathbf{x}|)$ has length 1, by the construction above. This vector is called the "unit vector in the direction of $\mathbf{x}$," or more briefly the "direction" of $\mathbf{x}$. The zero vector is the only one with length 0; every other vector is uniquely determined by the length and its direction since $\mathbf{x} = (|\mathbf{x}|)(\mathbf{x}[1/|\mathbf{x}|])$, the first factor being the length and the second the direction.

3. Linear dependence of vectors.

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be any collection of vectors. We shall say that they are collinear, or parallel, if when they are given respective representations $AB_1, \ldots, AB_n$ all starting at the same point, all of the points $A, B_1, \ldots, B_n$ lie in one straight line. Evidently the choice of beginning point A is immaterial. The vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are coplanar if when they are given the respective representations $AB_1, \ldots, AB_n$, all the points $A, B_1, \ldots, B_n$ lie in a plane. It is obvious that if at most one of the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is different from $\mathbf{O}$ the vectors are collinear; if at most two differ from $\mathbf{O}$ they are coplanar.

The geometric concepts defined in the preceding paragraph are closely related to the important concept of linear independence. A set $\mathbf{x}_1, \ldots, \mathbf{x}_n$ of vectors is linearly dependent if there is a set of real numbers $c_1, \ldots, c_n$ not all zero such that

$$c_1\mathbf{x}_1 + \ldots + c_n\mathbf{x}_n = \mathbf{O}; \tag{1}$$

otherwise they are linearly independent.

Two immediate corollaries are:

(2) Lemma. If some of the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly dependent, all of them are. Suppose, to be specific, that $\mathbf{x}_1, \ldots, \mathbf{x}_m$ are linearly dependent

where $m < n$. Then there are numbers $c_1, \ldots, c_m$ not all zero such that

$$c_1\mathbf{x}_1 + \cdots + c_m\mathbf{x}_m = \mathbf{O}.$$

But then

$$c_1\mathbf{x}_1 + \cdots + c_m\mathbf{x}_m + 0\mathbf{x}_{m+1} + \cdots + 0\mathbf{x}_n = \mathbf{O},$$

so $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly dependent.

(3) Lemma. The vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly dependent if and only if some one of them is expressible as a linear combination of the others.

Suppose that $\mathbf{x}_m$ is equal to a linear combination

$$c_1\mathbf{x}_1 + \cdots + c_{m-1}\mathbf{x}_{m-1} + c_{m+1}\mathbf{x}_{m+1} + \cdots + c_n\mathbf{x}_n$$

of the others. Then

$$c_1\mathbf{x}_1 + \cdots + c_{m-1}\mathbf{x}_{m-1} + (-1)\mathbf{x}_m + c_{m+1}\mathbf{x}_{m+1} + \cdots + c_n\mathbf{x}_n = \mathbf{O},$$

and the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly dependent. Conversely, suppose the vectors linearly dependent. Then there are numbers $c_1, \ldots, c_n$ not all zero such that (1) holds. Let c_m be a non-vanishing coefficient; then

$$\mathbf{x}_m = (-c_1/c_m)\mathbf{x}_1 + \cdots + (-c_{m-1}/c_m)\mathbf{x}_{m-1} + (-c_{m+1}/c_m)\mathbf{x}_{m+1} + \cdots + (-c_n/c_m)\mathbf{x}_n.$$

We now prove the following theorem.

(4) Theorem. One vector is linearly dependent if and only if it is $\mathbf{O}$. Two vectors are linearly dependent if and only if they are collinear. Three vectors are linearly dependent if and only if they are coplanar. Four vectors in three-dimensional space are always linearly dependent.

By definition, $\mathbf{O}$ is linearly dependent, since

$1(\mathbf{O}) = \mathbf{O}$. Conversely, if $c_1\mathbf{x}_1 = \mathbf{O}$ and $c_1 \neq 0$, then

$$\mathbf{O} = (1/c_1)c_1\mathbf{x}_1 = \mathbf{x}_1.$$

If $\mathbf{x}_1$ and $\mathbf{x}_2$ are collinear, either they are both $\mathbf{O}$, hence linearly dependent; or else one of them, say $\mathbf{x}_1$, is not $\mathbf{O}$. Then, as we have shown earlier, there is a number a such that $\mathbf{x}_2 = a\mathbf{x}_1$, so by Lemma (3) the vectors are linearly dependent. Conversely, if they are linearly dependent, by Lemma (3) one of them is a multiple of the other, say $\mathbf{x}_1 = a\mathbf{x}_2$. By construction of $a\mathbf{x}_2$, the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are collinear.

Let $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ be coplanar. If two of them are collinear (in particular if any one is $\mathbf{O}$), those two are linearly dependent by the preceding paragraph, so the three are linearly dependent by Lemma (2). If no two of them are collinear, construct representations AB_1, AB_2, AB_3 of $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ respectively. Through B_3 draw a line parallel to AB_1; it will intersect AB_2 in some point C, since AB_1 and AB_2 are not parallel. Then AC represents a multiple $a_2\mathbf{x}_2$ of $\mathbf{x}_2$ and CB_3 represents a multiple $a_1\mathbf{x}_1$ of $\mathbf{x}_1$, so $\mathbf{x}_3 = a_1\mathbf{x}_1 + a_2\mathbf{x}_2$, and by Lemma (3) the vectors are linearly dependent.

Conversely, if $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ are linearly dependent, one of them is a linear combination of the others, say $\mathbf{x}_3 = a_1\mathbf{x}_1 + a_2\mathbf{x}_2$. By construction, the right member of this equaton is coplanar with $\mathbf{x}_1$ and $\mathbf{x}_2$, so the three vectors are coplanar.

Let $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{x}_4$ be any four vectors. If any three are linearly dependent all four are, by Lemma (2). Otherwise, let AB_1, AB_2, AB_3, AB_4 be representations of $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{x}_4$ respectively. Through B_4 draw a line parallel to AB_3. This will meet the plane AB_1B_2 in a point C, since $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are not coplanar. Let $\mathbf{z}$ be the vector represented by AC. Since $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{z}$ are coplanar they are linearly dependent, and there are numbers c_1, c_2, c_3 not all zero such that

(5) $$c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + c_3 \mathbf{z} = \mathbf{0}.$$

It is not possible that $c_3 = 0$; if it were, (5) would express a linear dependence of $\mathbf{x}_1$ and $\mathbf{x}_2$. Hence this equation can be solved for $\mathbf{z}$;

(6) $$\mathbf{z} = a_1 \mathbf{x}_1 + a_2 \mathbf{x}_2.$$

The vector CB_4 represents a multiple of $\mathbf{x}_3$, say $a_3 \mathbf{x}_3$; and $\mathbf{x}_4 = \mathbf{z} + a_3 \mathbf{x}_3$. From this and (6) we have

$$\mathbf{x}_4 = a_1 \mathbf{x}_1 + a_2 \mathbf{x}_2 + a_3 \mathbf{x}_3.$$

By Lemma (3), the four vectors are linearly dependent.

Theorem (4) yields the following corollary.

(7) <u>Corollary</u>. <u>If $\mathbf{x}_1$ is a linearly independent (i.e., non-zero) vector, and $\mathbf{z}$ is collinear with $\mathbf{x}_1$, then $\mathbf{z}$ can be represented in exactly one way as a multiple of $\mathbf{x}_1$.</u>

<u>If $\mathbf{x}_1$, $\mathbf{x}_2$ are linearly independent (i.e., non-collinear) vectors, and $\mathbf{z}$ is coplanar with them, then $\mathbf{z}$ can be represented in exactly one way as a linear combination of them.</u>

<u>If $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are linearly independent (i.e., non-coplanar) vectors, every vector $\mathbf{z}$ in three-dimensional space can be represented in exactly one way as a linear combination of them.</u>

We prove the last statement; the proofs of the preceding statements are obtained by omitting all references to $\mathbf{x}_3$ or to $\mathbf{x}_2$ and $\mathbf{x}_3$. The vectors denoted $\mathbf{z}$, $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are linearly dependent by Theorem (4). Hence there are numbers c, c_1, c_2, c_3 not all zero such that

(8) $$c\mathbf{z} + c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + c_3 \mathbf{x}_3 = \mathbf{0}.$$

We cannot have $c = 0$; otherwise (8) would express a linear dependency among $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, contrary to hypothesis. Transposing and dividing by c yields

$$\mathbf{z} = (-c_1/c)\mathbf{x}_1 + (-c_2/c)\mathbf{x}_2 + (-c_3/c)\mathbf{x}_3,$$

so $\mathbf{z}$ can be represented in at least one way as a linear combination of $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$. If there were two different representations, say

$$\mathbf{z} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + a_3\mathbf{x}_3$$

and

$$\mathbf{z} = b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + b_3\mathbf{x}_3,$$

by subtraction we would obtain

$$(a_1 - b_1)\mathbf{x}_1 + (a_2 - b_2)\mathbf{x}_2 + (a_3 - b_3)\mathbf{x}_3 = \mathbf{O},$$

which contradicts the hypothesis that $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are linearly independent.

4. Components of vectors.

Let us choose some one point O of space and name it the "origin." Each point P determines a vector $\mathbf{x}$, represented by OP, which we shall call its "position vector." Conversely, each vector $\mathbf{x}$ has exactly one representation beginning at O; if Q is the end point of this representation, $\mathbf{x}$ is the position vector of the point Q. Thus the vectors are put in one-to-one correspondence with the points of three-dimensional space.

If $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$ are three linearly independent unit vectors we may use them to define a coordinate system. Each vector $\mathbf{x}$ may be written in one and only one way in the form $\mathbf{x} = x_1\mathbf{k}_1 + x_2\mathbf{k}_2 + x_3\mathbf{k}_3$. The numbers x_1, x_2, x_3 are called the components of $\mathbf{x}$ (with respect to the coordinate system $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$). Thus when a coordinate system has been selected each vector corresponds to a triple of numbers, and conversely. Each point P of space also has coordinates. These are defined to be the components of the position vector of P, which is represented by OP.

If representations of $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$ which begin at O are extended the resulting directed lines are called the OX_1- , OX_2- and OX_3-axes. When any vector $\mathbf{x}$ is represented as a linear combination

$$(1) \qquad \mathbf{x} = x_1\mathbf{k}_1 + x_2\mathbf{k}_2 + x_3\mathbf{k}_3$$

of the unit vectors $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, the three real numbers x_1, x_2, x_3 are called the <u>components</u> of $\mathbf{x}$ along the OX_1- , OX_2- , OX_3-axes respectively.

In order to avoid repetition, we shall adopt the following notational convention. When the coordinate vectors $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ have been chosen, the components of a vector denoted by a boldface letter will be designated by attaching subscripts 1, 2, 3 to the corresponding roman letter. Thus the components of $\mathbf{x}$ are (x_1, x_2, x_3); the components of $\bar{\mathbf{z}}^*$ are $(\bar{z}_1^*, \bar{z}_2^*, \bar{z}_3^*)$.

For any three vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, the equation $\mathbf{z} = \mathbf{x} + \mathbf{y}$ is equivalent to $\mathbf{x} + \mathbf{y} - \mathbf{z} = \mathbf{O}$, which with the help of (1) becomes

$$(2) \qquad \begin{aligned} &(x_1 + y_1 - z_1)\mathbf{k}_1 + (x_2 + y_2 - z_2)\mathbf{k}_2 \\ &\qquad + (x_3 + y_3 - z_3)\mathbf{k}_3 = \mathbf{O}. \end{aligned}$$

The vectors $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ being linearly independent by Theorem (3.4), this equation is true if and only if all three equations

$$(3) \quad z_1 = x_1 + y_1, \; z_2 = x_2 + y_2, \; z_3 = x_3 + y_3$$

are satisfied. Thus the single vector equation (1) is equivalent to the three equations (3) relating the components of the vectors.

Likewise the equation

$$(4) \qquad \mathbf{y} = c\,\mathbf{x}$$

is equivalent to

$$y_1\mathbf{k}_1 + y_2\mathbf{k}_2 + y_3\mathbf{k}_3 = c(x_1\mathbf{k}_1 + x_2\mathbf{k}_2 + x_3\mathbf{k}_3),$$

or

$(y_1 - cx_1)\mathbf{k}_1 + (y_2 - cx_2)\mathbf{k}_2 + (y_3 - cx_3)\mathbf{k}_3 = \mathbf{0}.$
By Theorem (3.4), this is true if and only if all three equations

(5) $$y_1 = cx_1, \quad y_2 = cx_2, \quad y_3 = cx_3$$

hold. Thus the single vector equation (4) is equivalent to the three equations (5) relating the components of the vectors.

Suppose that P: (x_1, x_2, x_3) and Q: (y_1, y_2, y_3) are any two points of three-dimensional space. Then OP and OQ represent the position vectors $\mathbf{x}$ and $\mathbf{y}$ of P and Q respectively. The ordered pair PQ represents a vector $\mathbf{z}$, and by definition we see that $\mathbf{x} + \mathbf{z} = \mathbf{y}$, or $\mathbf{z} = \mathbf{y} - \mathbf{x}$. Hence it follows that PQ represents $\mathbf{y} - \mathbf{x}$ whose components are $(y_1 - x_1, y_2 - x_2, y_3 - x_3)$. In other words, the line segment or ordered pair with first point (x_1, x_2, x_3) and last point (y_1, y_2, y_3) represents the vector whose components are

$$(y_1 - x_1, y_2 - x_2, y_3 - x_3).$$

5. Inner products and vector products.

It is not possible to define a process of multiplying vectors together in such a way as to preserve any close similarity to the ordinary multiplication of real numbers. However, two different expressions formed out of pairs of vectors occur frequently when one uses vectors in physics, and these two expressions have been given the names of the "inner product" and the "vector product" of the two vectors.

The inner product of $\mathbf{x}$ and $\mathbf{y}$ is not a vector, but a real number; it is the product of the length of $\mathbf{x}$, the length of $\mathbf{y}$ and the cosine of the angle $(\mathbf{x}, \mathbf{y})$ between them. We shall denote it by $\mathbf{x} \cdot \mathbf{y}$ (it is often called the "dot product" of $\mathbf{x}$ and $\mathbf{y}$). Thus

(1) $$\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}| \cdot |\mathbf{y}| \cdot \cos(\mathbf{x}, \mathbf{y}).$$

Let $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ be three mutually perpendicular unit vectors, and let $\mathbf{x}$ and $\mathbf{y}$ have the respective components (x_1, x_2, x_3) and (y_1, y_2, y_3). If a point O is chosen as origin, and P and Q are the points with coordinates (x_1, x_2, x_3) and (y_1, y_2, y_3) respectively, by definition $\mathbf{x}$ is the position vector of P and $\mathbf{y}$ is the position vector of Q. Let $\mathbf{z}$ be the vector represented by PQ; as we saw in the preceding section, $\mathbf{z} = \mathbf{y} - \mathbf{x}$, and the components of $\mathbf{z}$ are

$$(y_1 - x_1, y_2 - x_2, y_3 - x_3).$$

The length of $\mathbf{z}$ is the distance from P to Q, so by the distance formula,

$$(2) \qquad |\mathbf{z}|^2 = (y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2.$$

In the triangle OPQ the angle opposite PQ is $(\mathbf{x}, \mathbf{y})$, and the sides OP, OQ, PQ have lengths $|\mathbf{x}|$, $|\mathbf{y}|$, $|\mathbf{z}|$. respectively. So by the law of cosines, the distance formula and (1),

$$\begin{aligned} |\mathbf{z}|^2 &= |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2|\mathbf{x}||\mathbf{y}|\cos(\mathbf{x}, \mathbf{y}) \\ (3) \qquad &= x_1^2 + x_2^2 + x_3^2 \\ &\quad + y_1^2 + y_2^2 + y_3^2 - 2\,\mathbf{x}\cdot\mathbf{y}. \end{aligned}$$

Equating the expressions for $|\mathbf{z}|^2$ in (2) and (3) and simplifying yields

$$(4) \qquad \mathbf{x}\cdot\mathbf{y} = x_1y_1 + x_2y_2 + x_3y_3.$$

The left side of this equation is by its definition independent of the choice of $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$. Hence the right member is independent of this choice, in spite of the fact that each of the six numbers in the right member depends on the choice of the three unit vectors.

The vector $\mathbf{x} + \mathbf{y}$ has components

$$(x_1 + y_1, x_2 + y_2, x_3 + y_3).$$

Hence by (4),

$$(\mathbf{x} + \mathbf{y})\cdot\mathbf{z} = (x_1 + y_1)z_1 + (x_2 + y_2)z_2 + (x_3 + y_3)z_3$$

(5) $$= x_1z_1 + x_2z_2 + x_3z_3 + y_1z_1 + y_2z_2 + y_3z_3 = \mathbf{x}\cdot\mathbf{z} + \mathbf{y}\cdot\mathbf{z}.$$

From (4) it is evident that

(6) $$\mathbf{y}\cdot\mathbf{x} = \mathbf{x}\cdot\mathbf{y}.$$

Since $c\,\mathbf{y}$ has components (cy_1, cy_2, cy_3), from (4) we deduce

$$\mathbf{x}\cdot(c\,\mathbf{y}) = x_1cy_1 + x_2cy_2 + x_3cy_3 = c\,\mathbf{x}\cdot\mathbf{y}.$$

A useful corollary of (1) is

(7) $$\mathbf{x}\cdot\mathbf{x} = |\mathbf{x}|^2.$$

To inner multiplication there corresponds no form of division. The equation $\mathbf{x}\cdot\mathbf{y} = 0$ does not imply that either factor vanishes, but merely that either $\mathbf{x}$ or $\mathbf{y}$ is the zero vector or that $\cos(\mathbf{x}, \mathbf{y}) = 0$, that is, that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal. If we regard $\mathbf{0}$ as orthogonal to all vectors, the equation $\mathbf{x}\cdot\mathbf{y} = 0$ holds if and only if $\mathbf{x}$ is orthogonal to $\mathbf{y}$.

Any three non-coplanar vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, considered in that order, form a <u>right-handed</u> or a <u>left-handed</u> set according to the following test. Let OP, OQ, OR represent $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ respectively. In the OQR-plane it is possible to rotate OQ into the direction of OR by a turn of less than 180°. An observer stationed at P will report this rotation as clockwise or counterclockwise. In the former case the set is left-handed, in the latter it is right-handed. There is another way of describing such sets which accounts for the names. Keeping the thumb and index finger in the plane of the palm and the middle finger bent toward the palm, with one hand (but not both) it will be possible to point the thumb in the direction of $\mathbf{x}$, the index finger in the direction of $\mathbf{y}$ and the middle finger in the direction of $\mathbf{z}$. The hand with which this can be done gives its name to the system. (This is easy if the vectors are mutually perpendicular, but can be a trifle uncomfortable for some sets!)

It is easily seen that if $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ is a right-handed set, so are $\mathbf{y}$, $\mathbf{z}$, $\mathbf{x}$ and $\mathbf{z}$, $\mathbf{x}$, $\mathbf{y}$; while $\mathbf{x}$, $\mathbf{z}$, $\mathbf{y}$ and $\mathbf{z}$, $\mathbf{y}$, $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{x}$, $\mathbf{z}$ are left-handed. Henceforth we shall use only coordinate systems $OX_1X_2X_3$ in which the unit vectors $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ form a right-handed set.

The <u>vector product</u>, or <u>cross product</u>, $\mathbf{x} \times \mathbf{y}$ of the two vectors $\mathbf{x}$ and $\mathbf{y}$ is a vector, and is defined as follows. Let OP, OQ represent $\mathbf{x}$, $\mathbf{y}$ respectively. Then the length of $\mathbf{x} \times \mathbf{y}$ is the area of the parallelogram two of whose sides are OP and OQ, so that

$$
(8) \qquad \begin{aligned} |\mathbf{x} \times \mathbf{y}| &= |OP| \cdot |OQ| \cdot \sin \angle POQ \\ &= |\mathbf{x}| \cdot |\mathbf{y}| \cdot \sin(\mathbf{x}, \mathbf{y}). \end{aligned}
$$

This is evidently zero if and only if $\mathbf{x}$ and $\mathbf{y}$ are collinear. When it is not $\mathbf{0}$, $\mathbf{x} \times \mathbf{y}$ has direction perpendicular to the plane OPQ, and $\mathbf{x}$, $\mathbf{y}$, $\mathbf{x} \times \mathbf{y}$ form a right-handed system.

We shall now begin to prove the equation

$$
(9) \qquad \begin{aligned} \mathbf{x} \times \mathbf{y} &= \begin{vmatrix} \mathbf{k}_1 & \mathbf{k}_2 & \mathbf{k}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \\ &= (x_2y_3 - x_3y_2)\mathbf{k}_1 \\ &\quad + (x_3y_1 - x_1y_3)\mathbf{k}_2 + (x_1y_2 - x_2y_1)\mathbf{k}_3. \end{aligned}
$$

This is easy when the left member is **0**; for we have seen that this means that **x** and **y** are collinear, so that the triple (x_1, x_2, x_3) and (y_1, y_2, y_3) are proportional and the determinant is **0**. When the left member is not **0**, the proof of (9) can be made to rest upon the following well-known theorem of analytic geometry.

Let P: (x_1, x_2, x_3), Q: (y_1, y_2, y_3), R: (z_1, z_2, z_3) be any three points in three-dimensional space, the axis system being orthogonal and right-handed. Then the absolute value of the determinant

$$(10) \qquad \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

is equal to the volume of the parallelopiped whose edges are OP, OQ and OR: the determinant (10) is positive or negative according as OP, OQ, OR is a right-handed or a left-handed set.

Define

$$(11) \qquad \mathbf{z} = (x_2y_3 - x_3y_2)\mathbf{k}_1 + (x_3y_1 - x_1y_3)\mathbf{k}_2 + (x_1y_2 - x_2y_1)\mathbf{k}_3;$$

we wish to show that $\mathbf{z} = \mathbf{x} \times \mathbf{y}$. From the equation

$$\mathbf{z} \cdot \mathbf{x} = (x_2y_3 - x_3y_2)x_1 + (x_3y_1 - x_1y_3)x_2 + (x_1y_2 - x_2y_1)x_3 = 0$$

we see that **z** is perpendicular to **x**. In the same way we prove it perpendicular to **y**. Hence it is perpendicular to the plane of **x** and **y**.

Let V be the volume of the parallelopiped with edges OP, OQ, OR, where OP, OQ, OR are representations of **x**, **y**, **z** respectively. Since

$$(12)\quad \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = z_1(x_2y_3 - x_3y_2) + z_2(x_3y_1 - x_1y_3) + z_3(x_1y_2 - x_2y_1)$$
$$= z_1^2 + z_2^2 + z_3^2 = |\mathbf{z}|^2 > 0,$$

the above-mentioned theorem of geometry shows that the set **x**, **y**, **z** is right-handed, and

$$(13)\qquad V = |\mathbf{z}|^2.$$

But **z** is perpendicular to the plane OPQ, so the volume V is the product of the altitude $|\mathbf{z}|$ by the area of the base, which is the area of the parallelogram whose sides are OP and OQ:

$$(14)\qquad V = |\mathbf{z}| \text{ (area of parallelogram OP, OQ).}$$

By comparison of (13) and (14) we see that $|\mathbf{z}|$ is the area of the parallelogram with sides OP, OQ. But now **z** answers in all respects the description which defines $\mathbf{x} \times \mathbf{y}$, so $\mathbf{z} = \mathbf{x} \times \mathbf{y}$. This and (1) complete the proof of (9).

From (9) we immediately conclude that for any three vectors **x**, **y** and **z** and any real number a the equations

$$(15)\qquad \begin{aligned} \mathbf{x} \times \mathbf{x} &= 0, \\ \mathbf{x} \times \mathbf{y} &= -\mathbf{y} \times \mathbf{x}, \\ \mathbf{x} \times (a\,\mathbf{y}) = (a\,\mathbf{x}) \times \mathbf{y} &= a(\mathbf{x} \times \mathbf{y}), \\ (\mathbf{x} + \mathbf{y}) \times \mathbf{z} &= \mathbf{x} \times \mathbf{z} + \mathbf{y} \times \mathbf{z} \end{aligned}$$

are satisfied.

Expansion, as in (12), establishes the identities

$$(16)\qquad \mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}) = (\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}.$$

Vector products involving more than two factors can be simplified by use of the important formula

$$(17) \qquad \mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = (\mathbf{x} \cdot \mathbf{z})\mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\mathbf{z}.$$

This is established by two uses of (9) and a bit of algebraic manipulation, as follows:

$$\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = \begin{vmatrix} \mathbf{k}_1 & \mathbf{k}_2 & \mathbf{k}_3 \\ x_1 & x_2 & x_3 \\ y_2 z_3 - y_3 z_2 & y_3 z_1 - y_1 z_3 & y_1 z_2 - y_2 z_1 \end{vmatrix}$$

$$\begin{aligned}
&= \mathbf{k}_1 (x_2 y_1 z_2 - x_2 y_2 z_1 - x_3 y_3 z_1 + x_3 y_1 z_3) \\
&\quad + \mathbf{k}_2 (x_3 y_2 z_3 - x_3 y_3 z_2 - x_1 y_1 z_2 + x_1 y_2 z_1) \\
&\quad + \mathbf{k}_3 (x_1 y_3 z_1 - x_1 y_1 z_3 - x_2 y_2 z_3 + x_2 y_3 z_2) \\
&= \mathbf{k}_1 [\, y_1 (x_1 z_1 + x_2 z_2 + x_3 z_3) \\
&\qquad - z_1 (x_1 y_1 + x_2 y_2 + x_3 y_3)] \\
&\quad + \mathbf{k}_2 [\, y_2 (x_1 z_1 + x_2 z_2 + x_3 z_3) \\
&\qquad - z_2 (x_1 y_1 + x_2 y_2 + x_3 y_3)] \\
&\quad + \mathbf{k}_3 [\, y_3 (x_1 z_1 + x_2 z_2 + x_3 z_3) \\
&\qquad - z_3 (x_1 y_1 + x_2 y_2 + x_3 y_3)] \\
&= (\mathbf{k}_1 y_1 + \mathbf{k}_2 y_2 + \mathbf{k}_3 y_3)(x_1 z_1 + x_2 z_2 + x_3 z_3) \\
&\quad - (\mathbf{k}_1 z_1 + \mathbf{k}_2 z_2 + \mathbf{k}_3 z_3)(x_1 y_1 + x_2 y_2 + x_3 y_3) \\
&= \mathbf{y}(\mathbf{x} \cdot \mathbf{z}) - \mathbf{z}(\mathbf{x} \cdot \mathbf{y}).
\end{aligned}$$

Before leaving the subject of vector multiplication it should be remarked that the choice of a "right-handed rule" is rather arbitrary. It is quite feasible to begin with unit vectors $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ which form a left-handed system, and to use a lefthand rule for determining the direction of a vector product $\mathbf{x} \times \mathbf{y}$. Precisely the same algebraic formulas would hold for this system, although the geometric interpretation is different.

6. Continuity and differentiability of vector functions.

Let P be an independent variable, ranging over a set [P] in any kind of space. If to each value of P there corresponds a vector $\mathbf{x}(P)$, determined by any law whatever, then $\mathbf{x}(P)$ is a <u>vector</u> (or <u>vector-valued</u>) function of P. For example, a velocity can be represented by a vector. Hence the velocity $\mathbf{v}$ of the air at each point P of the earth's atmosphere is a vector function of the point P. At each time t the position of a moving point is specified by its coordinate vector $\mathbf{x}$ measured from an arbitrarily selected origin O. Hence $\mathbf{x}$ is a vector function $\mathbf{x}(t)$ of the time t.

The definitions of limit and of continuity for vector functions differ only in notation from the corresponding definitions for real-valued functions. Suppose that [P] is a set of points in a space of one, two or three dimensions and that $\mathbf{x}(P)$ is defined for all points P of the set [P]. Suppose that P_0 is a point such that points of [P] (other than P_0 itself) lie arbitrarily close to P_0. We then say that $\mathbf{x}(P)$ approaches a limit $\mathbf{k}$ as P tends to P_0, or in symbols

$$(1) \qquad \lim_{P \to P_0} \mathbf{x}(P) = \mathbf{k},$$

if

$$(2) \qquad \lim_{P \to P_0} |\mathbf{x}(P) - \mathbf{k}| = 0.$$

This last is a statement about real numbers. Since $|\mathbf{x}(P) - \mathbf{k}|$ is the length of the (vector) difference between $\mathbf{x}(P)$ and $\mathbf{k}$, and is also the distance between the points whose position vectors are $\mathbf{x}(P)$ and $\mathbf{k}$ respectively, we can interpret the last statement in either of two ways. To each positive number ϵ there corresponds a positive number δ such that for all points P (different from P_0) which belong to [P] and have distance less than δ from P_0, the vector difference $\mathbf{x}(P) - \mathbf{k}$ has length less than ϵ; or, alternatively, the point with position vector $\mathbf{x}(P)$ lies within distance ϵ of the point whose position vector is $\mathbf{k}$.

If we choose any rectangular coordinate system the vectors $\mathbf{x}(P)$ and $\mathbf{k}$ are each expressed by three components. It is rather easy to show that $\mathbf{x}(P)$ tends to $\mathbf{k}$ as P tends to P_0 if and only if all three equations

$$(3) \qquad \lim_{P \to P_0} x_i(P) = k_i \qquad (i = 1, 2, 3)$$

are satisfied. Consider first the expression

$$\sqrt{a^2 + b^2 + c^2}\,.$$

It is not increased by replacing any two of the letters a, b, c by zero; hence $|a|$, $|b|$ and $|c|$ cannot exceed the radical. On the other hand, the radical cannot exceed $|a| + |b| + |c|$, as is clear if we compare the square of this sum with the square of the radical. Hence

$$(4) \qquad \begin{aligned} &|x_i(P) - k_i| \\ &\le \sqrt{(x_1(P) - k_1)^2 + (x_2(P) - k_2)^2 + (x_3(P) - k_3)^2} \\ &= |\mathbf{x}(P) - \mathbf{k}| \\ &\le |x_1(P) - k_1| + |x_2(P) - k_2| + |x_3(P) - k_3|. \end{aligned}$$

Now if $\mathbf{x}(P)$ tends to $\mathbf{k}$ each of the three numbers $|x_i(P) - k_i|$ is caught between 0 and $|\mathbf{x}(P) - \mathbf{k}|$, which approaches 0. Hence each of the three approaches zero, and (3) is satisfied. Conversely, if (3) holds, each of the three functions $|x_i(P) - k_i|$ approaches 0 as P tends to P_0. Hence so does their sum. But $|\mathbf{x}(P) - \mathbf{k}|$ is caught between 0 and this sum, so it must approach zero. Therefore (1) is satisfied.

This result is still valid if the axes are oblique, but the proof is a little more complicated.

Equations (3) make it easy to prove that if

(5) $$\lim_{P\to P_o} \mathbf{x}(P) = \mathbf{k} \text{ and } \lim_{P\to P_o} \mathbf{y}(P) = \mathbf{l},$$

and u(P) is a real-valued function such that

(6) $$\lim_{P\to P_o} u(P) = m,$$

then the relations

(7) $$\lim_{P\to P_o} u(P)\ \mathbf{x}(P) = m\mathbf{k},$$

(8) $$\lim_{P\to P_o} [\ \mathbf{x}(P) + \mathbf{y}(P)\ ] = \mathbf{k} + \mathbf{l},$$

(9) $$\lim_{P\to P_o} \mathbf{x}(P) \cdot \mathbf{y}(P) = \mathbf{k}\cdot\mathbf{l},$$

(10) $$\lim_{P\to P_o} \mathbf{x}(P) \times \mathbf{y}(P) = \mathbf{k}\times\mathbf{l}$$

are satisfied. For example, (10) holds if and only if all three equations

$$\lim_{P\to P_o} [\,x_2(P)y_3(P) - x_3(P)y_2(P)\,] = k_2l_3 - k_3l_2,$$

$$\lim_{P\to P_o} [\,x_3(P)y_1(P) - x_1(P)y_3(P)\,] = k_3l_1 - k_1l_3,$$

$$\lim_{P\to P_o} [\,x_1(P)y_2(P) - x_2(P)y_1(P)\,] = k_1l_2 - k_2l_1$$

are satisfied. But these are consequences of (3) and its analogue for $\mathbf{y}$ (P), which in turn are consequences of (5).

Suppose now that the set [P] and the point P_o are as described in the second paragraph of this section, and that the point P_0 belongs to [P]. Then we say that $\mathbf{x}(P)$ is continuous at P_0 if the limit

$$\lim_{P \to P_0} \mathbf{x}(P)$$

exists and is equal to $\mathbf{x}(P_0)$. By the preceding proof, this is true if and only if each of the three real-valued functions $x_i(P)$, $(i = 1, 2, 3)$ is continuous at P_0. As usual, $\mathbf{x}(P)$ is said to be continuous on [P] if and only if it is continuous at each point P_0 of [P].

If $\mathbf{x}(t)$ is defined, say, for all t in the interval from t_1 to t_2, and t_0 is in that interval, the derivative $\mathbf{x}'(t_0)$ or $d\mathbf{x}/dt$ is defined to be the limit

$$\lim_{t \to t_0} \frac{\mathbf{x}(t) - \mathbf{x}(t_0)}{t - t_0}$$

provided that this limit exists. As we showed earlier in this section, the relation

$$\mathbf{x}'(t_0) = \lim_{t \to t_0} \frac{\mathbf{x}(t) - \mathbf{x}(t_0)}{t - t_0} \tag{11}$$

is equivalent to all three equations

$$x_i'(t_0) = \lim_{P \to P_0} \frac{x_i(t) - x_i(t_0)}{t - t_0} \quad (i = 1, 2, 3). \tag{12}$$

When the independent variable is interpreted as time, we shall usually write $\dot{\mathbf{x}}$ for the derivative, instead of $\mathbf{x}'$.

If u(t) is a real-valued function and $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are vector functions all defined on an interval and all having derivatives at a point t of the interval, the formulas

$$(13)\quad \frac{d}{dt}[u(t)\mathbf{x}(t)] = u'(t)\mathbf{x}(t) + u(t)\mathbf{x}'(t),$$

$$(14)\quad \frac{d}{dt}[\mathbf{x}(t) + \mathbf{y}(t)] = \mathbf{x}'(t) + \mathbf{y}'(t),$$

$$(15)\quad \frac{d}{dt}[\mathbf{x}(t)\cdot\mathbf{y}(t)] = \mathbf{x}'(t)\cdot\mathbf{y}(t) + \mathbf{x}(t)\cdot\mathbf{y}'(t),$$

$$(16)\quad \frac{d}{dt}[\mathbf{x}(t)\times\mathbf{y}(t)] = \mathbf{x}'(t)\times\mathbf{y}(t) + \mathbf{x}(t)\times\mathbf{y}'(t)$$

all hold. They can be established as (7), (8), (9) and (10) were, by use of components, or they can be established directly. Consider, for example, equation (16). Since

$$\frac{\mathbf{x}(t)\times\mathbf{y}(t) - \mathbf{x}(t_0)\times\mathbf{y}(t_0)}{t - t_0} = \frac{\mathbf{x}(t)\times[\mathbf{y}(t) - \mathbf{y}(t_0)]}{t - t_0} + \frac{[\mathbf{x}(t) - \mathbf{x}(t_0)]\times\mathbf{y}(t_0)}{t - t_0},$$

on letting t tend to t_0 and recalling (8) and (10) we obtain (16), apart from the triviality that we have t_0 in place of t.

From (15) and (5.7) it follows at once that

$$(17)\quad (|\mathbf{x}(t)|^2)' = [\mathbf{x}(t)\cdot\mathbf{x}(t)]' = 2\mathbf{x}(t)\cdot\mathbf{x}'(t).$$

The analogue for the cross product is a triviality because of (5.15). In (16) we must be careful to preserve the order of the factors, since cross-multiplication is not commutative (see 5.15).

7. Rigid motions.

Hitherto we have been discussing purely geometric quantities, involving points and line segments in three-dimensional space. Now we wish to discuss motions, thereby introducing a physical quantity, time. The concept of time is not a simple one. In relativity theory, time coordinates and position coordinates are inextricably bound together. But for experiments involving only terrestrial objects with velocities much less than that of light, it is entirely satisfactory to use the earth's rotation to define time. Let π_1 be a plane through the earth's axis and passing permanently through some identifiable point of the earth's surface (for example, the cross-hairs of the meridian circle at Greenwich Observatory). Let π_2 be a plane passing through the earth's axis and some distant star. The angle between π_1 and π_2 changes; a sidereal second is the time required for it to change 1/86,400 revolution. Due to the motion of the earth in its orbit, the solar day (between successive meridian-crossings of the sun) is longer than a sidereal day (between successive meridian-crossings of a fixed star) by roughly four minutes, on the average; enough to amount to one day per year. So a mean solar second is roughly 366.24/365.24 sidereal seconds; more precisely, it is 1.00273790 sidereal seconds.

As soon as we introduce different times into our discussion we encounter the need of what may be considered hair-splitting. Given a vector $\mathbf{x}$ at a time t_0 and a vector $\mathbf{y}$ of the same length at a different time t_1, how can we tell if these are the same or different? If t_0 were equal to t_1 the question would have an answer which is already given in Section 1. But suppose that A and B are two ends of a diameter of the earth, and that at time t_0 the line AB produced passes through a star S. Then at time t_0 we have

$$(1) \qquad \overrightarrow{AB}/|\overrightarrow{AB}| = \overrightarrow{AS}/|\overrightarrow{AS}|.$$

But at another time t_1 the points A, B, S will no longer be collinear, so the two members of (1) will be unequal. The question is, which (if either) is unchanged? To one who is religiously convinced that the earth stands still the answer seems evident; the left member is unchanged. And all of us tacitly treat the left member as unchanging when we say "the sun rises." If S is a distant star, as astronomer would usually be willing to proceed as though the right member of (1) is unchanged; this assumption is tacitly made whenever an astronomer determines time by the meridian passage of a star. But an astronomer trying to determine the parallax of S will assume neither one unchanged.

Once, not so many decades ago, we might have tried to get out of this difficulty by regarding a vector as unchanging if at all times it joined the same two points of the "ether." The Michelson-Morley experiment frustrates any such attempt. Instead, we abandon all efforts to give an absolute meaning to constancy of a vector, or immobility of a point, and find that we get along quite satisfactorily with a weaker substitute. Suppose that at each instant we choose three concurrent lines OX_1, OX_2, OX_3. Since the angle between OX_1 and OX_2 is a real number, no troubles arise if we require that this angle have the same value for all times t; and likewise for the angle between OX_1 and OX_3 and the angle between OX_2 and OX_3. For present purposes the manner in which the lines are chosen is unimportant; but in specific applications they will always be chosen as some recognizable lines related to a material object, for example, the earth. We can use these lines as the axes of a coordinate system, and we can introduce coordinate vectors $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ as before, for each separate instant of time. Now we can say that a point is fixed with respect to the $OX_1X_2X_3$-system if each of its three coordinates retains a constant value at all times; and we can say that a vector $\mathbf{x}$ is a constant vector with respect to the $OX_1X_2X_3$-system if each of its three components retains a constant value for all times.

In physics, a "particle" is ordinarily understood to mean a material object whose dimensions are too small to be of significance in the particular problem under discussion and whose spatial orientation is of no significance. Such a particle can then be regarded as merely an identifiable point of space, whose position with respect to a chosen coordinate system $OX_1X_2X_3$ is given at any time t by three coordinates $x_1(t)$, $x_2(t)$, $x_3(t)$. The velocity **v** at time t_0 of the particle, with respect to the $OX_1X_2X_3$-system, is defined as follows. Let P_t be the position of the particle at time t, and let P_0 be the point which is fixed with respect to the $OX_1X_2X_3$-system and which coincides at time t_0 with P. Then we define

$$\mathbf{v} = \lim_{t \to t_0} \frac{\overrightarrow{P_0 P_t}}{t - t_0}, \tag{2}$$

provided that the limit exists.

Since the point P_0 has constant coordinates

$$x_1(t_0),\ x_2(t_0),\ x_3(t_0),$$

the vector $\overrightarrow{P_0P_t}$ has components

$$x_1(t) - x_1(t_0),\ x_2(t) - x_2(t_0),\ x_3(t) - x_3(t_0),$$

and equation (2) is equivalent to the three equations

$$\begin{aligned} v_i &= \lim_{t \to t_0} \frac{x_i(t) - x_i(t_0)}{t - t_0} \\ &= \dot{x}_i(t_0),\ (i = 1, 2, 3). \end{aligned} \tag{3}$$

This permits us to write

$$\mathbf{v} = \dot{\mathbf{x}}(t), \tag{4}$$

wherein we must keep in mind that neither side of the equation has any meaning except with reference to some specific coordinate system, and if $x_1(t)$, $x_2(t)$, $x_3(t)$ are the components of $\mathbf{x}(t)$ in a coordinate system with coordinate vectors $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ then for this system we have

$$(5) \qquad \dot{\mathbf{x}}(t) = \dot{x}_1(t)\mathbf{k}_1 + \dot{x}_2(t)\mathbf{k}_2 + \dot{x}_3(t)\mathbf{k}_3.$$

Thus in finding the velocity $\dot{\mathbf{x}}$ of a particle with respect to a coordinate system we treat the coordinate vectors of the system as constants.

The velocity $\mathbf{v}$ of a particle should be carefully distinguished from its speed, which by definition is the absolute value $|\mathbf{v}|$ of its velocity. Thus, properly speaking, an airplane cannot have a velocity of 300 miles per hour (with respect, say, to the ground); it can have a speed of 300 miles per hour, and it will have that speed if its velocity is, say, 300 miles per hour due west and horizontal.

If $O^*X_1^*X_2^*X_3^*$ is another coordinate system, a point P_0 fixed with respect to the new system may have a velocity $\mathbf{w}$, with respect to the original system, which need not be $\mathbf{O}$. Moreover, this velocity $\mathbf{w}$ may vary with the point P_0 and with the time t. It will be called the velocity, at time t and place P_0, of the $O^*X_1^*X_2^*X_3^*$-system with respect to the $OX_1X_2X_3$-system. In order to avoid repeated re-statements of a hypothesis, we shall always assume, when changing from one coordinate system to another, that the velocity of the new system with respect to the old one exists; we rule out non-differentiable motions.

A particle P which has a velocity $\mathbf{v}$ with respect to the $OX_1X_2X_3$-system will ordinarily have a different velocity $\mathbf{v}^*$ with respect to the $O^*X_1^*X_2^*X_3^*$-system. These velocities are related as follows:

If at time t_0 a particle P has velocity $\mathbf{v}^*$ with respect to the $O^*X_1^*X_2^*X_3^*$-system, and at the time t_0 and at the place occupied by P at time t_0 the $O^*X_1^*X_2^*X_3^*$-system has velocity $\mathbf{w}$ with respect to the $OX_1X_2X_3$-system, then at time t_0 the particle P has a velocity $\mathbf{v}$ with respect to the $OX_1X_2X_3$-system, which satisfies the equation

$$(6) \qquad \mathbf{v} = \mathbf{v}^* + \mathbf{w}.$$

Let P_o^* be the point, in the $O^*X_1^*X_2^*X_3^*$-system, whose coordinates are constantly equal to

$$(x_1^*(t_o),\ x_2^*(t_o),\ x_3^*(t_o));$$

and let P_o be the point, fixed in the $OX_1X_2X_3$-system, whose coordinates are constantly equal to

$$(x_1(t_o),\ x_2(t_o),\ x_3(t_o)).$$

At time t_o the points P, P_o, P_o^* all coincide. Clearly

$$\overrightarrow{P_oP} = \overrightarrow{P_oP_o^*} + \overrightarrow{P_o^*P}.$$

If we divide by $t - t_o$ and let t tend to t_o, the three terms tend by definition to $\mathbf{v}$, $\mathbf{v}^*$ and $\mathbf{w}$ respectively, establishing (6).

A set of particles, finite or infinite in number, will be called a body. A body is rigid if each pair of its particles keeps the same distance apart at all times. Thus if $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are the position vectors of two particles of the body in any coordinate system we have

(7) $$|\mathbf{x}(t) - \mathbf{y}(t)| = \text{const.},$$

or, what amounts to the same thing,

(8) $$(\mathbf{x}(t) - \mathbf{y}(t))\cdot(\mathbf{x}(t) - \mathbf{y}(t)) = \text{const.}$$

If $\mathbf{x}(t)$, $\mathbf{y}(t)$, $\mathbf{z}(t)$ are the position vectors of any three points of a rigid body, from the identity

$$\begin{aligned}|\mathbf{x} - \mathbf{y}|^2 &= [(\mathbf{x} - \mathbf{z}) - (\mathbf{y} - \mathbf{z})]\cdot[(\mathbf{x} - \mathbf{z}) - (\mathbf{y} - \mathbf{z})]\\ &= |\mathbf{x} - \mathbf{z}|^2 - 2(\mathbf{x} - \mathbf{z})\cdot(\mathbf{y} - \mathbf{z}) + |\mathbf{y} - \mathbf{z}|^2,\end{aligned}$$

together with (7), we deduce

(9) $$\begin{aligned}&(\mathbf{x} - \mathbf{z})\cdot(\mathbf{y} - \mathbf{z})\\ &= \tfrac{1}{2}\left[|\mathbf{x} - \mathbf{z}|^2 + |\mathbf{y} - \mathbf{z}|^2 - |\mathbf{x} - \mathbf{y}|^2\right]\\ &= \text{const.}\end{aligned}$$

By differentiation this yields the useful identity

(10) $(\dot{x} - \dot{z})\cdot(y - z) = -(x - z)\cdot(\dot{y} - \dot{z})$,

valid at all times for every three particles of a rigid body, provided of course that $\dot{x}$, $\dot{y}$ and $\dot{z}$ are well defined. In particular, if $y = x$ this yields

(11) $$(\dot{x} - \dot{z})\cdot(x - z) = 0.$$

In a later theorem of this section we shall wish to choose four points of a rigid body with position vectors x_0, x_1, x_2, x_3 such that $x_1 - x_0$, $x_2 - x_0$, and $x_3 - x_0$ are mutually perpendicular unit vectors. Clearly this may be impossible, for example, when the body is contained in a sphere of diameter less than 1, or when the body contains less than four points. However, these cases offer no real difficulty. For we can show that an axis system can be attached to a rigid body; that is, it is possible to define a coordinate system in such a way that

(a) Every point of the body has constant coordinates in the system, and

(b) Every two points with constant coordinates in the system keep a constant distance apart.

Thus by (b) all the points with constant coordinates can be thought of as the particles of a rigid body extending throughout space, and by (a) this extended rigid body contains the one we started with.

In order to avoid some mathematical complexities we shall disregard the case in which all the particles of the body lie on a line, and we shall prove

(12) Lemma. Let B be a rigid body containing at least three non-collinear particles. Then there exists a coordinate system having properties (a) and (b) of the preceding paragraph.

Let P_0, P_1 and P_2 be three non-collinear points of the body, let x be the vector represented by P_0P_1 and

$\mathbf{y}$ the vector represented by P_oP_2. Then the three vectors $\mathbf{k}_1 = \mathbf{x}/|\mathbf{x}|$, $\mathbf{k}_2 = \mathbf{x} \times \mathbf{y}/|\mathbf{x} \times \mathbf{y}|$ and $\mathbf{k}_3 = \mathbf{k}_1 \times \mathbf{k}_2$ are mutually orthogonal unit vectors. We select P_o as the origin 0 of a coordinate system for the points of space, the coordinates of any point P being the components in the directions $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ of the position vector of P. It is clear that this coordinate system satisfies the requirement (b). It remains to prove that (a) also holds.

Let $\mathbf{z}$ be the position vector of a point of the body. We may write

(13) $$\mathbf{z} = z_1 \cdot \mathbf{k}_1 + z_2 \cdot \mathbf{k}_2 + z_3 \cdot \mathbf{k}_3.$$

From this equation and the fact that $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$ are mutually orthogonal it follows that $z_1 = \mathbf{z} \cdot \mathbf{k}_1$, $z_2 = \mathbf{z} \cdot \mathbf{k}_2$ and $z_3 = \mathbf{z} \cdot \mathbf{k}_3$. It now must be shown that these three are constant. We first notice that $\mathbf{z} \cdot \mathbf{k}_1 = \mathbf{z} \cdot \mathbf{x}/|\mathbf{x}|$ is constant, by reason of (9). Further, the angle between $\mathbf{x}$ and $\mathbf{y}$ is constant, for it may be obtained from $\mathbf{x} \cdot \mathbf{y}$, $|\mathbf{x}|$ and $|\mathbf{y}|$. Hence $|\mathbf{x} \times \mathbf{y}|$ is constant. We now use (5.17) to write

(14) $$\begin{aligned} \mathbf{k}_3 &= \mathbf{x} \times (\mathbf{x} \times \mathbf{y})/|\mathbf{x}|\,|\mathbf{x} \times \mathbf{y}| \\ &= [(\mathbf{x} \cdot \mathbf{y})\,\mathbf{x} - (\mathbf{x} \cdot \mathbf{x})\,\mathbf{y}]/|\mathbf{x}|\,|\mathbf{x} \times \mathbf{y}|. \end{aligned}$$

From this expression, since $\mathbf{z} \cdot \mathbf{x}$, $\mathbf{z} \cdot \mathbf{y}$, $\mathbf{x} \cdot \mathbf{y}$, $|\mathbf{x}|$ and $|\mathbf{x} \times \mathbf{y}|$ are constant, we see that $\mathbf{z} \cdot \mathbf{k}_3$ is constant by again using (9). Finally, since $|\mathbf{z}|$ is constant, $\mathbf{z} \cdot \mathbf{k}_2$ is determined except for sign. Hence $|\mathbf{z} \cdot \mathbf{k}_2|$ is constant. Since $\mathbf{z}$ and $\mathbf{k}_2$ are continuous functions of time, $\mathbf{z} \cdot \mathbf{k}_2$ must then be constant. The lemma is thereby proved.

With respect to a given coordinate system, the motion of a rigid body at an instant t_o is a <u>translation</u> if at that instant all particles of the body have the same velocity, so that the equation

(15) $$\dot{\mathbf{x}}(t_o) = \dot{\mathbf{y}}(t_o)$$

holds for the position vectors $\mathbf{x}$, $\mathbf{y}$ of every pair of particles of the body. The motion at time t_0 is a rotation about a point P_0 if P_0, being rigidly attached to the body, has velocity $\mathbf{0}$ at time t.

From these definitions it is easy to prove the following theorem.

(16) Theorem. Every motion of a rigid body can be regarded as the sum of a translation and a rotation about an arbitrarily selected point attached to the body.

Let $OX_1X_2X_3$ be the chosen reference system, with coordinate vectors $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, and let $\mathbf{x}_0(t)$ be the position vector of an arbitrarily selected point P_0 rigidly attached to the body. Let $O^*X_1{}^*X_2{}^*X_3{}^*$ be a new coordinate system with origin at P_0, but with the same coordinate vectors $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ as the original system. If a point P is fixed with respect to the new system, having constant position vector $\mathbf{y}$, its position vector with respect to the original system is $\mathbf{x}_0(t) + \mathbf{y}$, so its velocity with respect to the original system is $\dot{\mathbf{x}}_0(t)$. This is the same for all points P, so at every time the motion of the new system with respect to the original system is a translation. The motion of the body with respect to the new system is a rotation, since the point P_0 of the body remains fixed at the origin in the new system.

The next theorem is of great importance in studying the motions of rigid bodies.

(17) Theorem. Let B be a rigid body and $OX_1X_2X_3$ a coordinate system. Then at each time t_0 there is a vector $\boldsymbol{\omega}$ (called the angular velocity of the body B with respect to the coordinate system $OX_1X_2X_3$) such that the position vectors $\mathbf{x}(t)$, $\mathbf{x}_0(t)$ of any two particles of B satisfy the equation

(18) $$\dot{\mathbf{x}}(t_0) - \dot{\mathbf{x}}_0(t_0) = \boldsymbol{\omega} \times [\,\mathbf{x}(t_0) - \mathbf{x}_0(t_0)\,].$$

The following proof is a modification of one suggested by A. P. Morse. Let $\mathbf{x}_0(t)$ be the position vector of an arbitrary point P_0 of the body, and let us choose three points P_1, P_2, P_3 rigidly attached to the body and so located that the three vectors

$$\overrightarrow{P_oP_1},\ \overrightarrow{P_oP_2},\ \overrightarrow{P_oP_3}$$

each have length 1 and form a right-handed orthogonal system. We denote the position-vectors of P_1, P_2, P_3 by $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ respectively, and for notational convenience we define

$$(19)\quad \mathbf{y}_1 = \mathbf{x}_1 - \mathbf{x}_o,\ \mathbf{y}_2 = \mathbf{x}_2 - \mathbf{x}_o,\ \mathbf{y}_3 = \mathbf{x}_3 - \mathbf{x}_o.$$

Then by the definition of vector product we find

$$(20)\quad \mathbf{y}_1 = \mathbf{y}_2 \times \mathbf{y}_3,\ \mathbf{y}_2 = \mathbf{y}_3 \times \mathbf{y}_1,\ \mathbf{y}_3 = \mathbf{y}_1 \times \mathbf{y}_2.$$

Since the $\mathbf{y}_i$ are linearly independent, any vector $\mathbf{z}$ whatever can be expressed as a linear combination of the $\mathbf{y}_i$, in the form $\mathbf{z} = r_1\mathbf{y}_1 + r_2\mathbf{y}_2 + r_3\mathbf{y}_3$. But the $\mathbf{y}_i$ are orthogonal and have length 1; so by taking dot products we find $r_i = \mathbf{z}\cdot\mathbf{y}_i$, $i = 1, 2, 3$. In other words, we have shown

(21) <u>If</u> $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$ <u>are three mutually perpendicular unit vectors, then an arbitrary vector</u> $\mathbf{z}$ <u>satisfies the equation</u>

$$(22)\quad \mathbf{z} = (\mathbf{z}\cdot\mathbf{y}_1)\mathbf{y}_1 + (\mathbf{z}\cdot\mathbf{y}_2)\mathbf{y}_2 + (\mathbf{z}\cdot\mathbf{y}_3)\mathbf{y}_3.$$

We shall first show that there can be at most one vector ω with which (18) is satisfied, and we shall find an expression for this one possible vector. By (20), (5.16), (18) and (19),

$$(23)\quad \begin{aligned} \omega\cdot\mathbf{y}_1 &= \omega\cdot(\mathbf{y}_2\times\mathbf{y}_3) \\ &= (\omega\times\mathbf{y}_2)\cdot\mathbf{y}_3 \\ &= \dot{\mathbf{y}}_2\cdot\mathbf{y}_3. \end{aligned}$$

In a like manner,

$$(24)\quad \omega\cdot\mathbf{y}_2 = \dot{\mathbf{y}}_3\cdot\mathbf{y}_1,\ \omega\cdot\mathbf{y}_3 = \dot{\mathbf{y}}_1\cdot\mathbf{y}_2.$$

Substituting in (22), with x replaced by ω, yields

(25) $\omega = (\dot{y}_2 \cdot y_3) y_1 + (\dot{y}_3 \cdot y_1) y_2 + (\dot{y}_1 \cdot y_2) y_3.$

This is the only possible vector with which (18) can be true. It remains, however, to show that (18) does hold with this particular ω.

We first make a preliminary calculation; namely we compute $\omega \times y_1$. Referring to the form (25) for ω, we have from (20)

(26) $\omega \times y_1 = -(\dot{y}_3 \cdot y_1) y_3 + (\dot{y}_1 \cdot y_2) y_2.$

On the other hand, from (10), $\dot{y}_1 \cdot y_3 = -y_1 \cdot \dot{y}_3$, and $\dot{y}_1 \cdot y_1 = 0$. Thus we may write (26) in the form:

(27)
$$\omega \times y_1 = (\dot{y}_1 \cdot y_1) y_1 + (\dot{y}_1 \cdot y_2) y_2 + (\dot{y}_1 \cdot y_3) y_3.$$

As has been shown above in (22), the right-hand side of this equation is simply $\dot{y}_1$. Of course, the same argument can be applied to y_2 and y_3, so that it is clear that

(28) $$\omega \times y_i = \dot{y}_i \text{ for } i = 1, 2, 3.$$

Now let $x(t)$ be the position vector of an arbitrarily chosen point rigidly attached to the body. There are numbers a_1, a_2, a_3 such that

(29)
$$\begin{aligned} y(t) &= x(t) - x_o(t) \\ &= a_1 y_1(t) + a_2 y_2(t) + a_3 y_3(t). \end{aligned}$$

By Lemma (12) the numbers a_i are constants. Hence, at all times,

(30) $$\dot{y} = a_1 \dot{y}_1 + a_2 \dot{y}_2 + a_3 \dot{y}_3.$$

Substituting from (28) the values of $\dot{y}_i$ we have

(31)
$$\begin{aligned} \dot{y} &= a_1 \omega \times y_1 + a_2 \omega \times y_2 + a_3 \omega \times y_3 \\ &= \omega \times (a_1 y_1 + a_2 y_2 + a_3 y_3) \\ &= \omega \times y. \end{aligned}$$

When the original variables x and x_0 are substituted back by means of (29) this becomes the desired relation and the theorem is proved.

It is to be noted that the argument is perfectly valid for left-handed coordinate systems, if the left-handed vector product is used. In either case, we may regard (25) as a formal definition of ω. As long as only right-handed coordinate systems are used ω may be treated as a vector which obeys the same rules for change under change of coordinate system as do the velocity or position vectors. In the only case (Chapter II) where it is necessary to consider both right- and left-handed systems we shall return to the definition (25).

It is evident that if $\omega = O$ the motion at time t_0 is a translation with respect to the coordinate system, for then by (20) we have $\dot{x}(t_0) = \dot{x}_0(t_0)$ for every two points of the body B. If the motion of the body is a rotation, by definition it is possible to select a point P_0 whose velocity at time t_0 is O. Then, unless $\omega = O$ (in which case all particles of B have velocity O at time t_0), the points with position vectors $x = x_0(t_0) + s\omega$, $-\infty < s < \infty$, form a line passing through P_0. By (20), all points on this line have velocity O at time t_0, for

$$\begin{aligned} \dot{x} &= \dot{x}_0 + \omega \times [x - x_0] \\ &= O + \omega \times [s\omega] \\ &= O, \end{aligned}$$

by (5.15). This line of stationary points is called the <u>instantaneous</u> <u>axis</u> <u>of</u> <u>rotation</u>. We thus see that if a rigid body rotates, it has a whole line of stationary points, its instantaneous axis, and moreover its angular velocity vector ω has the direction of this axis of rotation.

Theorem (17) makes it easy to prove that the physical operation of superposing rotations about the same point corresponds to the mathematical operation of adding the corresponding angular velocity vectors.

Let P be a point of a rigid body which at time t_0 has velocity $\mathbf{O}$ with respect to each of two systems $OX_1X_2X_3$ and $O^*X_1^*X_2^*X_3^*$. Then at place P and time t_0 the $O^*X_1^*X_2^*X_3^*$-system has velocity $\mathbf{O}$ with respect to the $OX_1X_2X_3$-system. The motion of the body with respect to the $O^*X_1^*X_2^*X_3^*$-system is a rotation about P; let ω_1 denote its angular velocity. Likewise the motion of the $O^*X_1^*X_2^*X_3^*$-system with respect to the $OX_1X_2X_3$-system is a rotation about P; let ω^* be its angular velocity. Consider any point Q of the body. There is a point Q^* fixed with respect to the $O^*X_1^*X_2^*X_3^*$-system which at time t_0 coincides with Q, and there is a point Q_0 fixed with respect to the $OX_1X_2X_3$-system which at time t_0 coincides with Q. Using Theorem (17)

velocity of Q with respect to $O^*X_1^*X_2^*X_3^*$-system

$$= \frac{d}{dt}\overrightarrow{Q^*Q} = \omega_1 \times \overrightarrow{PQ},$$

velocity of Q^* with respect to $OX_1X_2X_3$-system

$$= \frac{d}{dt}\overrightarrow{Q_0Q^*} = \omega^* \times \overrightarrow{PQ^*}.$$

Therefore by (6), recalling that Q^* and Q coincided at time t_0,

velocity of Q with respect to $OX_1X_2X_3$-system

$$= \omega_1 \times \overrightarrow{PQ} + \omega^* \times \overrightarrow{PQ}$$
$$= (\omega_1 + \omega^*) \overrightarrow{PQ},$$

and the motion of the body with respect to the $OX_1X_2X_3$-system is a rotation about P with angular velocity $\omega_1 + \omega^*$.

It is clear from (17) that the angular velocity vector ω is determined by the body B and the coordinate system, and does not depend on the choice of any particular point of the body B. In the next section we shall introduce the concept of "inertial frames," which are a certain class of coordinate systems each having a motion of uniform translation with respect to each of the others. It is easy to show that a body B will have the same angular velocity with respect to all such systems. For let $O^*X_1^*X_2^*X_3^*$ be a system which has at time t a motion of translation with respect to the $OX_1X_2X_3$-system. Let $\mathbf{x}^*$, $\mathbf{x}$ be the position vectors of a point P with respect to these systems. Then at time t_0 the differences $\mathbf{x}^* - \mathbf{x}$ and $\dot{\mathbf{x}}^* - \dot{\mathbf{x}}$ are independent of P. So in (18), neither the left member nor the coefficient of ω is changed when we change from one coordinate system to the other, and therefore the same ω serves in both systems.

8. Mass, momentum, and force.

In order to introduce the concept of mass it is convenient to describe an imaginary experiment. Suppose that a small car is placed on a frictionless table which rests on the earth, and that a string tied to this car goes over the edge of the table and is tied to a material object. These conditions cannot actually be realized, but can be approximated by mounting the car on wheels with good ball bearings and leading the string over a pulley with similar bearings. If the car is held stationary and then released, it will begin to move; and if properly held, its motion will be a translation in the direction of the string. At time t each particle of the car will have the same velocity $\mathbf{v}(t)$, and therefore the same acceleration

$$\mathbf{a}(t) = \dot{\mathbf{v}}(t).$$

Assign an arbitrary positive number m_0 to the car-and-string-and-weight system. This will be called the "mass" of the system.

Now let A_1, A_2, ... be any collection of material objects. If object A_n is placed on the car and the car released properly, the car and object will move in translation, and will have an acceleration α_n at the instant of release. Like the acceleration α_o of the empty car at release, this will be in the direction of the string. Let the object A_n be labeled with the number m_n which satisfies the equation,

$$m_o|\alpha_o| = (m_o + m_n)|\alpha_n|.$$

The number m_n will be called the "mass" of the object A_n. Then experiment will reveal (within experimental error) the following facts. The numbers m_n are all positive. If two objects A_n, A_p are placed simultaneously on the car and the car is released the acceleration $\alpha_{n,p}$ at release will satisfy the equation

$$(m_o + m_n + m_p)|\alpha_{n,p}| = m_o|\alpha_o|;$$

and likewise if more than two objects are placed on the car simultaneously. If the car-and-string-and-weight system is replaced by another such system and the experiment is repeated, a mass m_o^* can be assigned to the new system in such a way that each object A_n is assigned the same mass m_n by the new experiment as it was by the old one.

Since the number m_o can be chosen arbitrarily, the mass of any one selected object (say A_1) can be made equal to any chosen number. Thus the masses of all other objects are determined. Most of what is referred to as the civilized world has agreed that a certain lump of platinum (called the standard kilogram) safeguarded at Sèvres, in France, is the selected object. If this lump of platinum is assigned mass 1000, then all masses are expressed in grams. If it is assigned mass 2.2046, all masses are expressed in pounds. If it is assigned mass 2.2046/32.2, all masses are expressed in slugs. A gram is by definition one thousandth of the mass of this lump. Incidentally, a gram is very nearly the same as the mass of one cubic centimeter of distilled water at 4^o Centigrade.

Fundamental in the Newtonian mechanics are certain coordinate systems called "inertial frames." A philosophical discussion of inertial frames would lead us too far afield. Fortunately such a discussion is unnecessary for the purposes of this book. It will be quite adequate for the mechanics of terrestrial objects to consider that one example of an inertial frame is the system whose origin is at the center of mass* of the solar system and whose three coordinate axes are non-coplanar and pass through three distant stars. Other systems are inertial frames if and only if their motion with respect to the system just described is a translation with constant velocity. A system rigidly attached to the earth is not an inertial frame. However, in many (but not all) problems of mechanics it may be regarded as an inertial frame without introducing serious errors.

If a particle of mass m has velocity $\mathbf{v}$ with respect to an inertial frame, the vector $m\mathbf{v}$ is called the momentum of the particle with respect to that inertial frame. The rate of change of the momentum $m\mathbf{v}$ is given the name of the force acting on the particle. Since m is constant, the rate of change of $m\mathbf{v}$ is $m\dot{\mathbf{v}}$, or $m\mathbf{a}$, where $\mathbf{a} = \dot{\mathbf{v}}$ is the acceleration of the particle. This definition of force may seem to be an evasion, but actually it is motivated by experiences common to all mankind and others. Everyone has a rather vague concept of force derived from the feeling of muscular tension, and everyone knows that it requires more muscular effort to throw a large stone than to throw a small stone with the same speed. The definition of force as rate of change of momentum gives mathematical precision to this vague concept.

From the definition it is clear that a force is represented by a vector. However, it is also clear that the action of a force cannot be predicted without

*The term "center of mass" will be defined in Section 9.

knowing one more fact, namely, the point at which the force acts. This is quite apparent if we think of a non-rigid body made of several disconnected particles. A force acting on this body will affect the motion of only the single particle on which it acts. In fact, even babies learn quite early that a door opens more readily if a push is exerted against the door-knob side than if the same push is exerted against the hinge side.

Once the units of mass, length and time have been chosen the unit of magnitude of force (usually abbreviated to "unit of force") is specified by the equation $\mathbf{f} = m\mathbf{a}$. If the units of length, mass and time are the centimeter, gram and second respectively (this is called the c.g.s., or centimeter-gram-second, system) the unit of force is the dyne, which is the force which will give a mass of one gram an acceleration of one centimeter per second per second. If the units of length, mass and time are the foot, the pound and the second respectively (the "f.p.s." system), the unit of force is the <u>poundal</u>, which is the force required to give a mass of one pound an acceleration of one foot per second per second. If the units of length, mass and time are the foot, the slug and the second, the unit of force is the <u>pound force</u>. This is the "engineering system" of units. The pound force is the gravitational force acting on a pound mass at a point of the earth's surface where the gravitational acceleration is 32.2 feet per second per second. Most of the work in exterior ballistics at present being done in the United States is in terms of the f.p.s. system.

The usefulness of the notion of force in mechanics stems largely from the fact that it is frequently possible, as a result of some experiments, to predict the forces that will act on a particle under certain known circumstances. For example, any particle released in a vacuum chamber at a fixed place on the surface of the earth will have an acceleration $\mathbf{g}$, the same for all particles. (The direction of $\mathbf{g}$ is named "vertically downwards"; its magnitude is usually called the "gravitational acceleration," or "g," for the locality). Thus we can predict that a particle of mass m placed in that vacuum chamber will be acted on by a force $m\mathbf{g}$. The magnitude of this force is called the <u>weight</u> of the particle.

Again, imagine a spring of very small mass with one end attached to a point on the surface of a level, frictionless table. If the spring is stretched until the free end is at a point P of the table, any particle which we choose to attach to the end of the spring will, on release, have a certain rate of change of momentum, independent of the particle. We can then conclude that the spring exerts the same force on all particles which may be attached to it at P.

Suppose there are several springs (say n of them) attached to the table, and that if the i-th spring is stretched until its free end is at P it exerts a force $\mathbf{f}_i$. (For the moment we have no interest in the effects on the table produced by the spring). A particle of mass m will then move with acceleration

$$\mathbf{a}_i = \mathbf{f}_i/m$$

if attached to the end of the i-th spring at P. Experiment will show that if all the springs are attached simultaneously to the particle at P it will move with acceleration $\mathbf{a}_1 + \mathbf{a}_2 + \ldots + \mathbf{a}_n$. The force acting on it is then by definition

$$\begin{aligned} \mathbf{f} &= m(\mathbf{a}_1 + \ldots + \mathbf{a}_n) \\ &= m\mathbf{a}_1 + \ldots + m\mathbf{a}_n \\ &= \mathbf{f}_1 + \ldots + \mathbf{f}_n. \end{aligned}$$

In other words, the effect of the simultaneous action of forces $\mathbf{f}_1, \ldots, \mathbf{f}_n$ on a particle is the same as the effect of the single force $\mathbf{f}_1 + \ldots + \mathbf{f}_n$. The mathematical operation of vectorial addition of forces corresponds to the physical operation of letting the several forces act simultaneously on a particle.

Returning to our frictionless table, let two particles A_1 and A_2, with respective masses m_1 and m_2, be joined by a spring stretched beyond its normal length, and not touching any particle other than A_1 and A_2. It will be found that A_1 and A_2 exhibit accelerations, which we may denote by $\mathbf{a}_1$ and $\mathbf{a}_2$ respectively, such that

$$m_1 \mathbf{a}_1 = - m_2 \mathbf{a}_2.$$

That is, the force acting on A_1 because of the interconnection of the particles is the negative of the force acting on A_2 because of the interconnection. Even more than this is true; the accelerations $\mathbf{a}_1$ and $\mathbf{a}_2$ will have directions lying along a line joining the two particles.

Summing up, we have stated the following fundamental relations.

(1) If a force $\mathbf{f}$ acts on a particle of mass m, the acceleration $\mathbf{a}$ of the particle with respect to an inertial frame satisfies the equation $\mathbf{f} = m\mathbf{a}$.

(2) If particles A_1 and A_2 are interconnected in any way, and $\mathbf{f}_{12}$ is the force acting on A_1 and $\mathbf{f}_{21}$ the force acting on A_2 because of the interconnection, then $\mathbf{f}_{21} = - \mathbf{f}_{12}$. Moreover, vectors $\mathbf{f}_{12}$ and $\mathbf{f}_{21}$ are collinear with the vector A_1A_2.

These are the statements in vector symbolism of Newton's three laws of motion:

I. Every body continues in its state of rest, or of uniform motion in a straight line, except insofar as it is compelled by an impressed force to change that state.

II. The rate of change of momentum is proportional to the impressed force, and is in the direction of the line in which the force acts.

III. <u>To every action corresponds an equal and opposite reaction</u>.

These laws have proved themselves adequate for the mechanics of terrestrial objects with velocities much less than that of light, and have been corroborated by a multitude of experiments under such conditions. In particular, they form the basis of exterior ballistics.

9. Center of mass.

Let $A_1, A_2, \ldots, A_n$ be a collection of particles with respective masses $m_1, m_2, \ldots, m_n$, and let their respective position vectors in an inertial frame be $X_1, X_2, \ldots, X_n$. Suppose that exterior forces $F_1, F_2, \ldots$ act on $A_1, A_2, \ldots$ respectively. In addition to the external forces there will be forces due to interrelations between the particles. Let $f_{i,j}$ be the force acting on A_i because of its interrelation with A_j. Then the total force acting on A_i is $F_i + f_{i,1} + \ldots + f_{i,n}$, so its rate of change of momentum is

$$(1) \quad m_i \ddot{X}_i = F_i + f_{i,1} + \ldots + f_{i,i-1} + f_{i,i+1} + \ldots + f_{i,n}.$$

Now we add these n equations member by member. Each pair of distinct subscripts i, j occurs twice, in opposite orders; for example, $f_{3,1}$ occurs in the equation for $m_3 \ddot{X}_3$, while $f_{1,3}$ occurs in the equation for $m_1 \ddot{X}_1$. Because of (8.2), when one adds the equations these terms eliminate each other, and all that remains is

$$(2) \quad \Sigma\, m_i \ddot{X}_i = \Sigma\, F_i.$$

This equation assumes an especially useful form if we define the <u>center of mass</u> $\bar{X}$ of the collection of particles as follows:

(3) $$\bar{x} = (\Sigma\ m_i x_i)/(\Sigma\ m_i).$$

Then

$$\Sigma\ m_i x_i = (\Sigma\ m_i)\ \bar{x},$$

whence

$$\Sigma\ m_i \dot{x}_i = (\Sigma\ m_i)\ \dot{\bar{x}},$$

$$\Sigma\ m_i \ddot{x}_i = (\Sigma\ m_i)\ \ddot{\bar{x}}.$$

Substitution in (2) yields

(4) $$(\Sigma\ m_i)\ \ddot{\bar{x}} = \Sigma\ F_i.$$

In words, the motion of the center of mass is the same as it would be if the total mass of the particles were concentrated at the center of mass and all the forces acted at that point.

As an important special case, if no external forces act on the system the acceleration of the center of mass will be 0, and the center of mass will move with constant speed along a straight line; its momentum will be unchanged, no matter what internal forces act in the system. This is the principle of conservation of momentum.

For example, the relative motions of the members of the solar system are highly complicated. But apart from the very small gravitational attraction of the other stars, there is no external force acting on the system. So apart from this very minute correction, the center of mass of the solar system moves, in any inertial frame, with constant speed along a straight line. For another example, consider the interplanetary rocket so dear to some writers of fiction. Imagine it moving with motors cut off; its velocity will be constant. If the motor is started some gases will be ejected to the rear with high velocity. But the center of mass of the rocket and the ejected gases will continue to have the same velocity. Because the ejected gases are separating from the rocket, the center of mass moves to the rear of the rocket; that is, the rocket goes ahead of the center of mass.

10. Work and energy.

Suppose that a particle moves with constant velocity $\dot{x}(t)$ from a point with position vector $x(t_1)$ to a point with position vector $x(\tau)$, and at each point is acted on by a constant force f. Then by definition the work done by the force on the particle is

$$(1) \qquad W = f\cdot[x(\tau) - x(t_1)].$$

This can clearly be written also in the forms

$$(2) \qquad W = [f\cdot \dot{x}][\tau - t_1],$$

and

$$(3) \qquad W = \int_{t_1}^{\tau} f \cdot \dot{x}\, dt.$$

Suppose next that the particle moves along a path $x = x.(t)$, where $t_1 \leq t \leq \tau$ and at each time t is acted on by a force $f(t)$, where $x(t)$, $\dot{x}(t)$ and $f(t)$ are all continuous. We wish to define the work done by the force on the particles in such a way as to satisfy the following two self-suggesting conditions. (1) If $t_1 \leq t_3 \leq t_4 \leq t_5 \leq \tau$, the work done between times t_3 and t_5 is the sum of the work done between t_3 and t_4 and the work done between t_4 and t_5. (2) The work done between t_3 and t_4 lies between the least and the greatest values (inclusive) of $f(t)\cdot\dot{x}(t)[t_4 - t_3]$ for $t_3 \leq t \leq t_4$. If the interval $[t_1, \tau]$ is cut into subintervals by points $t_1 < t_2 < \dots < t_n = \tau$, and m_i and M_i are respectively the least and greatest values of $f(t)\cdot\dot{x}(t)$ for $t_i \leq t \leq t_{i+1}$, these conditions require that W must lie between

$$\sum_{i=1}^{n-1} m_i[t_{i+1} - t_i] \text{ and } \sum_{i=1}^{n-1} M_i[t_{i+1} - t_i]$$

inclusive. It is well known that there is exactly one number W which satisfies this requirement for all methods of subdividing the interval $[t_1, \tau]$ and that is

$$(4) \qquad W = \int_{t}^{\tau} f(t) \cdot \dot{x}(t)\, dt.$$

Accordingly this will be taken as the definition of the total work done by the force $f(t)$ on the particle as it traverses the path $x = x(t)$, $t_1 \leq t \leq \tau$.

According to the law of motion (8.2) the internal force acting on each particle of a rigid body due to the presence of another particle always lies in the direction of the line joining the particles. From this it follows that the total work done by the internal forces in a rigid body undergoing any motion is always zero. For let $x_1(t)$ and $x_2(t)$ be the position vectors of two particles of a rigid body. Let f_{12} and f_{21} be the forces acting on the first and second particles respectively, due to the other of the two. Then by (8.2) we have $f_{21} = -f_{12}$. But since f_{12} has the direction of the line joining the particles, there is a number k such that

$$f_{12} = k(x_2 - x_1).$$

Hence

$$f_{21} = -k(x_2 - x_1).$$

By (7.11),

$$\dot{x}_1 \cdot (x_1 - x_2) = \dot{x}_2 \cdot (x_1 - x_2).$$

On multiplying by k, this becomes

$$f_{12} \cdot \dot{x}_1 = - f_{21} \cdot \dot{x}_2.$$

Now, on integrating between any limits, we find that the sum of the work done on the first particle by f_{12} and the work done on the second particle by f_{21} is zero. Applying this to all possible pairs of particles shows that the total work done is zero.

The kinetic energy, with respect to an inertial frame, of a particle with mass m and velocity $\dot{\mathbf{x}}$, is by definition

(5) $$T = \tfrac{1}{2}m\,|\dot{\mathbf{x}}|^2 = \tfrac{1}{2}m\,\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}.$$

The kinetic energy of a body, rigid or not, is the sum of the kinetic energies of all its particles.

If a particle moves in an inertial frame and is acted on by a force, the increase in its kinetic energy between any two times t_1 and t_2 is equal to the work done on the particle by the force. Let $\mathbf{f}(t)$ be the force acting at time t, and let $\mathbf{x}(t)$ be the position vector at time t. If m is the mass of the particle, then

$$m\,\ddot{\mathbf{x}}(t) = \mathbf{f}(t).$$

Hence

$$m\,\dot{\mathbf{x}}\cdot\ddot{\mathbf{x}} = \mathbf{f}(t)\cdot\dot{\mathbf{x}}(t).$$

But

$$\frac{d}{dt}|\dot{\mathbf{x}}|^2 = \frac{d}{dt}\dot{\mathbf{x}}\cdot\dot{\mathbf{x}} = 2\,\dot{\mathbf{x}}\cdot\ddot{\mathbf{x}},$$

so the preceding equation can be written

$$\frac{d}{dt}\left[\tfrac{1}{2}m\,|\dot{\mathbf{x}}|^2\right] = \mathbf{f}\cdot\dot{\mathbf{x}}.$$

Integration from t_1 to t_2 yields

$$\tfrac{1}{2}m\,|\dot{\mathbf{x}}(t_2)|^2 - \tfrac{1}{2}m\,|\dot{\mathbf{x}}(t_1)|^2 = \int_{t_1}^{t_2} \mathbf{f}\cdot\dot{\mathbf{x}}\,dt,$$

establishing our statement.

Since we have already seen that the total work done on a rigid body by its internal forces is zero, by applying the preceding paragraph to each particle of a rigid body and summing we find that

(6) <u>The increase in kinetic energy, between time</u> t_1 <u>and time</u> t_2, <u>of a rigid body moving in an inertial</u>

frame is equal to the total work done on the body by the external forces acting on it.

This leads us into an apparent paradox. Suppose that an airplane, resting on its wheels on a runway, fires a one-pound projectile forward. The powder charge gives the shell a muzzle velocity* of 2000 feet per second, hence a kinetic energy of

$$\tfrac{1}{2}(2000)^2 = 2{,}000{,}000 \text{ foot-poundals.}$$

Later the same airplane is in flight at 400 feet per second, and a similar projectile is fired with a similar charge. It is plausible, and in fact correct, that the muzzle velocity is again 2000 feet per second. Hence the speed of the projectile is 2400 feet per second with respect to the earth, and its kinetic energy is $\frac{1}{2}(2400)^2 = 2{,}880{,}000$ foot-poundals. The same charge has given more kinetic energy to the projectile in the second instance than it did in the first.

Again, consider a jet-propelled airplane cruising at 300 feet per second. The pilot feeds more fuel; this burns at the rate of five pounds per second, and is ejected at the rate of 4000 feet per second through the nozzles. Thus in each second the change of momentum of the burned fuel has magnitude (5)(4000) foot-pounds per second, and the rate of change of momentum is 20,000 foot-pounds per second per second, or 20,000 poundals. The work done on the airplane in one second, at a speed of 300 feet per second, is then 6,000,000 foot-poundals. But suppose that with this propulsion the airplane speeds up to 500 feet per second. Then the same rate of feeding of fuel will cause work to be done on the airplane at the rate of 10,000,000 foot-poundals per second.

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*We have been using the word velocity to denote the vector $\dot{x}$ and the word speed to denote $|\dot{x}|$. In speaking of muzzle velocity as a scalar we bow to ballistic tradition.

The paradox disappears when we observe that in each case we have started with a certain system and then ignored a part of the system we started with. In the first example we forgot about the airplane. Let M be its mass. The same force that gave the projectile a speed v forward, when the plane was on the ground, gave the plane a speed V in the opposite direction. Since the total momentum was 0 to begin with, by the principle of the conservation of momentum the final momentum $1v - MV$ was also 0. Also $v + V = 2000$, by hypothesis. Hence

$$v + (1/M)v = 2000,$$

or

$$v = 2000\,M/(M + 1),$$

and

$$V = 2000/(M + 1).$$

The total increase in kinetic energy is not 2,000,000 foot-poundals, but is

$$\tfrac{1}{2}\cdot 1\cdot[2000M/(M + 1)]^2 + \tfrac{1}{2}\cdot M\cdot[2000/(M + 1)]^2$$
$$= 2{,}000{,}000\,M/(M + 1) \text{ foot-poundals.}$$

When the airplane speed was 400 feet per second, the momentum of airplane and shell before firing equaled 400 (M + 1) foot-pounds per second. By conservation of momentum, the velocities v_1 and V_1 of shell and plane after firing satisfy $1v_1 + MV_1 = 400\,(M + 1)$, and also $v_1 - V_1 = 2000$. Hence

$$v_1 = 400 + 2000M/(M + 1),$$

and

$$V_1 = 400 - 2000/(M + 1).$$

The kinetic energy before firing was

$$\tfrac{1}{2}(M + 1)\,400^2 = 80{,}000\,(M + 1) \text{ foot-poundals.}$$

The kinetic energy after firing was

$$\tfrac{1}{2}[400 + 2000M/(M + 1)]^2 + \tfrac{1}{2}M[400 - 2000/(M + 1)]^2$$
$$= 80,000\ (M + 1) + 2,000,000\ M/(M + 1)$$

foot-poundals, so this time, too, the increase in kinetic energy was 2,000,000 $M/(M + 1)$ foot-poundals.

In the example of the jet-propelled airplane, when the airplane had speed 300 feet per second, each five pounds of fuel had kinetic energy $(5/2)(300)^2 = 225,000$ foot-poundals before burning and had kinetic energy $(5/2)(300-4000)^2 = 34,225,000$ foot-poundals after burning, an increase of 34,000,000 foot-poundals. So in each second the total work done on airplane and ejected gases was 6,000,000 + 34,000,000 = 40,000,000 foot-poundals. At speed 500 feet per second each five pounds of fuel had kinetic energy $(5/2)(500)^2 = 625,000$ foot-poundals before burning and had kinetic energy $(5/2)(500-4000)^2 = 30,625,000$ after burning, an increase of 30,000,000. So in each second the total work done was 10,000,000 + 30,000,000 = 40,000,000 foot-poundals.

Besides resolving the paradox, the preceding paragraph illustrates two peculiarities of jet-propelled airplanes. While the numbers do not apply to any specifie airplane, they are not absurd in magnitude, and indicate that most of the energy of the fuel is wasted in the ejected gases. Second, the higher the speed the more of the fuel energy goes into the airplane, and the less into the ejected gases, so that jet-propelled airplanes are most efficient at high speeds.

The two examples just considered can be used to show the usefulness of a decomposition of the kinetic energy which we now describe. Let a body be composed of particles $P_1, P_2, \ldots$, which have respective masses $m_1, m_2, \ldots$ and respective position vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots$ with respect to an inertial frame. The center of mass will have position vector $\mathbf{X}$, where

(7) $$(\Sigma m_i)X = \Sigma m_i x_i.$$

We define the kinetic energy of translation of the body to be the kinetic energy which the body would have if each particle had the velocity $\dot{X}$ which is actually possessed by the center of mass and we define the kinetic energy of motion about the center of mass to be the kinetic energy with respect to a frame which moves without rotation in such a way as to keep its origin at the center of mass.

(8) K. E. of translation $= \frac{1}{2} \Sigma m_i \, |\dot{X}|^2.$

(9) K. E. of motion about center of mass

$$= \frac{1}{2} \Sigma m_i |\dot{x}_i - \dot{X}|^2.$$

It is now easy to establish the following theorem.

(10) Theorem. The kinetic energy of a body with respect to an inertial frame is the sum of its kinetic energy of translation and its kinetic energy of motion about the center of mass.

For the kinetic energy is

(11)
$$\begin{aligned} T &= \tfrac{1}{2} \Sigma m_i \dot{x}_i \cdot \dot{x}_i \\ &= \tfrac{1}{2} \Sigma m_i [(\dot{x}_i - \dot{X}) + \dot{X}] \cdot [(\dot{x}_i - \dot{X}) + \dot{X}] \\ &= \tfrac{1}{2} \Sigma m_i |\dot{x}_i - \dot{X}|^2 + \Sigma m_i (\dot{x}_i - \dot{X}) \cdot \dot{X} \\ &\quad + \tfrac{1}{2} \Sigma m_i |\dot{X}|^2. \end{aligned}$$

But by differentiating both members of the identity (7) we find

$$\Sigma m_i (\dot{x}_i - \dot{X}) = 0.$$

Substituting this in (11) and recalling (8) and (9) establishes the theorem.

To apply this to the examples, we first observe that in each instance the velocity of the center of mass is unchanged by the internal forces, as stated in (9.4) and the following sentences. Hence the kinetic energy of translation is unchanged in both instances. The motion about the center of mass undergoes the same change due to firing the gun at airplane speed 400 feet per second as it does when the gun is fired with the airplane at rest. This represents the entire change in kinetic energy which therefore is unaffected by the velocity of the airplane. Likewise the motion of jet-propelled airplane and ejected gas about the center of mass is the same at 300 feet per second as it is at 500 feet per second, so again the total increase in kinetic energy is the same at both speeds.

11. Kinetic energy of rotation of a rigid body.

The kinetic energy of translation of a body is quite a simple thing; as (10.8) shows, it is the same as a single particle of mass Σm_i would have if it had the motion of the center of mass of the body. Even for rigid bodies the kinetic energy of motion about the center of mass (which in this case is a rotation about the center of mass, as (7.16) shows) is more complicated.

The kinetic energy of rotation about the center of mass is kinetic energy with respect to a non-rotating frame in which the center of mass is at rest. There is some profit in first considering a somewhat more general case, in which a rigid body rotates about some point, not necessarily its center of mass. Take the origin at this point, and let the particles of the body have masses m_1, m_2, ... and position vectors $\mathbf{x}_1$, $\mathbf{x}_2$, ..., respectively. Let $\boldsymbol{\omega}$ be the angular velocity vector. By (7.18),

$$(1) \qquad \dot{\mathbf{x}}_i = \boldsymbol{\omega} \times \mathbf{x}_i.$$

Hence the kinetic energy satisfies the equation

$$(2) \qquad 2T = \Sigma m_i (\boldsymbol{\omega} \times \mathbf{x}_i)\cdot(\boldsymbol{\omega} \times \mathbf{x}_i).$$

By (5.16) and (5.17) this transforms into

$$
\begin{aligned}
2T &= \Sigma m_i \omega \cdot [\, x_i \times (\omega \times x_i)\,] \\
&= \Sigma m_i \omega \cdot [\, \omega (x_i \cdot x_i) - x_i(\omega \cdot x_i)\,] \\
&= \Sigma m_i [(\omega_1^2 + \omega_2^2 + \omega_3^2)(x_{i1}^2 + x_{i2}^2 + x_{i3}^2) \\
&\qquad - (\omega_1 x_{i1} + \omega_2 x_{i2} + \omega_3 x_{i3})^2] \\
&= \Sigma m_i [\omega_1^2 (x_{i2}^2 + x_{i3}^2) + \omega_2^2 (x_{i1}^2 + x_{i3}^2) \\
&\qquad + \omega_3^2 (x_{i1}^2 + x_{i2}^2) - 2\omega_1\omega_2 x_{i1} x_{i2} \\
&\qquad - 2\omega_1\omega_3 x_{i1} x_{i3} - 2\omega_2\omega_3 x_{i2} x_{i3}].
\end{aligned} \tag{3}
$$

If we define

$$
\begin{aligned}
I_1 &= \Sigma m_i (x_{i2}^2 + x_{i3}^2), \\
I_2 &= \Sigma m_i (x_{i1}^2 + x_{i3}^2), \\
I_3 &= \Sigma m_i (x_{i2}^2 + x_{i3}^2), \\
I_{12} &= \Sigma m_i x_{i1} x_{i2}, \\
I_{23} &= \Sigma m_i x_{i2} x_{i3}, \\
I_{13} &= \Sigma m_i x_{i1} x_{i3},
\end{aligned} \tag{4}
$$

this can be written in the form

$$
\begin{aligned}
2T &= I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 \\
&\quad - 2I_{12}\omega_1\omega_2 - 2I_{23}\omega_2\omega_3 - 2I_{13}\omega_1\omega_3.
\end{aligned} \tag{5}
$$

A superficial glance at equation (5) would lead one to regard the situation as more complicated than it really is. To begin with, it looks as though the "moments of inertia," I_1, I_2, and I_3, and the "products of inertia" I_{12}, I_{13}, and I_{23} would have to be calculated for every possible point about which the body might revolve. Theorem (10.10) shows that this is superfluous; if we know the moments and products of inertia for an axis system with origin at the center of mass we can find the kinetic energy of rotation about the center of mass, and then by Theorem (10.10) all else we need to know to find the kinetic energy is the total mass and the motion of the center of mass. Moreover, even if we wish to find the moments and products of inertia for some coordinate system with origin not at the center of mass, we can easily compute them if we know the moments and products of inertia for a coordinate system with origin at the center of mass and axes parallel to the given axes. Suppose that a body has center of mass at $\mathbf{X}$. If the coordinate system is translated so that its origin moves to the center of mass, a point with position vector $\mathbf{x}$ in the old system will have position vector $\mathbf{y} = \mathbf{x} - \mathbf{X}$ in the new system. In the old system let the moments and products of inertia I_1, etc., be defined by (4); in the new system the moments and products of inertia will have values I_1^*, etc., defined by (4) with the x's in the right member replaced by y's. Then

$$
(6) \qquad \begin{aligned} I_1 &= I_1^* + (\Sigma m_i)(X_2^2 + X_3^2), \\ I_{12} &= I_{12}^* + (\Sigma m_i)X_1X_2, \end{aligned}
$$

with analogous equations for I_2, I_3, I_{13} and I_{23}, for

$$
\begin{aligned} I_1 &= \Sigma m_i(x_{i2}^2 + x_{i3}^2) \\ &= \Sigma m_i[(y_{i2} + X_2)^2 + (y_{i3} + X_3)^2] \\ &= \Sigma m_i(y_{i2}^2 + y_{i3}^2) + 2X_2\Sigma m_i y_{i2} + 2X_3\Sigma m_i y_{i3} \\ &\qquad + (\Sigma m_i)(X_2^2 + X_3^2). \end{aligned}
$$

But since the origin of the new system is at the center of mass we have $\Sigma m_i y_{i2} = \Sigma m_i y_{i3} = 0$, and the first of equations (6) is established. The second is proved similarly.

Thus if we know the moments and products of inertia of a rigid body with respect to one set of orthogonal axes, we can compute the kinetic energy of the body under any rotation about any point. It should be stressed here that it is permissible to consider the axes as fixed in the body. In the foregoing discussion, ω was the angular velocity with respect to an inertial frame, but no statement whatever was made about the motion of the coordinate system. All that was asked was that the origin should be at the point about which the body is rotating at the instant at which the kinetic energy was to be found. The components x_{11}, x_{12}, x_{13} of the position vectors x_1 at that instant entered the discussion, but their derivatives did not.

Evidently it would be quite convenient if we could get rid of the products of inertia in some manner. It is an important theorem that this is always possible. Let us say that a line L through the center of mass is a <u>principal axis of inertia</u> if, for some rectangular coordinate system with origin at the center of mass and x_1-axis along L, the two products of inertia I_{12} and I_{13} are zero. We can then prove

(7) <u>Theorem. Every rigid body possesses three mutually perpendicular principal axes of inertia.</u>

Without reference to any particular coordinate system, the kinetic energy T of rotation about the center of mass is a continuous function of the angular velocity ω, as shown by (2). Consequently, among all unit vectors ω there is one which gives T its greatest value, M. Choose this vector as coordinate vector x_1, and let x_2 and x_3 be any unit vectors which with x_1 form a right-handed orthogonal system. By hypothesis, for every unit vector ω we have $T \leq M$, equality holding if $\omega = x_1$. So by (5) the inequality

$$\begin{aligned} &\tfrac{1}{2}\{I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 - 2I_{12}\omega_1\omega_2 \\ (8) \qquad &- 2I_{13}\omega_1\omega_3 - 2I_{23}\omega_{23}\} \\ &- M(\omega_1^2 + \omega_2^2 + \omega_3^2) \leq 0 \end{aligned}$$

holds for all unit vectors ω, equality holding for $\omega = x_1$. But then (8) holds for all vectors ω, for it holds for the unit vector $\omega/|\omega|$, and $|\omega|^{-2}$ is a common factor to all terms. Thus the left member of (8) reaches its maximum value 0 when $\omega = x_1$, that is when $\omega_1 = 1, \omega_2 = 0, \omega_3 = 0$. At this point its three partial derivatives must vanish, since this is a necessary condition for a maximum. Hence we find

$$
(9) \qquad
\begin{aligned}
I_1 - 2M &= 0, \\
- I_{12} &= 0, \\
- I_{13} &= 0,
\end{aligned}
$$

the last two of which show that x_1 is a principal axis of inertia.

We have thus shown that every rigid body has at least one principal axis of inertia. The proof of Theorem (7) will be complete as soon as we prove

(10) Theorem. If L is a principal axis of inertia of a rigid body, there are two other principal axes of inertia perpendicular to L and to each other.

Since L is a principal axis of inertia, we can choose a right-handed system of three orthogonal axes $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ with $\mathbf{x}_1$ along L and such that two products of inertia vanish:

$$
\Sigma m_i x_{i1} x_{i2} = 0, \quad \Sigma m_i x_{i1} x_{i3} = 0.
$$

For any angle ϑ the three vectors

$$
(11) \qquad
\begin{aligned}
&\mathbf{x}_1, \\
\mathbf{y}_2 &= \mathbf{x}_2 \cos\vartheta - \mathbf{x}_3 \sin\vartheta, \\
\mathbf{y}_3 &= \mathbf{x}_2 \sin\vartheta + \mathbf{x}_3 \cos\vartheta.
\end{aligned}
$$

also form a right-handed system of orthogonal unit vectors, as is easily verified. With respect to these new axes the products of inertia are:

$$
(12) \qquad
\begin{aligned}
\Sigma m_i x_{i1} y_{i2} &= (\Sigma m_i x_{i1} x_{i2}) \cos\vartheta - (\Sigma m_i x_{i1} x_{i3}) \sin\vartheta = 0, \\
\Sigma m_i x_{i1} y_{i3} &= (\Sigma m_i x_{i1} x_{i2}) \sin\vartheta + (\Sigma m_i x_{i1} x_{i3}) \cos\vartheta = 0, \\
\Sigma m_i y_{i2} y_{i3} &= \Sigma m_i (x_{i2} \cos\vartheta - x_{i3} \sin\vartheta)(x_{i2} \sin\vartheta + x_{i3} \cos\vartheta) \\
&= \Sigma m_i (x_{i2}^2 - x_{i3}^2) \sin\vartheta \cos\vartheta - \Sigma m_i x_{i2} x_{i3} (\cos^2\vartheta - \sin^2\vartheta) \\
&= \tfrac{1}{2} \{ \Sigma m_i (x_{i2}^2 - x_{i3}^2) \sin 2\vartheta + 2 \Sigma m_i x_{i2} x_{i3} \cos 2\vartheta \}.
\end{aligned}
$$

By proper choice of ϑ the last of these can evidently be made to vanish, and the theorem is proved.

Incidentally, we have shown that if x_1 lies along a principal axis of inertia, so that by definition I_{12} and I_{13} vanish for <u>some</u> choice of x_2 and x_3, then they vanish for <u>every</u> choice of x_2 and x_3. For the most general change of coordinates from the original axes to new orthogonal axes x_1, y_2, y_3 is given by (11), and first two equations of (12) show that two new products of inertia also vanish.

The usefulness of the principal axes of inertia makes it worth while to seek methods of identifying them easily in special circumstances. Let us say that a body has symmetry with respect to a plane Π if to each particle P of the body corresponds another particle Q of the body having the same mass and so situated that the plane Π is the perpendicular bisector of the line segment joining P and Q. The body has "θ-degree symmetry" about a line L if $0 < \theta < 360^\circ$ and a rotation of θ degrees about L brings each particle of the body to a place previously occupied by another particle of the same mass.

If a body has a plane of symmetry, its center of mass lies in that plane. For let Π be a plane of symmetry. Choose a coordinate system with origin in Π and x_1-vector perpendicular to Π. Then if P is a particle of the body, with mass m and coordinates (x_1, x_2, x_3), there is a particle Q of the body with mass m and coordinates $(-x_1, x_2, x_3)$. In the sum $\Sigma m_i x_{i1}$ which defines $(\Sigma m_i)X_1$ the contributions of these particles cancel each other. This is true for all particles, so $X_1 = 0$, and the center of mass lies in the plane $x_1 = 0$ which is Π. Likewise, if a body has θ-degree symmetry about a line L, its center of mass lies on L. For let the body be rotated by θ degrees about L. The center of mass moves rigidly with the body, so rotates θ degrees about L. On the other hand, every place formerly occupied by a particle of the body is again occupied by a particle of the same mass, so the center of mass remains where it was. The only points that can be rotated θ degrees about L and still be where they were are the points on L itself, so the center of mass must be on L.

Bodies with either of these types of symmetry have at least one easily recognized principal axis of inertia, as the following theorems show.

(13) Theorem. If a body has a plane of symmetry Π, the line through the center of mass and perpendicular to Π is a principal axis of inertia.

Choose a coordinate system with origin at the center of mass and coordinate vector $\mathbf{x}_1$ perpendicular to Π. Then if P is a particle of the body with coordinates (x_1, x_2, x_3), there is a particle of the body with the same mass and with coordinates $(-x_1, x_2, x_3)$. Hence the contributions of P and Q to the products of inertia $I_{12} = \Sigma m_i x_{i1} x_{i2}$ and $I_{13} = \Sigma m_i x_{i1} x_{i3}$ cancel each other. This being true for all points P of the body, I_{12} and I_{13} both vanish, and the x_1-axis is the principal axis of inertia.

(14) Theorem. If a body has θ-degree symmetry about a line L, the line L is a principal axis of inertia. Moreover if θ is not 180° every line through the center of mass and perpendicular to L is also a principal axis of inertia, and the body has the same moment of inertia about all these lines.

Choose a right-handed orthogonal coordinate system with origin at the center of mass and coordinate vector $\mathbf{x}_1$ in the direction of L; among all lines perpendicular to L we choose one about which the moment of inertia is a minimum, and let $\mathbf{x}_2$ have the direction of this line. If the body is rotated through an angle α about L, the particle whose coordinates had been (x_1, x_2, x_3) moves to the place (x_1', x_2', x_3'), where

$$(15)\qquad \begin{aligned} x_1' &= x_1, \\ x_2' &= x_2 \cos \alpha - x_3 \sin \alpha, \\ x_3' &= x_2 \sin \alpha + x_3 \cos \alpha. \end{aligned}$$

The new moment of inertia about x_2 is

$$(16)\qquad \begin{aligned} I_2' &= \Sigma m_i[(x_{i1}')^2 + (x_{i3}')^2] \\ &= \Sigma m_i x_{i1}^2 \\ &\quad + \Sigma m_i x_{i2}^2 \sin^2\alpha + \Sigma m_i x_{i3}^2 \cos^2\alpha \\ &\quad + 2\,\Sigma\, m_i x_{i2} x_{i3} \cos\alpha \sin\alpha; \end{aligned}$$

while the new product of inertia I_{12}' is

$$(17)\qquad \begin{aligned} I_{12}' &= \Sigma\, m_i x_{i1}' x_{i2}' \\ &= \Sigma\, m_i x_{i1} x_{i2} \cos\alpha - \Sigma\, m_i x_{i1} x_{i3} \sin\alpha, \\ &= I_{12} \cos\alpha - I_{13} \sin\alpha, \end{aligned}$$

and similarly

$$(18)\qquad I_{13}' = I_{12} \sin\alpha + I_{13} \cos\alpha.$$

By the choice of x_2, I_2' has a minimum when $\alpha = 0$. So the derivative of I_2' with respect to α vanishes when $\alpha = 0$; that is,

$$(19)\qquad \Sigma m_i x_{i2} x_{i3} = 0.$$

If we choose $\alpha = \vartheta$, because of symmetry each particle has taken the place of another particle of the same mass, so I_2', I_{13}' and I_{12}' have the same value for $\alpha = \vartheta$ as for $\alpha = 0$. Thus by (17), (18), (16) and (19),

$$(20)\qquad \begin{aligned} I_{12}(1 - \cos\vartheta) + I_{13} \sin\vartheta &= 0, \\ - I_{12} \sin\vartheta + I_{13}(1 - \cos\vartheta) &= 0; \end{aligned}$$

$$(21)\qquad (\Sigma m_i x_{i3}^2) \sin^2\vartheta = (\Sigma m_i x_{i2}^2) \sin^2\vartheta.$$

The determinant of coefficients of I_{12} and I_{13} in (20) is $(1 - \cos\vartheta)^2 + \sin^2\vartheta = 2(1 - \cos\vartheta)$, which is not 0 because $0 < \vartheta < 360^\circ$. Hence the only solution of (20) is $I_{12} = I_{13} = 0$, and so L is a principal axis of inertia.

Suppose next that ϑ is not 180°. Then $\sin \vartheta \neq 0$, and we can divide both members of (21) by $\sin^2\vartheta$. Substituting the result in (16) yields

$$I_2' = \Sigma m_i[x_{i1}^2 + x_{i2}^2],$$

for all values of α. Thus I_2' is independent of α, and the moment of inertia is the same about all lines through the origin and perpendicular to L. But then each one of these lines minimizes the moment of inertia, and x_2 could have been chosen along any one of them, and (19) would still be valid; that is, I_{23} would be 0. Since I_{12} is already known to be 0, this proves that every line through the center of mass and perpendicular to L is a principal axis of inertia.

Most of the projectiles studied in exterior ballistics have θ-degree symmetry for some θ between 0° and 180° (exclusive). Artillery shell have θ-degree symmetry for all ϑ. All aircraft bombs in service at the time of writing this manuscript have 90°-symmetry. By Theorem (14), for all these projectiles the axis of symmetry is a principal axis of inertia, and so is every line which is perpendicular to the axis of symmetry and passes through the center of mass. Moreover, the projectile has the same moment of inertia about all of those lines.

12. Angular momentum.

From experiments in elementary physics we know that if a body is hinged at a point O and a force $\mathbf{f}$ is applied at a point P, the effect of the force in turning the body is proportional to the length of OP and to the component of $\mathbf{f}$ perpendicular to OP. If θ is the angle between OP and $\mathbf{f}$, the effect of the force is proportional to

$$|OP| \cdot |\mathbf{f}| \cdot \sin \theta.$$

But this is the magnitude of $\overrightarrow{OP} \times \mathbf{f}$. This suggests that the vector product $\overrightarrow{OP} \times \mathbf{f}$ may be a quantity worth studying. More generally, if $\mathbf{v}$ is a vector associated with a point P (as, for example, a force is associated with its point of application, or the velocity of a particle is associated with the position of the particle), and O is any point, the moment of $\mathbf{v}$ about O is defined to be the quantity

(1) $$\text{Moment of } \mathbf{v} \text{ about } O = \overrightarrow{OP} \times \mathbf{v}.$$

Thus, for example, if a force $\mathbf{f}$ is applied at a point with position vector $\mathbf{x}$, its moment about the point with position vector $\mathbf{x}_0$ is $(\mathbf{x} - \mathbf{x}_0) \times \mathbf{f}$. If a particle has position vector (t), its velocity $\dot{\mathbf{x}}$ has moment $(\mathbf{x} - \mathbf{x}_0) \times \dot{\mathbf{x}}$ about $\mathbf{x}_0$. If m is the mass of the particle, its momentum is $m\dot{\mathbf{x}}$; so the moment about $\mathbf{x}_0$ of the momentum is $m(\mathbf{x} - \mathbf{x}_0) \times \dot{\mathbf{x}}$. An alternative name for the moment of the momentum of a particle about a point $\mathbf{x}_0$ is the angular momentum of the particle about $\mathbf{x}_0$. The angular momentum about a point O, of a body (rigid or not) is by definition the sum of the angular momenta of its several particles.

The following lemma is quite easy to prove.

(2) Lemma. Let $\mathbf{x}(t)$ be the position vector of a particle of mass m in an inertial frame, and let $\mathbf{x}_0$ be the position vector of a point Q fixed in the frame. If the particle is acted on by a force $\mathbf{f}$, its rate of change of angular momentum about Q is equal to the moment of $\mathbf{f}$ about Q.

The rate of change of angular momentum is

(3) $$\begin{aligned} \frac{d}{dt} m(\mathbf{x} - \mathbf{x}_0) \times \dot{\mathbf{x}} &= m\dot{\mathbf{x}} \times \dot{\mathbf{x}} + m(\mathbf{x} - \mathbf{x}_0) \times \ddot{\mathbf{x}} \\ &= (\mathbf{x} - \mathbf{x}_0) \times (m\ddot{\mathbf{x}}) \\ &= (\mathbf{x} - \mathbf{x}_0) \times \mathbf{f}, \end{aligned}$$

by (5.15), (6.16) and (8.1). This establishes the lemma.

By addition it follows that the rate of change of the angular momentum about Q of any body is equal to the sum of the moments about Q of all the forces, external and internal, acting on the particles of the body. We now show that the sum of the moments about Q of the internal forces is always $\mathbf{0}$. Let $\mathbf{x}_1$, $\mathbf{x}_2$ be the position vectors of two particles P_1, P_2 of the body. Let $\mathbf{f}_{12}$ be the force acting on P_1, because of the presence of P_2, and let $\mathbf{f}_{21}$ be the force acting on P_2 because of the presence of P_1. By (8.2), $\mathbf{f}_{21} = -\mathbf{f}_{12}$, and $\mathbf{f}_{12}$ and $\mathbf{f}_{21}$ are multiples of the vector P_1P_2, so that

$$(4) \qquad \mathbf{f}_{21} = k\,\overrightarrow{P_1P_2},$$

where k is a real number. The moment of $\mathbf{f}_{21}$ about Q is

$$(5) \qquad \begin{aligned} \overrightarrow{QP}_2 \times \mathbf{f}_{21} &= (\overrightarrow{QP}_1 + \overrightarrow{P_1P_2}) \times \mathbf{f}_{21} \\ &= \overrightarrow{QP}_1 \times \mathbf{f}_{21} + \overrightarrow{P_1P_2} \times (k\overrightarrow{P_1P_2}) \\ &= \overrightarrow{QP}_1 \times \mathbf{f}_{21} \\ &= -\overrightarrow{QP}_1 \times \mathbf{f}_{12}, \end{aligned}$$

and is therefore the negative of the moment of $\mathbf{f}_{12}$ about Q. Thus the sum of the moments of the two internal forces is $\mathbf{0}$. By adding over all pairs of points of the body we find that the sum of the moments of all internal forces is $\mathbf{0}$.

From the first and last sentence of the preceding paragraph we see that

(6) Theorem. The rate of change of the angular momentum of a body about a point Q, fixed in an inertial frame, is equal to the sum of the moments about Q of all the external forces acting on the body.

For example, let us make the plausible assumption that the forces acting on the bodies of the solar system due to the attraction of bodies not in that system are too small ever to be discernible. Then we may regard the solar system as a non-rigid body acted upon by no external forces. By Theorem (6) the angular momentum of the solar system remains constant, and consequently a plane passed through the center of mass of the solar system and perpendicular to the angular momentum vector remains invariant in direction relative to an inertial frame. This plane is known as the "invariable plane" of the solar system.

For a second example, consider a system consisting of two particles P_1, P_2 with the respective masses m_1, m_2 and acted upon by no external forces. Concerning the internal forces $\mathbf{f}_{12}$ and $\mathbf{f}_{21}$ we make no assumptions beyond the standing hypothesis (8.2). By the paragraph following (9.4), with respect to any inertial frame the center of mass of the system moves with constant velocity along a straight line. Therefore a system with origin at the center of mass and axes parallel to those of the original inertial frame is again an inertial frame, and we have shown that there exists an inertial frame in which the center of mass remains fixed at the origin.

Let $\mathbf{x}_1$, $\mathbf{x}_2$ be the position vectors of P_1, P_2 in this particular inertial frame. Since the center of mass is at $\mathbf{O}$ we have $m_1\mathbf{x}_1 + m_2\mathbf{x}_2 = \mathbf{O}$, whence,

$$\mathbf{x}_1 = -(\mathbf{x}_2 - \mathbf{x}_1)m_2/(m_1 + m_2),$$
$$\mathbf{x}_2 = (\mathbf{x}_2 - \mathbf{x}_1)m_1/(m_1 + m_2).$$

From this and (8.2) we see that $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{f}_{12}$ and $\mathbf{f}_{21}$ are all multiples of $\mathbf{x}_2 - \mathbf{x}_1$. Therefore the moment about the origin O of $\mathbf{f}_{12}$ is $\mathbf{x}_1 \times \mathbf{f}_{12} = \mathbf{O}$, and likewise the moment of $\mathbf{f}_{21}$ is $\mathbf{O}$. By Lemma (2) each particle individually retains constant angular momentum about O, so that

$$\frac{d}{dt}[m_1\mathbf{x}_1 \times \dot{\mathbf{x}}_1] = \mathbf{O}, \quad \frac{d}{dt}[m_2\mathbf{x}_2 \times \dot{\mathbf{x}}_2] = \mathbf{O}.$$

Hence there is a constant vector $\mathbf{c}$ such that

$$\mathbf{x}_1 \times \dot{\mathbf{x}}_1 = \mathbf{c},$$

whence

$$\mathbf{c} \cdot \mathbf{x}_1 = (\mathbf{x}_1 \times \dot{\mathbf{x}}_1) \cdot \mathbf{x}_1 = 0.$$

Since $\mathbf{x}_2 = -(m_1/m_2)\mathbf{x}_1$, it is also true that

$$\mathbf{c} \cdot \mathbf{x}_2 = -(m_1/m_2)\mathbf{c} \cdot \mathbf{x}_1 = 0.$$

Hence both P_1 and P_2 remain in the plane $\mathbf{c} \cdot \mathbf{x} = 0$. (We are assuming that at some instant P_1 has motion not in the line P_1P_2, so that $\mathbf{c} \neq \mathbf{0}$.)

Between times t and t + dt the particle P_1 moves from $\mathbf{x}_1(t)$ to $\mathbf{x}_1(t + dt)$, and, except for an error of higher order than dt, the latter is $\mathbf{x}_1(t) + \dot{\mathbf{x}}_1(t)dt$. The area of the parallelogram with vertices at the origin, at $\mathbf{x}(t)$ and at $\mathbf{x}(t) + \dot{\mathbf{x}}(t)dt$ is

$$|\mathbf{x}(t) \times \dot{\mathbf{x}}(t)dt|.$$

But in the time interval from t to t + dt the line segment from 0 to P_1 (the "radius vector") sweeps over a sector which, except for an error of higher order than dt, is a triangle with vertices at 0, at $\mathbf{x}_1(t)$ and at $\mathbf{x}_1(t) + \dot{\mathbf{x}}_1(t)dt$. This triangle consists of one-half of the parallelogram just mentioned. Hence

$$\begin{aligned} &\text{Area swept out between t and t + dt} \\ &= \tfrac{1}{2}|\mathbf{x}(t) \times \dot{\mathbf{x}}(t)dt| \\ &= \tfrac{1}{2}|\mathbf{c}|dt. \end{aligned}$$

So the rate at which the "radius vector" $\overrightarrow{OP_1}$ sweeps out area in the plane is the constant $|\mathbf{c}|/2$. Likewise the "radius vector" $\overrightarrow{OP_2}$ also sweeps over area at a constant rate.

According to the definition, the angular momentum of a body can be found by adding the angular momenta of its several particles. However, for rigid bodies an easier method can be used; the angular momentum can be computed from the components of angular velocity and the moments and products of inertia defined in (11.4). Let B be a rigid body rotating about a point, which for notational convenience we take to be the origin. Let $\mathbf{x}_1$, $\mathbf{x}_2$, ... be the position vectors of the particles P_1, P_2, ... of the body, and let m_1, m_2, ... be their masses. If $\boldsymbol{\omega}$ is the angular velocity of the body, its angular momentum $\mathbf{h}$ is given by

$$(7) \qquad \mathbf{h} = \Sigma m_i \mathbf{x}_i \times \dot{\mathbf{x}}_i.$$

By (7.8) and (5.17), this implies

$$(8) \qquad \begin{aligned} \mathbf{h} &= \Sigma m_i \mathbf{x}_i \times (\boldsymbol{\omega} \times \mathbf{x}_i) \\ &= \Sigma m_i [(\mathbf{x}_i \cdot \mathbf{x}_i) \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \mathbf{x}_i) \mathbf{x}_i]. \end{aligned}$$

If we select an arbitrary set of coordinate vectors $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ forming a right-handed orthogonal system, equation (8) is equivalent to the three "scalar" equations

$$(9) \qquad \begin{aligned} h_1 = & [\Sigma m_i(x_{i2}^2 + x_{i3}^2)] \omega_1 \\ & - [\Sigma m_i x_{i2} x_{i1}] \omega_2 - [\Sigma m_i x_{i3} x_{i1}] \omega_3, \\ h_2 = & - [\Sigma m_i x_{i1} x_{i2}] \omega_1 \\ & + [\Sigma m_i(x_{i1}^2 + x_{i3}^2)] \omega_2 - [\Sigma m_i x_{i3} x_{i2}] \omega_3, \\ h_3 = & - [\Sigma m_i x_{i1} x_{i3}] \omega_1 \\ & - [\Sigma m_i x_{i2} x_{i3}] \omega_2 + [\Sigma m_i(x_{i1}^2 + x_{i2}^2)] \omega_3. \end{aligned}$$

By (11.4), this can be written

$$(10) \qquad \begin{aligned} h_1 &= I_1 \omega_1 - I_{12} \omega_2 - I_{13} \omega_3, \\ h_2 &= - I_{12} \omega_1 + I_2 \omega_2 - I_{23} \omega_3, \\ h_3 &= - I_{13} \omega_1 - I_{23} \omega_2 + I_3 \omega_3. \end{aligned}$$

Since the motion of the center of mass satisfies the very simple equation (9.4), it may reasonably be expected that it will often be important to compute the angular momentum about the center of mass. But in this case we recall that the coordinate vectors

$\mathbf{k}_i$ can be so chosen that the products of inertia all vanish, which simplifies (10). The result is

(11) Theorem. Let a rigid body B have angular velocity ω about its center of mass. Let coordinate vectors $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ be chosen so as to lie along principal axes of inertia. If ω_1, ω_2, ω_3 denote the components of ω, so that

$$\omega = \omega_1 \mathbf{k}_1 + \omega_2 \mathbf{k}_2 + \omega_3 \mathbf{k}_3,$$

the angular momentum is

(12) $$\mathbf{h} = \omega_1 I_1 \mathbf{k}_1 + \omega_2 I_2 \mathbf{k}_2 + \omega_3 I_3 \mathbf{k}_3.$$

13. Centrifugal and Coriolis forces.

Often it is convenient to refer positions of particles to a coordinate system which is not an inertial frame, but has constant angular velocity with respect to an inertial frame. The usual reason for choosing such a system is of course our customary preference for referring terrestrial events to coordinates fixed in the earth, rather than using coordinate systems specified in terms of distant stars. However, if we use such a system we can no longer expect that momentum relative to the system will obey the laws we have listed.

Let O be a point about which the second, or "moving," system rotates. For notational convenience we shall use O as the origin of both the inertial $OX_1X_2X_3$-system and the rotating $OY_1Y_2Y_3$-system. Let ω denote the constant angular velocity of the new system with respect to the old. If $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ are the coordinate vectors of the new system, the coordinate vector $\mathbf{x}(t)$ of any point can be written in the form

(1) $$\mathbf{x}(t) = x_1(t)\mathbf{k}_1 + x_2(t)\mathbf{k}_2 + x_3(t)\mathbf{k}_3.$$

The velocity of the point with respect to the new system is obtained by differentiating (1) under the assumption that the $\mathbf{k}_i$ are constant. This velocity will be denoted by the symbol

$$(2) \quad \mathbf{x}^{(\cdot)}(t) = \dot{x}_1(t)\mathbf{k}_1 + \dot{x}_2(t)\mathbf{k}_2 + \dot{x}_3(t)\mathbf{k}_3.$$

Likewise the acceleration with respect to the new system is

$$(3) \quad \mathbf{x}^{(\cdot\cdot)}(t) = \ddot{x}_1(t)\mathbf{k}_1 + \ddot{x}_2(t)\mathbf{k}_2 + \ddot{x}_3(t)\mathbf{k}_3.$$

On the other hand, the acceleration of the point with respect to the inertial system is, by (2) and (7.16),

$$(4) \quad \begin{aligned} \ddot{\mathbf{x}}(t) = {} & \ddot{x}_1(t)\mathbf{k}_1 + \ddot{x}_2(t)\mathbf{k}_2 + \ddot{x}_3(t)\mathbf{k}_3 \\ & + 2\dot{x}_1(t)\dot{\mathbf{k}}_1 + 2x_2(t)\dot{\mathbf{k}}_2 + 2\dot{x}_3(t)\dot{\mathbf{k}}_3 \\ & + x_1(t)\ddot{\mathbf{k}}_1 + x_2(t)\ddot{\mathbf{k}}_2 + x_3(t)\ddot{\mathbf{k}}_3. \end{aligned}$$

By (7.28),

$$(5) \quad \dot{\mathbf{k}}_i = \omega \times \mathbf{k}_i, \qquad (1 = 1, 2, 3)$$

whence, ω being constant,

$$(6) \quad \begin{aligned} \ddot{\mathbf{k}}_i &= \omega \times \dot{\mathbf{k}}_i \\ &= \omega \times (\omega \times \mathbf{k}_i) \\ &= (\omega \cdot \mathbf{k}_i)\,\omega - (\omega \cdot \omega)\mathbf{k}_i \quad (i = 1, 2, 3). \end{aligned}$$

Substituting (2), (3), (5) and (6) in (4) yields

$$(7) \quad \begin{aligned} \ddot{\mathbf{x}}(t) = {} & \mathbf{x}^{(\cdot\cdot)} + 2\,\omega \times \mathbf{x}^{(\cdot)}(t) \\ & + (\omega \cdot \mathbf{x})\,\omega - (\omega \cdot \omega)\mathbf{x}. \end{aligned}$$

If $\mathbf{x}(t)$ is the position vector of a particle P with mass m, and the particle is acted on by a force $\mathbf{f}$, then $m\ddot{\mathbf{x}}(t) = \mathbf{f}$. So on multiplying both members of (7) by m and transposing we find

$$(8) \quad \begin{aligned} m\mathbf{x}^{(\cdot\cdot)}(t) = {} & \mathbf{f} + m\,[(\omega \cdot \omega)\mathbf{x}(t) - (\omega \cdot \mathbf{x}(t))\omega\,] \\ & - 2m\,\omega \times \mathbf{x}^{(\cdot)}(t). \end{aligned}$$

The first term added to $\mathbf{f}$ in this equation has a simple interpretation. Temporarily, let us denote by $\mathbf{u}$ the unit vector in the direction of ω, so that

$\omega = |\omega| \mathbf{u}$. Then

$$(9) \quad \begin{aligned} &m[(\omega \cdot \omega)\mathbf{x} - (\omega \cdot \mathbf{x})\omega] \\ &= m|\omega|^2[\mathbf{x} - (\mathbf{u}\cdot\mathbf{x})\mathbf{u}]. \end{aligned}$$

It is easy to see that the vectors $\mathbf{x}_1 = (\mathbf{u}\cdot\mathbf{x})\mathbf{u}$ and $\mathbf{x}_2 = \mathbf{x} - (\mathbf{u}\cdot\mathbf{x})\mathbf{u}$ constitute a decomposition of $\mathbf{x}$ into vectors respectively parallel and perpendicular to $\mathbf{u}$; for clearly $\mathbf{x}_1 + \mathbf{x}_2 = \mathbf{x}$, and $\mathbf{x}_1$ is parallel to $\mathbf{u}$ because it is a multiple of $\mathbf{u}$, and $\mathbf{x}_2$ is orthogonal to $\mathbf{u}$ because

$$\mathbf{u}\cdot\mathbf{x}_2 = \mathbf{u}\cdot\mathbf{x} - (\mathbf{u}\cdot\mathbf{x})\mathbf{u}\cdot\mathbf{u} = 0.$$

Hence the quantity in square brackets in the right member of (9) is the component of $\mathbf{x}$ perpendicular to $\mathbf{u}$, or what is the same thing, perpendicular to ω. The length of this component, which we shall denote by r, is the distance from the particle P to the axis of rotation. So the term

$$m[(\omega \cdot \omega)\mathbf{x}(t) - (\omega \cdot \mathbf{x}(t))\omega]$$

in (8) represents a vector of magnitude $m|\omega|^2 r$ in the direction of that perpendicular to the axis of rotation which passes through P. This is the familiar "centrifugal force."

The last term in (8) is less familiar. It is known as the <u>Coriolis</u> <u>force</u>. The introduction of this and the "centrifugal force" may be thought of as the price which we must pay for the luxury of basing our measurements of position on a rotating coordinate system instead of transforming to an inertial frame. To exemplify this, imagine a level merry-go-round rotating counterclockwise with an angular speed Ω about its center O. A circular track runs around this merry-go-round; its center is at O, and its radius will be denoted by r. A car runs counterclockwise about this track with speed v relative to the track. We wish to find the inward force exerted by the track on the wheels of the car; and we shall calculate this from three different points of view. Let us choose the x_3-axis vertically upward. At the instant $t = 0$ at which the force is to be computed, the car C is on a point P of the rails, and also is at a point Q of the inertial frame whose origin is at O. The x_3-axis will pass through O vertically upward. An observer using the inertial frame as a basis for his computations will choose an x_1-axis passing at all times through Q; an observer in the car will choose an x_1-axis passing at all times through C; and an observer who prefers to refer positions to the merry-go-round will choose an x_1-axis passing at all times through P.

The first observer will calculate the force as follows. The line OC makes at time t an angle $(v/r)t$ with OP, which in turn makes an angle Ωt with OQ; so the car has coordinates

$$(r \cos (\Omega + v/r)t,\ r \sin (\Omega + v/r)/t,\ 0.)$$

at time t. Its acceleration vector is

$$\begin{aligned}(&- r(\Omega + v/r)^2 \cos (\Omega + v/r)t, \\ &- r(\Omega + v/r)^2 \sin (\Omega + v/r)t,\ 0)\end{aligned}$$

and if the car's mass is m, the force acting on the car at time $t = 0$ is $(-mr(\Omega + v/r)^2, 0, 0)$.

The second observer will calculate the force as follows. The line OC has angular speed $\Omega + v/r$ about the x_3-axis. In the reference system in which OC is the x_1-axis the car C has constant position vector $\mathbf{x} = (r, 0, 0)$. Also $\boldsymbol{\omega} = (0, 0, \Omega + v/r)$. So (8) reduces to

$$m\mathbf{0} = \mathbf{f} + m[(\Omega + v/r)^2 \mathbf{x} - \mathbf{0}] - 2m\,\boldsymbol{\omega} \times \mathbf{0},$$

whence

$$\mathbf{f} = (-m(\Omega + v/r)^2 r, 0, 0).$$

The third observer will have to make use of the Coriolis force as well as the centrifugal force. His system has angular velocity $\omega = (0, 0, \Omega)$, and in it C has position vector

$$\mathbf{x} = (r \cos (v/r)t, r \sin (v/r)t, 0).$$

Hence (8) takes the form at time $t = 0$

$$m(-v^2/r, 0, 0) = \mathbf{f} + m[\Omega^2(r, 0, 0) - \mathbf{0}] - 2m\,\omega \times (0, v, 0)$$
$$= \mathbf{f} + (m\Omega^2 r, 0, 0) + (2m\Omega v, 0, 0),$$

whence

$$\mathbf{f} = (-m\Omega^2 r - 2m\Omega v - mv^2/r, 0, 0)$$
$$= (-m(\Omega + v/r)^2 r, 0, 0).$$

So all three observers arrive at the same value for $\mathbf{f}$, as clearly they must. But the observer in the car, who used a rotating reference system in which the car is fixed, must compute the centrifugal force; and the observer on the merry-go-round, who uses a rotating coordinate system in which the car is moving, must compute both centrifugal force and Coriolis force.

14. Dimensional analysis.

Most of the quantities of physics are measured by making direct measurements of certain lengths, times and masses, and then forming a certain function of the numbers obtained by the direct measurements. For instance, a velocity is found by directly measuring a length and the time required for a body to traverse that length, and then dividing the number expressing the length by the number expressing the time. This can be expressed symbolically by stating that a velocity has the "dimensions" $[L]/[T]$, or $[L][T]^{-1}$, meaning that a velocity is found by dividing the number expressing a certain length by the number expressing a certain time. An angle, determined by constructing a circle with center at its vertex and dividing the length of subtended arc by the radius, would have dimensions $[L]/[L]$, or $[L][L]^{-1}$.

Henceforth we shall consider only physical quantities expressible in terms of measurements of length, time and mass. For other possible fundamental measurements we refer the reader to P. W. Bridgman's Dimensional Analysis (New Haven, Conn.: Yale University Press, 1931) to which the following exposition is greatly indebted. (The present section and the next, however, contain as much dimensional analysis as we shall need in this book.)

A first, rather simple, use of dimension theory occurs in connection with change of units. Suppose that we have chosen certain units of length, time and mass, and that a physical quantity is determined by the following procedure.

a. Measure the lengths of a certain set of lines; let $\ell_1, \ldots, \ell_m$ be the numbers expressing these lengths in terms of the chosen unit of length.

b. Measure a certain set of time intervals; let $t_1, \ldots, t_n$ be the numbers expressing these time intervals in terms of the chosen unit of time.

c. Measure a certain set of masses; let $m_1, \ldots, m_p$ be the numbers expressing these masses in terms of the chosen unit of mass.

d. Form the power-product

$$(1) \quad \ell_1^{a_1} \ldots \ell_m^{a_m} t_1^{b_1} \ldots t_n^{b_n} m_1^{c_1} \ldots m_p^{c_p},$$

where the a_i, b_i, c_i are preassigned constant exponents, not necessarily integral and not necessarily positive.

Now let us change to a new system of units in which the new length unit is $1/\lambda$ times the old length unit, the new time unit is $1/\tau$ times the old time unit, and the new mass unit is $1/\mu$ times the old mass unit, τ, μ, λ being positive numbers. The first measured length was χ_1 of the old units, hence $\chi_1 \lambda$ of the new, and so on; the power-product corresponding to (1) in terms of the new numbers is

$$\{ (\chi_1\lambda)^{a_1} \dots (\chi_m\lambda)^{a_m} (t_1\tau)^{b_1} \dots (t_n\tau)^{b_n}$$
$$(m_1\mu)^{c_1} \dots (m_p\mu)^{c_p} \}$$

$$(2) \quad = \lambda^{a_1 + \dots + a_m} \tau^{b_1 + \dots + b_n} \mu^{c_1 + \dots + c_p}$$
$$\cdot(\chi_1^{a_1} \dots \chi_m^{a_n} t^{b_1} \dots t_n^{b_n} m^{c_1} \dots m_p^{c_p}).$$

Thus the effect of the change of units is to multiply the quantity (1) by

$$(3) \quad \lambda^{a_1 + \dots + a_m} \tau^{b_1 + \dots + b_n} \mu^{c_1 + \dots + c_p}$$

According to the notational convention in the first paragraph of this section, the quantity (1) would have the dimensions

$$(4) \quad [L]^{a_1} \dots [L]^{a_m} [T]^{b_1} \dots [T]^{b_n} [M]^{c_1} \dots [M]^{c_p}.$$

But by (3), we would arrive at the same factor for change of units if we ascribed it the dimensions

$$(5) \quad [L]^{a_1 + \dots + a_m} [T]^{b_1 + \dots + b_n} [M]^{c_1 + \dots + c_p}.$$

The condensed form (5) is less informative than the expanded form (4), for the latter sets forth something of the history of the operations (a), (b), (c) involved in its computation. But it is equally serviceable for finding the effect of change of units, and in fact for all the needs of the present book; and so it will be used henceforth. For example, the symbol [L]/[L] for the dimensions of angle, which tells us that an angle is found by dividing a length by a length, will be abandoned in favor of the symbol $[L]^0$, which tells us a little more obviously that the number expressing the measure of an angle does not change when the unit of length is changed. If we wish, we may also write the dimensions of an angle as $[L]^0[T]^0[M]^0$, or we may say that angle has dimensions zero, or is dimensionless.

Since by adopting the symbolism (5) we abandon the operational viewpoint in favor of convenience in changing units, we shall be consistent and re-phrase the definition of dimension too. A physical quantity will be said to have dimensions $[L]^a[T]^b[M]^c$ if, when the old units of length, time and mass are replaced by new units respectively $1/\lambda$, $1/\tau$ and $1/\mu$ times as great. the number expressing the physical quantity is multiplied by $\lambda^a\ \tau^b\ \mu^c$. This leads to some obvious theorems.

(6) *Theorem. If a physical quantity Q is the sum of a finite number or of a convergent infinite series of quantities each of the same dimension $[L]^a[T]^b[M]^c$, then Q also has those dimensions.* For under the change of units just specified each summand is multiplied by $\lambda^a\ \tau^b\ \mu^c$, so their sum is also multiplied by this factor.

(7) *Theorem. If P and Q are physical quantities having the respective dimensions $[L]^a[T]^b[M]^c$ and $[L]^x[T]^y[M]^z$, then the product PQ has the dimensions*

$$[L]^{a+x}[T]^{b+y}[M]^{c+z},$$

the quotient P/Q has the dimensions

$$[L]^{a-x}[T]^{b-y}[M]^{c-z},$$

and the power P^k has the dimensions

$$[L]^{ak}[T]^{bk}[M]^{ck}.$$

For under the change of units previously specified, P is multiplied by $\lambda^a \tau^b \mu^c$ and Q by $\lambda^x \tau^y \mu^z$, so PQ is multiplied by $\lambda^{a+x} \tau^{b+y} \mu^{c+z}$ and P/Q is multiplied by $\lambda^{a-x} \tau^{b-y} \mu^{c-z}$. The discussion of P^k is equally obvious.

(8) <u>Theorem</u>. <u>If</u> P <u>is</u> <u>the</u> <u>limit</u> <u>of</u> <u>a</u> <u>sequence</u> <u>of</u> <u>physical</u> <u>quantities</u>,

$$P = \lim_{n\to\infty} P_n$$

<u>and</u> <u>each</u> P <u>has</u> <u>dimensions</u> $[L]^a [T]^b [M]^c$, <u>then</u> P <u>also</u> <u>has</u> <u>those</u> <u>same</u> <u>dimensions</u>. For under the change of units previously specified each P_n is multiplied by the same factor $\lambda^a \tau^b \mu^c$, hence the limit P_n is also multiplied by this factor.

It is profitable to consider certain examples now before proceeding further with the theory. The area of a rectangle is found by multiplying the numerical measures of its base and its altitude, hence by Theorem (7) has dimensions $[L]^2$. (We thus condense the fuller form $[L]^2[T]^0[M]^0$). The area of a region consisting of a finite number of rectangles also has dimensions $[L]^2$, by Theorem (6). The area of an arbitrary region of the plane can be found as the limit of the areas of a sequence of regions approximating it from within, each of these regions being a finite sum of rectangles. So this too has dimensions $[L]^2$. The area of a polyhedron is the sum of the areas of its plane faces, so has dimensions $[L]^2$. The area of a general curved surface is the limit of the areas of a sequence of approximating polyhedra, so has dimensions $[L]^2$.

By a similar argument we find that volumes have dimension $[L]^3$.

If the position vector of a particle changes from $\mathbf{x}$ at time t to $\mathbf{x} + \Delta\, \mathbf{x}$ at time $t + \Delta t$, the ratio of $|\Delta \mathbf{x}|/\Delta t$ has dimensions $[L][T]^{-1}$. By Theorem (8), so has its limit as $\Delta t \to 0$, which is the speed of the particle at time t. The same is true, by a similar argument, for each of the components $\dot{x}_1$, $\dot{x}_2$, $\dot{x}_3$ of the velocity. If we wish, we may say that the velocity $\dot{\mathbf{x}}$ has dimensions $[L][T]^{-1}$, since each component has these dimensions. In fact, we could perfectly well have extended the definition before Theorem (6) to include vector-valued functions too; we would have obtained the same dimensions for $\dot{\mathbf{x}}$. Since momentum is mass times velocity, its dimensions are $[L][T]^{-1}[M]$.

If a particle has velocity $\dot{\mathbf{x}}$ at time t and velocity $\dot{\mathbf{x}} + \Delta \dot{\mathbf{x}}$ at time $t + \Delta t$, each ratio $\Delta \dot{x}_i / \Delta t$ has dimensions $[L][T]^{-2}$, by Theorem (7). So has the limit $\ddot{x}_i$ as $\Delta t \to 0$, by Theorem (8). Thus each component of the acceleration $\ddot{\mathbf{x}}$ has dimensions $[L][T]^{-2}$. We may say that $\ddot{\mathbf{x}}$ has these dimensions.

Since force is mass times acceleration, its dimensions are $[L][T]^{-2}[M]$.

The mean density of a material object is the ratio of its mass to its volume, hence by Theorem (7) has dimensions $[L]^{-3}[M]$. The density at a point is the limit of the mean density in a cube centered at the point as the edge of the cube approaches zero, so by Theorem (8) it has dimensions $[L]^{-3}[M]$.

In the definition of work, each dot product $\mathbf{f}_i \cdot \Delta \mathbf{x}_i$ has dimensions $[L]^2[T]^{-2}[M]$, by Theorem (7). By Theorem (6) the scalar product has these dimensions, and so has the sum $(\Sigma\, \mathbf{f}_i \cdot \Delta \mathbf{x}_i)$. By Theorem (8) so has the limit, which is defined to be the work.

Since the kinetic energy T is $mv^2/2$, by Theorem (7) its dimensions are $[L]^2[T]^{-2}[M]$, which are the same as those of work. We could have foreseen this from Theorem (10.6).

The rate of flow of a fluid is found by measuring the volume Δv which passes a cross-section between time t and time $t + \Delta t$, dividing Δv by Δt and letting $\Delta t \to 0$. Hence it has dimensions $[L]^1[T]^{-1}$.

Suppose that a plane surface of area A slides in its own plane with velocity v, the plane being parallel to a flat bounding surface of a vessel filled with a fluid; the other distance to this surface will be called d, and the other boundaries will be supposed far enough away to produce no perceptible effect. The force f exerted by the fluid on the moving surface is known to be proportional to the area A and the velocity v, and inversely proportional to the distance d; $f = \mu Av/d$. The coefficient of proportionality μ depends on the fluid, and is called the coefficient of viscosity. Since $\mu = fd/Av$, it has dimension

$$([L]\,[T]^{-2}[M])([L])/[L]^2([L]\,[T]^{-1}) = [L]^{-1}[T]^{-1}[M].$$

Every physical quantity with which we shall deal is intimately associated with a real-valued function of several real variables,

$$F(\ell_1, \ldots, \ell_m, t_1, \ldots, t_n, m_1, \ldots, m_p).$$

For instance, a velocity is associated with $F = \ell_1/t_1$, distance and a certain time (the time taken by a body in moving that distance). In the simplest case, a length ℓ is measured (along the path of a body) and a time t is measured (the time the body takes to move through the distance). Then ℓ is substituted for ℓ_1 and t for t_1 in F; the corresponding value of F is the mean velocity of the body during the time of the experiment. This simplest case is in fact all that will ever actually occur. But in our mathematical model of the universe we idealize the situation by imagining that the mean velocity can be measured over each of an infinite sequence of intervals containing a point P and with lengths approaching zero. The limit of the mean velocities — that is, the limit of the values of F — is the instantaneous velocity at P. As another example, area is associated with $F = \ell_1\ell_2$. The area of a rectangle is found by measuring its two dimensions and substituting in F. The area of a figure consisting of finitely many rectangles is obtained by adding the several values of F, that is, by adding their several areas. The area of a general plane figure is found as the limit of areas of figures consisting each of a finite number of rectangles; hence it is the limit of a sequence of numbers, each of which is a sum of finitely many values of F. We shall make the following assumption.

(9) To each physical quantity P hereafter mentioned there corresponds to a real-valued function

$$F(\ell_1, \ldots, \ell_m, t_1, \ldots, t_n, m_1, \ldots, m_p),$$

continuous and having continuous first partial derivatives for all positive values of $\ell_1, \ldots, m_p$; and the value of the physical quantity P corresponding to a physical system is found by one of these three methods:

(i) A specified set of m lengths, n times and p masses are measured, and their values substituted as arguments in the function

$$F(\ell_1, \ldots, \ell_m, t_1, \ldots, t_n, m_1, \ldots, m_p).$$

(ii) Finitely many sets of measurements of m lengths, n times and p masses are made, each as specified in (i); each set separately is used as arguments in F, and the several values of F thus obtained are added.

(iii) The operations described in (i) and (ii) are performed on each of an infinite sequence of physical systems (which may all be copies of one system). The corresponding values of F form a sequence of numbers; the limit of this sequence is found.

Clearly we could extend this list if we wished. For example we could envisage an infinite sequence of infinite sequences of systems, apply (9iii) to each system, and take the limit of the resulting sequence of limits. But we do not need this. In fact, in proving the two principal theorems of dimension theory ((10) and (15.8)) we need refer only to (9i). Only (9i) and (9ii) can actually be carried out by an experimenter; (9iii) is an idealization from the world of experiment to the idealized mathematical model which is the basis of classical mechanics.

If the function F is a product of powers of the ℓ_i, t_i and m_i (for which we use the notation (1)), then as already shown, the physical quantity P will have dimensions

$$[L]^{a_1 + \dots + a_m} [T]^{b_1 + \dots + b_n} [M]^{c_1 + \dots + c_p},$$

provided that the evaluation is effected as in (9i). But then by (6) and (8), P will still have these dimensions if evaluated by (9ii) or (9iii). Here we have used detailed knowledge of the structure of the function F to show that the physical quantity P has dimensions of the type $[L]^a [T]^b [M]^c$. Our next theorem will show that whenever the quantity P is of a kind that we might vaguely call "pure" (we shall be less vague in a moment) the same conclusion may be drawn, even if we know nothing of the structure of the function F. As an example of a physical quantity which is not "pure," but is a "hybrid," we take the quantity (area + perimeter) of a geometric figure. This is formed according to (9), and yet is clearly an inconvenient sort of quantity to deal with. The vague distinction between "pure" and "hybrid" quantities can be made quite precise by means of a criterion which refers only to the values of P, and not at all to the analytic nature of the function F. This criterion is called <u>absolute significance of ratio</u>. Consider, for example, a particle moving with respect to a reference system. The number expressing its speed depends on the units chosen; if its speed is 60 miles per hour, it is 88 feet per second. But if one body has twice the speed of another when one unit system is used, it has twice the speed of the other when any other system is used. The ratio of speeds 2:1 has "absolute significance" in that it is unchanged when units change. The hybrid "area + perimeter" lacks this property; two squares of side 1 foot and 2 feet have measures 5 and 12 when the unit of length is the foot, 192 and 672 when the unit of length is the inch, and the ratios 5:12 and 192:672 are different.

The physical quantities having absolute significance of ratio also possess a highly important and useful property; the dimensions must be expressible as powers, as the next theorem shows.

(10) Theorem. *If P is a physical quantity formed according to the rules (9) and possessing absolute significance of ratio, there are constants* a, b, c *such that P has dimensions* $[L]^a[T]^b[M]^c$.

If P has absolute significance of ratio for all entities measured by (9), in particular it has absolute significance of ratio for those entities which can be measured by (9i) alone. For all positive values of ℓ_i, t_i and m_i we can define a new function

$$(11)\quad \begin{aligned} &f(\ell, t, m, \alpha_2, \ldots, \alpha_m, \beta_2, \ldots, \beta_n, \gamma_2, \ldots, \gamma_p) \\ &= F(\ell_1, \ldots, \ell_m, t_1, \ldots, t_n, m_1, \ldots, m_p), \end{aligned}$$

where

$$(12)\quad \begin{aligned} &\ell = \ell_1,\ t = t_1,\ m = m_1, \\ &\alpha_i = \ell_i/\ell_1,\ \beta_i = t_i/t_1,\ \gamma_i = m_i/m_1. \end{aligned}$$

Suppose now that measurements of the physical quantity P are carried out according to (9i) on two entities, the first leading to numbers $\ell_1, \ldots, m_p$ and the second to numbers $\bar{\ell}_1, \ldots, \bar{m}_p$. There is a number k such that

$$(13)\quad \begin{aligned} &F(\ell, t, m, \alpha_2, \ldots, \alpha_m, \beta_2, \ldots, \beta_n, \gamma_2, \ldots, \gamma_p) \\ &= kf(\bar{\ell}, \bar{t}, \bar{m}, \bar{\alpha}_2, \ldots, \bar{\alpha}_m, \\ &\qquad \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{\gamma}_2, \ldots, \bar{\gamma}_n). \end{aligned}$$

(We assume we are working in the region $f \neq 0$.) If we change to new units of length, time and mass which are respectively $1/\lambda$, $1/\tau$, $1/\mu$ times the old units, the numerical measures χ_i, t_i, m_i are replaced by $\lambda\chi_i$, τt_i, μm_i respectively. By (12), χ, t, and m are replaced by $\lambda\chi$, τt, μm respectively, while the α_i, β_i, γ_i are unchanged. But by hypothesis, the ratio k must not change. Hence

$$f(\lambda\chi, \tau t, \mu m, \alpha_2, \ldots, \alpha_m, \beta_2, \ldots, \beta_n, \gamma_2, \ldots, \gamma_p)$$

$$(14) \quad = kf(\lambda\bar{\chi}, \tau\bar{t}, \mu\bar{m}, \bar{\alpha}_2, \ldots, \bar{\alpha}_m, \bar{\beta}_2, \ldots, \bar{\beta}_n, \bar{\gamma}_2, \ldots, \bar{\gamma}_p).$$

Let us denote the partials of the function f in (12) with respect to first, second and third arguments by f_χ, f_t, f_m respectively. Since (14) holds for all λ, τ and μ, we can differentiate with respect to λ and then set $\lambda = \tau = \mu = 1$, obtaining

$$(15) \qquad \chi f_\chi = k \bar{\chi} \bar{f}_\chi,$$

where for brevity f_χ means $f_\chi(\chi, t, m, \ldots, \gamma_p)$ and $\bar{f}_\chi$ means the corresponding derivatives for the barred arguments. But if we solve (13) for k and substitute in (15) we obtain

$$(16) \qquad \chi f_\chi/f = \bar{\chi} \bar{f}_\chi/\bar{f},$$

In a similar way, by differentiation in (14) with respect to τ and μ we find

$$(17) \qquad tf_t/f = \bar{t}\bar{f}_t/\bar{f},$$

$$(18) \qquad mf_m/f = \bar{m}\bar{f}_m/\bar{f}.$$

In other words, for all entities for which the physical quantity P can be measured by (9i), the expression $\chi f_\chi/f$ has one and the same value, which we may denote by a; and analogously for tf_t/f and mf_m/f.

Thus there are three constants a, b, c such that for all entities for which the physical quantity P can be formed by (9i), the equations

$$\chi f_\chi / f = a, \qquad (19)$$
$$t f_t / f = b,$$
$$m f_m / f = c$$

hold.

The quantum theory warns us that we had better not assume that the entities for which P can be formed constitute a continuous array. But in (19) we made no such assumption; these equations hold whenever P has meaning. If there are entities for which P has no meaning, there is no harm in making the mathematical assumption that (19) holds for them, too; this will never lead to contradiction, since the physical situation will never occur anyhow. So we may assume that (19) holds for all positive values of χ, t, m, α_2, etc.

Now we define

$$\phi(\chi, t, m, \alpha_2, \ldots, \gamma_p) = \log f(\chi, t, m, \alpha_2, \ldots, \gamma_p). \qquad (20)$$

Then (19) implies

$$\phi_\chi = a/\chi, \quad \phi_t = b/t, \quad \phi_m = c/m. \qquad (21)$$

Let χ_0, t_0, m_0, α_2, ..., γ_p be any positive numbers. To be specific, we shall choose $\chi_0 = t_0 = m_0 = 1$. For any other positive numbers χ, t, m (the α_2, etc., being unchanged) the points

$$(1, 1, 1, \alpha_2, \ldots, \gamma_p)$$

and

$$(\chi, t, m, \alpha_2, \ldots, \gamma_p)$$

can be joined by a smooth curve in (m + n + p)-dimensional space along which the first three coordinates χ, t, m stay positive and α_2, etc., remain constant.

Then by integrating along the curve

$$\phi(\ell, t, m, \alpha_2, \ldots, \gamma_p) - \phi(1, 1, 1, \alpha_2, \ldots, \gamma_p)$$

$$= \int_{(1,\,1,\,1)}^{(\ell,\,t,\,m)} \{\phi_\ell\, d\ell + \phi_t\, dt + \phi_m\, dm\}$$

$$= \int_{(1,\,1,\,1)}^{(\ell,\,t,\,m)} \{a\, d\ell/\ell + b\, dt/t + c\, dm/m\}$$

$$= a \log \ell + b \log t + c \log m,$$

whence

$$\phi(\ell, t, m, \alpha_2, \ldots, \gamma_p) = \log(\ell^a t^b m^c) - \phi(1, 1, 1, \alpha_2, \ldots, \gamma_p).$$

So by (20)

$$f(\ell, t, m, \alpha_2, \ldots, \alpha_m, \beta_2, \ldots, \beta_n, \gamma_2, \ldots, \gamma_p) = \psi(\alpha_2, \ldots, \gamma_p)\ell^a t^b m^c, \tag{22}$$

where ψ is a function

$$\psi(\alpha_2, \ldots, \gamma_p) = e^{-\phi(1,\,1,\,1,\,\alpha_2,\,\ldots\,\gamma_p)},$$

of the ratios $\alpha_2, \ldots, \gamma_p$ alone.

Suppose now that we change to new units of length, time and mass which are respectively $1/\lambda$, $1/\tau$ and $1/\mu$ times as great as the old ones. Then, as previously remarked, ℓ, t and m are replaced by $\lambda\ell$, τt, μm respectively, while α_2, etc., are unchanged. Hence in (22) the factor ψ is unchanged, and f is multiplied by $\lambda^a \tau^b \mu^c$. This proves that the quantity P, whose numerical measure is f, has dimensions $[L]^a[T]^b[M]^c$.

As yet, this applies only to entities for which P can be formed by (9i) alone. But it remains true for those whose formation requires (9ii, iii) also, as we see at once with the help of Theorems (6) and (8).

Nothing has been said about the nature of the function ψ. Nothing can be said, apart from obvious remarks about continuity, without further hypotheses. Specifically, it does not have to be a power-product. For example, if we are dealing with triangles their areas may be calculated by the formula

$$A = \tfrac{1}{4}\{(\chi_1 + \chi_2 + \chi_3)(\chi_1 + \chi_2 - \chi_3)(\chi_1 - \chi_2 + \chi_3)(\chi_2 + \chi_3 - \chi_1)\}^{\frac{1}{2}}$$

$$= \tfrac{1}{4}\chi^2\{(1 + \alpha_2 + \alpha_3)(1 + \alpha_2 - \alpha_3)(1 - \alpha_2 + \alpha_3)(\alpha_2 + \alpha_3 - 1)\}^{\frac{1}{2}}.$$

where $\chi = \chi_1$, $\alpha_2 = \chi_2/\chi_1$, $\alpha_3 = \chi_3/\chi_1$, and χ_1, χ_2, χ_3 are the three sides of the triangle. Here ψ is the expression given in the form of a square root, and is not a power-product. Generally, the form (22) for f is enough to insure absolute significance of ratio, without further hypotheses.

It is entirely possible to write valid equations in which not all terms are of the same dimensions. For instance, an ellipse E and a rectangle R may have the same perimeter and also the same area, in which case the equation

perimeter of E + area of E = perimeter of R + area of R

is valid for all systems of units. However, any such equation can be made to yield several dimensionally homogeneous equations; in the example just considered,

perimeter of E = perimeter of R

and

area of E = area of R.

For let all terms of the equation be transposed to the left and all terms of like dimensions grouped together. We then have an equation of the form

$$P_1 + \dots + P_n = 0.$$

Under the usual change of units this becomes

$$\lambda^{a_1} \tau^{b_1} \mu^{c_1} P_1 + \dots + \lambda^{a_n} \tau^{b_n} \mu^{c_n} P_n = 0,$$

where $[L]^{a_i} [T]^{b_i} [M]^{c_i}$ are the dimensions of P_i. If we divide through by the lowest powers of λ, τ and μ occurring in this equation we obtain a polynomial whose value is identically zero. Hence, by a well-known theorem, all the coefficients $P_1, \dots, P_n$ must be zero, and we obtain the separate dimensionally homogeneous equations

$$P_1 = 0, \dots, P_n = 0.$$

This result is often useful in furnishing a quick check on computations. For example, if in the course of a computation we find that we have written

$$dx/dt = v_0 + gt^2,$$

we know that we have made a mistake somewhere, for dx/dt and v_0 have dimensions $[L]/[T]$, while gt^2 has dimensions $[L]$.

15. The Buckingham Π-theorem.

It frequently happens that a physical quantity of known dimensions can be shown to have a numerical value which is a certain function of the numerical values of certain other physical quantities. The usefulness of this will be shown in Theorem (8), but first we will look at an example.

Suppose that a pendulum, consisting of a particle of mass m attached to the end of a weightless string of length l, is displaced, held at rest and dropped.

We shall accept as known, to save trouble, that its motion is in a plane, and we shall locate the particle by measuring its distance s along the arc it traverses, $s = 0$ being the lowest point and s being positive on one side and negative on the other. Let s_0 be the location of release. Since the tension in the string does no work, being perpendicular to the motion, (see (10.4)), the gain in kinetic energy $(m\dot{s}^2/2) - 0$ must equal the loss in potential energy

$$(m\dot{s}^2/2) = mg\ell(\cos s/\ell - \cos s_0/\ell),$$

or

(1) $$\dot{s} = \pm \sqrt{2g\ell\left[\cos (s/\ell) - \cos (s_0/\ell)\right]}.$$

This equation possesses a solution $s = \phi(t, g, \ell, s_0)$, and it is not difficult to show that it must be periodic. Taking time $t = 0$ at release, the period T is the least positive solution of the equation

(2) $$s_0 = \phi(T, G, \ell, s_0)$$

hence depends on g, ℓ and s_0:

(3) $$T = F(g, \ell, s_0).$$

The equation (1) has two distinct kinds of invariance. Since s/ℓ and s_0/ℓ are dimensionless, by Theorems (14.7) and (14.8) both members of (1) have the same dimensions $[L][T]^{-1}$. Hence if a given set of physical quantities $\dot{s}$, s, s_0, ℓ, g have numerical values in one system of units which satisfy (1), under a change of units the same physical quantities will have new numerical values which still satisfy (1). Here we have kept the actual physical entities unchanged, but the numbers expressing their measures have changed because of the change of units. On the other hand, (1) is a relation between the <u>numbers</u> expressing the speed $\dot{s}$, the lengths s, s_0 and ℓ, and the acceleration g. If, for example, two pendulums of different lengths are swinging in different places, and for the first pendulum we have $s_0 = \pi/3$, $\ell = 3$, $s = 0$, $g = 12$ in some system of units, we shall have $\dot{s} = 6$ in that system of units; and if for the second pendulum we have $s_0 = \pi/3$, $\ell = 3$, $\dot{s} = 0$, $g = 12$ in some system of units different from the first, we will again have $\dot{s} = 6$, this time in terms of the new unit system. As a consequence, the solution (2) of (1) is also a function of the numerical measures of t, g, ℓ and s_0, and the numerical measure T of the period is a function of the numerical measures g, ℓ, s_0 of gravitational acceleration, length and initial displacement, the functional relation (3) being independent of the unit system chosen.

The content of Theorem (8) can be better understood if we first discuss power-products of dimensional quantities. Suppose that $v_1, \ldots, v_p$ are physical quantities, and that v_i has dimensions

$$[L]^{a_i}[T]^{b_i}[M]^{c_i}.$$

Then if $k_1, \ldots, k_p$ are real numbers, by Theorem (14.7) the power-product

$$v_1^{k_1} v_2^{k_2} \ldots v_p^{k_p}$$

has dimensions

$$[L]^{a_1k_1+\ldots+a_pk_p}[T]^{b_1k_1+\ldots+b_pk_p}[M]^{c_1k_1+\ldots+c_pk_p}.$$

In particular this power-product will be dimensionless if the three equations

$$(4) \qquad \begin{aligned} a_1k_1 + \ldots + a_pk_p &= 0, \\ b_2k_2 + \ldots + b_pk_p &= 0, \\ c_3k_3 + \ldots + c_pk_p &= 0 \end{aligned}$$

are satisfied.

Equations (4) always have the trivial solution $k_1 = \dots = k_p = 0$. They may also have other (non-trivial) solutions. If they have a non-trivial solution they have infinitely many, for if $k_1, \dots, k_p$ is a solution of (4) so is $hk_1, \dots, hk_p$ for every real number h. However, here again the concept of linear independence (explained in Section 3) is useful; its extension from three-dimensional space to p-dimensional space being entirely straightforward. The number of linearly independent solutions of (4) is known by virtue of a well-known theorem of algebra.* If we select from the array

$$
\begin{array}{l}
a_1, a_2, \dots, a_p \\
b_1, b_2, \dots, b_p \\
c_1, c_2, \dots, c_p
\end{array}
\tag{5}
$$

the minor of highest order whose determinant is different from zero, the order r of this determinant is called the <u>rank</u> of the matrix of coefficients. Then it is well known that it is possible to find p - r linearly independent solutions of (4), but that no larger set of linearly independent solutions can be found. For example, suppose that v_1, v_2, v_3 have the dimensions

$$[L]^2[T]^{-2}[M]^1, \quad [L]^3[T]^{-1}[M]^0, \quad \text{and } [L]^{-1}[T]^{-1}[M]^1$$

respectively. (As we saw in Section 14, these are the dimensions of work, flow and coefficient of viscosity respectively.) Then

$$
\begin{array}{lll}
a_1 = 2, & a_2 = 3, & a_3 = -1 \\
b_1 = -2, & b_2 = -1, & b_3 = -1 \\
c_1 = 1, & c_2 = 0, & c_3 = 1.
\end{array}
$$

---

*See, for example, M. Bôcher, <u>Introduction to Higher Algebra</u> (New York: The Macmillan Company, 1938), p.50.

The matrix of coefficients has non-vanishing minors of order 2, for example the lower-left minor

$$\begin{vmatrix} -2 & -1 \\ 1 & 0 \end{vmatrix},$$

but the (one and only) minor of order three has value 0, so the rank is 2. Hence equations (4) have a linearly independent solution, for example

$$k_1 = -1,\ k_2 = 1,\ k_3 = 1,$$

but no more. By Theorem (3.4) and Corollary (3.7) every solution of (4) is a multiple of this one. In terms of power-products, $v_1^{-1} v_2 v_3$ is a dimensionless power-product, and every dimensionless power-product is a power of this one.

A set of p - r linearly independent solutions of (4) is called a maximal set; every solution of (4) is a linear combination of these. If

$$(6)\qquad \begin{array}{c} k_{1,1}, \ldots, k_{p,1}, \\ \cdots\cdots\cdots\cdots\cdots, \\ k_{1,p-r}, \ldots, k_{p,p-r} \end{array}$$

constitute such a maximal set of linearly independent solutions of (4), each of the power-products

$$(7)\qquad \begin{array}{c} \Pi_1 = v_1^{k_{1,1}} \cdots v_p^{k_{p,1}}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \Pi_{p-r} = v_1^{k_{1,p-r}} \cdots v_p^{k_{p,p-r}} \end{array}$$

has dimensions zero, and every zero-dimensional power-product of the v_i can be represented as a power-product of $\Pi_1, \ldots, \Pi_{p-r}$. Such a set of power-products will be called a maximal independent set of power-products of dimension zero.

We can now state the Buckingham Π-theorem.

(8) Theorem. Let P be a physical quantity having absolute significance of ratio, whose numerical measure is a continuously differentiable function of the numerical measures of certain other physical quantities $v_1, \ldots, v_p$ which also possess absolute significance of ratio. Then

(i) there exists a power-product

(9) $$\Pi_o = v_1^{k_{1,o}} \cdots v^{k_{p,o}}$$

of $v_1, \ldots, v_p$ which has the same dimensions as P;

(ii) given any set of power-products

(10) $$\Pi_i = v_1^{k_{1,i}} \cdots v_p^{k_{p,i}} \quad (i = 0, 1, \ldots, p - r)$$

in which Π_o has the dimensions of P and

$$\Pi_1, \ldots, \Pi_{p-r}$$

form a maximal independent set of dimensionless power-products of the v_i, there is a function G of the dimensionless power-products $\Pi_1, \ldots, \Pi_{p-r}$ alone such that

(11) $$P = \Pi_o G(\Pi_1, \ldots, \Pi_{p-r}).$$

By hypothesis, there is a continuously differentiable function f such that

(12) $$P = f(v_1, \ldots, v_p).$$

Let r be the rank of the array (15.5), where as before we understand that v_i has dimensions

(13) $$[L]^{a_i}[T]^{b_i}[M]^{c_i}.$$

Then, as we have already seen, there is a set of p - r power-products forming a maximal independent set of power-products of dimension zero; and every such set will contain exactly p - r power-products. We let $\Pi_1, \ldots, \Pi_{p-r}$ be any such set, using the notation (7) or (10).

Since $\Pi_1, \ldots, \Pi_{p-r}$ form an independent set, the equations

$$(14)\qquad \begin{array}{c} k_{1,1}c_1 + \cdots + k_{1,p-r}c_{p-r} = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ k_{p,1}c_1 + \cdots + k_{p,p-r}c_{p-r} = 0 \end{array}$$

have no solution except $c_1 = \cdots = c_{p-r} = 0$. That is, these equations have 0 linearly independent solutions. So by the theorem of algebra already used, the rank of the matrix

$$(15)\qquad \begin{array}{c} k_{1,1}, \ldots, k_{1,p-r} \\ \cdots\cdots\cdots\cdots\cdots \\ k_{p,1}, \ldots, k_{p,p-r} \end{array}$$

must be the same as the number p - r of unknowns c_i. That is, the array must contain a non-vanishing minor of order p - r. There is no loss of generality in supposing that this minor consists of the last p - r rows of the array (15), since the order of these rows can be changed by merely interchanging subscripts on the v_i. Then by writing the last r columns of the array

$$(16)\qquad \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0 \end{array}$$

to the right of array (15) (remember that r, the rank of (5), must be either 0, 1, 2 or 3) we obtain a square array

$$
(17) \qquad \begin{matrix} k_{1,1}, \ldots, k_{1,p} \\ \ldots\ldots\ldots\ldots\ldots \\ k_{p,1}, \ldots, k_{p,p} \end{matrix}
$$

with determinant different from zero.

If we define

$$
(18) \qquad \Pi_i = v_1^{k_{1,i}} \ldots v_p^{k_{p,i}} \quad (i = 1, \ldots, p)
$$

we have a set of power-products including the set (7) and r others. Evidently the values of these Π_i are determined by the values of the v_i. The converse is also true. Every power-product

$$
(19) \qquad v_1^{n_1} \ldots v_p^{n_p}
$$

(in particular, each v_i) is representable as a power-product

$$
(20) \qquad \Pi_1^{h_1} \ldots \Pi_p^{h_p}.
$$

For this requires only that the h_i be so chosen that the product

$$
\Pi_1^{h_1} \ldots \Pi_p^{h_p}
$$

$$
= v_1^{k_{1,1}h_1 + \ldots + k_{1,p}h_p} \ldots v_p^{k_{p,1}h_1 + \ldots + k_{p,p}h_p}
$$

be identically the same as (19), and this is always possible, since the equations

$$
\begin{matrix} k_{1,1}h_1 + \ldots + k_{1,p}h_p = n_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ k_{p,1}h_1 + \ldots + k_{p,p}h_p = n_p \end{matrix}
$$

have solutions, the determinant of (17) being different from zero. Hence the v_i determine and are determined by the values of the Π_i, and P, which, by (12) is a function of the v_i, is also a function H of the Π_i:

(21) $$P = H(\Pi_1, \ldots, \Pi_p).$$

By (13) and (18), Π_i has dimensions

(22) $$[L]^{\alpha_i}[T]^{\beta_i}[M]^{\gamma_i},$$

where

(23) $$\begin{aligned}\alpha_i &= k_{1,i}a_1 + \ldots + k_{p,i}a_p,\\ \beta_i &= k_{1,i}b_1 + \ldots + k_{p,i}b_p,\\ \gamma_i &= k_{1,i}c_1 + \ldots + k_{p,i}c_p \quad (i = 1, \ldots, p).\end{aligned}$$

Since P has absolute significance of ratio, by Theorem (14.10) it too has certain dimensions, say

(24) $$[L]^a[T]^b[M]^c.$$

Now let us make the usual change of units, replacing the units of length, time and mass by new units respectively $1/\lambda$, $1/\tau$ and $1/\mu$ times as great. The power-product Π_i is multiplied by the factor

$$\lambda^{\alpha_i}\tau^{\beta_i}\mu^{\gamma_i}$$

by (22) while P is multiplied by $\lambda^a\tau^b\mu^c$. So by (21)

(25) $$\begin{aligned}&H(\lambda^{\alpha_1}\tau^{\beta_1}\mu^{\gamma_1}\Pi_1, \ldots, \lambda^{\alpha_p}\tau^{\beta_p}\mu^{\gamma_p}\Pi_p)\\ &= \lambda^a\tau^b\mu^c H(\Pi_1, \ldots, \Pi_p).\end{aligned}$$

If we differentiate both members of this equation with respect to λ and then set $\lambda = \tau = \mu = 1$, we obtain the first of the following equations, wherein for compactness we have written H_i for $\partial H/\partial \Pi_i$:

$$H_1(\Pi_1, \ldots, \Pi_p)\alpha_1 + \ldots + H_p(\Pi_1, \ldots, \Pi_p)\alpha_p = aH(\Pi_1, \ldots, \Pi_p),$$

$$(26) \quad H_1(\Pi_1, \ldots, \Pi_p)\beta_1 + \ldots + H_p(\Pi_1, \ldots, \Pi_p)\beta_p = bH(\Pi_1, \ldots, \Pi_p),$$

$$H_1(\Pi_1, \ldots, \Pi_p)\gamma_1 + \ldots + H_p(\Pi_1, \ldots, \Pi_p)\gamma_p = cH(\Pi_1, \ldots, \Pi_p).$$

The second and third of these equations are obtained similarly by differentiating with respect to τ and μ respectively.

The H, H_i are in general variables, depending on the Π_i. But if we choose and fix some one set of values of the Π_i for which H is not zero, say $\Pi_i = \Pi_i{}^*$, the quantities

$$e_i = H_i(\Pi_1{}^*, \ldots, \Pi_p{}^*)/H(\Pi_1{}^*, \ldots, \Pi_p{}^*)$$

are constants, and by (26) the equations

$$e_1\alpha_1 + \ldots + e_p\alpha_p = a,$$

$$e_1\beta_1 + \ldots + e_p\beta_p = b,$$

$$e_1\gamma_1 + \ldots + e_p\gamma_p = c$$

are satisfied. From this and (22) it follows at once that the power-product

$$(27) \qquad \Pi_o = \Pi_1^{e_1} \ldots \Pi_p^{e_p}$$

has the same dimensions $[L]^a [T]^b [M]^c$ as P itself. But by substitution from (18) in (27) we see that Π_o is a power-product of the v_i, and conclusion (i) is established.

To prove (ii) we first make the restrictive assumption that P is dimensionless. Then the right members of equations (26) are zero, and by substitution from (23) we find

$$
\begin{aligned}
&(k_{1,1}H_1 + \dots + k_{1,p}H_p)a_1 \\
&\qquad + \dots + (k_{p,1}H_1 + \dots + k_{p,p}H_p)a_p = 0, \\
&(k_{1,1}H_1 + \dots + k_{1,p}H_p)b_1 \\
(28) \quad &\qquad + \dots + (k_{p,1}H_1 + \dots + k_{p,p}H_p)b_p = 0, \\
&(k_{1,1}H_1 + \dots + k_{1,p}H_p)c_1 \\
&\qquad + \dots + (k_{p,1}H_1 + \dots + k_{p,p}H\ )c_p = 0.
\end{aligned}
$$

Thus the quantities enclosed in parenthesis are solutions $k_1, \dots, k_p$ of equations (4). But since $\Pi_1, \dots, \Pi_{p-r}$ constitute a maximal independent set of zero-dimensional power-products, the particular solutions exhibited in (28) must be linear combinations of the maximal set (6). That is, there are numbers $c_1, \dots, c_{p-r}$ (in general depending on the Π_i) such that

$$
\begin{aligned}
&k_{1,1}H_1 + \dots + k_{1,p}H_p = c_1k_{1,1} + \dots + c_{p-r}k_{1,p-r}, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\
&k_{p,1}H_1 + \dots + k_{p,p}H_p = c_1k_{p,1} + \dots + c_{p-r}k_{p,p-r}.
\end{aligned}
$$

By transposition this yields

$$
\begin{aligned}
&(H_1 - c_1)k_{1,1} + \dots + (H_{p-r} - c_{p-r})k_{1,p-r} \\
&\qquad + H_{p-r+1}k_{1,p-r+1} + \dots + H_pk_{1,p} = 0, \\
(29) \quad &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\
&(H_1 - c_1)k_{p,1} + \dots + (H_{p-r} - c_{p-r})k_{p,p-r} \\
&\qquad + H_{p-r+1}k_{p,p-r+1} + \dots + H_pk_{p,p} = 0.
\end{aligned}
$$

But the determinant of the array (17) is not zero, so (29) implies

$$H_1 - c_1 = 0, \ldots, H_{p-r} - c_{p-r} = 0, \tag{30}$$
$$H_{p-r+1} = 0, \ldots, H_p = 0.$$

It is the last line of these equations that is important. It tells us that the partial derivatives of H with respect to $\Pi_{p-r+1}, \ldots, \Pi_p$, vanish identically. Hence H is actually independent of these Π_i, and is a function G of the other arguments, $\Pi_1, \ldots, \Pi_{p-r}$, alone; and so (21) takes the form

$$P = G(\Pi_1, \ldots, \Pi_{p-r}).$$

This completes the proof of the theorem for the special case in P is dimensionless, Π_0 being simply 1.

If P is not dimensionless, by (i) we know that there is a power-product Π_0 of the v_i with the same dimensions as P. If we define H^* to be H/Π_0, (21) can be written

$$P/\Pi_0 = H^*(\Pi_1, \ldots, \Pi_p). \tag{31}$$

The left member is dimensionless, so by the preceding proof H^* is independent of $\Pi_{p-r+1}, \ldots, \Pi_p$, and is therefore a function $G(\Pi_1, \ldots, \Pi_{p-r})$ of the first $p - r$ power-products. Substitution in (31) and multiplication by Π_0 completes the proof.

Let us apply this to the period of the pendulum (3). Here $p = 3$, the quantities v_i being g, ℓ, and s. Hence

$$a_1 = 1,\ a_2 = 1,\ a_3 = 1,$$
$$b_1 = -2,\ b_2 = 0,\ b_3 = 0, \tag{32}$$
$$c_1 = 0,\ c_2 = 0,\ c_3 = 0.$$

This has rank $r = 2$, since a second-order minor (upper left) has value $2 \neq 0$, but the one and only third-order minor (the whole array) has value 0. Hence $p - r = 1$, and a single independent power-product of dimension 0 is a maximal set. We choose $\Pi_1 = s_0/\ell$ as such a power-product. For Π_0 we must choose a power-product $g^a \ell^b s_0^c$ of dimensions $[T]$. Since this product has dimensions

$$[L]^{a+b+c}[T]^{-2a}[M]^0,$$

we must have $-2a = 1$ and $a + b + c = 0$, and therefore $b + c = \frac{1}{2}$. We choose $a = -\frac{1}{2}$, $b = \frac{1}{2}$, $c = 0$, so that $\Pi_0 = \sqrt{\ell/g}$. Then Theorem (8) tells us that there is a dimensionless function of s_0/ℓ such that

$$T = \sqrt{\ell/g}\, G(s_0/\ell).$$

Had we chosen to solve equations (1) in full detail, we would have found that G is an elliptic function.

Theorem (8) gave us the privilege of using any dimensionless power-product for Π_1. We could have taken Π_1 to be $s_0^{-7}\ell^7$. But it is natural for us to exercise this privilege in such a way that the dimensionless power-products are convenient and familiar combinations, insofar as this can be done. Since s_0/ℓ is the amplitude of the oscillation, it is natural that we select this for Π_1. Likewise for Π_0 we could have chosen the product of $\sqrt{\ell/g}$ and any power of s_0/ℓ. In general, in Theorem (8) if we have any one Π_0 of correct dimensions we could replace it by its product with any power-product of $\Pi_1, \ldots, \Pi_{p-r}$; this in fact is all the freedom we have. The choice of Π_0, apart from having the correct dimensions, is dictated purely by convenience. Usually it is convenient to choose Π_0 in such a way that G does not vary much in value when $\Pi_1, \ldots, \Pi_{p-r}$ change, provided of course that this is possible. The choice $\Pi_0 = \sqrt{\ell/g}$ answers this description for the next paragraph will show that it makes G nearly constant when s_0/ℓ is small.

Suppose now that we recall that Galileo discovered that the period of a pendulum making small oscillations is (almost) independent of the amplitude. Then in (3) the function F does not depend on s_0, and we can write

$$T = F(g, l).$$

For Π_0 we seek a combination $g^a l^b$ of dimensions [T]. The only such combination is $a = -\frac{1}{2}$, $b = \frac{1}{2}$, so that $\Pi_0 = \sqrt{l/g}$. The matrix (5) consists of the first two columns of (32), and still has rank 2. But now p = 2, since there are two arguments g and l of F. Hence the number p - r of independent power-products of dimension zero is 0, and in Theorem (8) there are no power-products Π_1. That is, G is a dimensionless function of no variables — in other words, a dimensionless constant c. By Theorem (8),

$$T = c\sqrt{l/g}.$$

If in (1) we replace $\cos(s/l)$ by $1 - s^2/2l^2$ and analogously for s_0/l, we would obtain an approximate form, adequate for small amplitudes. The detailed solution of this equation would yield $T = 2\pi\sqrt{l/g}$, agreeing with the result of the dimensional analysis and adding the information $c = 2\pi$.

Now we shall make a deliberate mistake, to show a possible careless misuse of the Π-theorem. Suppose that a particle is travelling in a circle of radius r about the sun. By Newton's law of gravitation, the force on the particle is directly proportional to the mass m of the particle and the mass M of the sun, and inversely proportional to the square of the distance r. The acceleration, being force divided by m, is proportional to Mr^{-2}. From knowledge of the acceleration we could compute the motion, hence the period T. Therefore we attempt to write

$$T = F(M, r). \tag{33}$$

Now we try to form the power-product $\Pi_0 = M^a r^b$ of dimensions [T]. But this is impossible; neither M nor r involves time. Looking back, we observe that (33) cannot satisfy the hypothesis of Theorem (8) that the numerical measure of T is a function of the numerical measures of M and r, for under change of time-unit the left member changes and the right member does not. Going back still further, the statement that the force is directly proportional to m and M and inversely proportional to r^2 can be written

$$f = kmM/r^2.$$

This implies $k = fr^2/mM$, and so k has dimensions $[L]^3[T]^{-2}[M]^{-1}$. The detailed solution of the equations would necessarily contain this "dimensional constant" k, as well as M and r. Hence (33) should have been written as

$$(34) \qquad T = F(M, r, k).$$

Now we find that the power-product Π_0, of dimensions [T], must have the form

$$\Pi_0 = k^{-1/2}\, r^{3/2}\, M^{-1/2},$$

and no dimensionless power-product exists. Hence in Theorem (8) the function G must be a dimensionless constant c, and T is given by

$$T = c\sqrt{r^3/kM}.$$

In any particular system of units the constant k and the sun's mass M have fixed numerical values, and we find that the period of a particle travelling in a circular orbit is proportional to the 3/2 power of the radius.

This example illustrates Bridgman's statement that dimensional analysis is an analysis of an analysis. We can only apply it when we know enough about a physical situation to be able to say what quantities would be involved in a complete analysis. These quantities, both variables and constants, are the ones which must appear in the dimensional analysis.

Let us now consider another example, of importance in ballistics. If a body B is translated at constant velocity through a uniform fluid which is "at rest at infinity" (i.e., has velocity approaching zero as the distance from B increases), there will be a force acting on the body. Let D denote the component of the force (the drag) opposite in direction to the velocity. The structure of the body can be specified by giving a certain number of its linear dimensions $\ell_1, \ldots, \ell_n$. For instance, in the case of a cylinder these could be diameter and length; in the case of a shell they could be diameter, length of head, radius of ogive, distance from head to front edge of rotating band, etc. If we know enough hydrodynamics we can write the equations of the flow of the fluid about the body. These equations are partial differential equations involving the density ρ of the fluid, its viscosity μ, the speed v of the body, and the elasticity of the fluid. At a given density, the elasticity determines and is determined by the speed of sound in the fluid, so we can replace elasticity in the list by speed of sound v_s. The fluid will adhere to the surface of the body, so its velocity relative to the body is zero at the boundary. Thus the boundary conditions involve the numbers $\ell_1, \ldots, \ell_n$ which specify the outlines of the body. Moreover, as is standard in physical discussions, the equations are relations between the numerical measures of

$$\rho, \mu, v, v_s, \ell_1, \ldots, \ell_n.$$

So even though we might not be able to solve the equations, even approximately, we know that the solution gives the numerical measures of the components of velocity of the fluid at each point outside B, expressing them as functions of the numerical measures of ρ, μ, etc. Proceeding a step further, from this flow we can compute the drag D, which again is a function of the numerical measures of ρ, μ, etc.:

$$(35) \qquad D = F(\rho, \mu, v, v_s, \ell_1, \ldots, \ell_n).$$

The power-product $\rho^{\alpha}\mu^{\beta}v^{\gamma}v_s^{\delta}\chi_1^{\varepsilon_1}\ldots\chi_n^{\varepsilon_n}$ has dimensions

$$(36)\quad [L]^{-3\alpha-\beta+\gamma+\delta+\varepsilon_1+\ldots+\varepsilon_n}[T]^{-\beta-\gamma-\delta}[M]^{\alpha+\beta}$$

In order that this have the dimension $[L][T]^{-2}[M]$ of the force, D, we must have

$$(37)\quad \begin{aligned} -3\alpha-\beta+\gamma+\delta+\varepsilon_1+\ldots+\varepsilon_n &= 1,\\ -\beta-\gamma-\delta &= -2,\\ \alpha+\beta &= 1. \end{aligned}$$

For arbitrarily chosen α this leads to

$$(38)\quad \begin{aligned} \varepsilon_1+\ldots+\varepsilon_n &= 1+\alpha,\\ \beta &= 1-\alpha,\\ \gamma+\delta &= 1+\alpha. \end{aligned}$$

We choose $\delta = 0$, and also shall single out one of the dimensions χ_1 as especially important; for instance, in shell and bombs the diameter plays this role. We may suppose χ_1 is this selected dimension, and we shall set $\varepsilon_2 = \ldots = \varepsilon_n = 0$. Then

$$(39)\quad \Pi_o = \rho^{\alpha}\mu^{1-\alpha}v^{1+\alpha}\chi_1^{1+\alpha}.$$

To find the dimensionless power-products we amend (37) by replacing the right members by 0. The matrix of coefficients is

$$(40)\quad \begin{matrix} -3, & -1, & 1, & 1, & 1, & \ldots, & 1,\\ 0, & -1, & -1, & -1, & 0, & \ldots, & 0,\\ 1, & 1, & 0, & 0, & 0, & \ldots, & 0, \end{matrix}$$

which has rank 3 (the determinant of the first three columns has the value -1). The number of arguments of F is $4+n$, so we need $4+n-3 = n+1$ independent dimensionless power-products to form a maximal set.

Evidently $\ell_2/\ell_1, \ldots, \ell_n/\ell_1$ afford us n - 1 such. Another is v/v_s, and still another is $\rho v \ell_1/\mu$. These last two are familiar combinations in hydrodynamics, and are named the "Mach number" and the "Reynolds number" respectively.

Suppose first that we decide to choose $\alpha = 0$. By (39), D has the form

(41) $D = \mu v \ell_1 G(v/v_s, \rho v \ell_1/\mu, \ell_2/\ell_1, \ldots, \ell_n/\ell_1)$.

If instead we choose $\alpha = 1$ we obtain

(42) $D = \rho v^2 \ell_1^2 K_D(v/v_s, \rho v \ell_1/\mu, \ell_2/\ell_1, \ldots, \ell_n/\ell_1)$,

where we have called the dimensionless function K_D instead of G because this notation is customary in ballistics. There is nothing in the theory which makes one of these superior to the other. However, it is desirable to choose the one in which the dimensionless function is most nearly constant. No single choice covers all possibilities. If the Reynolds number is very small, for instance in the settling of sediment through water, (41) is to be preferred, for then G is very nearly constant. However, in the aerodynamics of airplanes and in ballistics (42) is more desirable. For most projectiles of military interest K_D will vary only a few per cent between, say, Mach numbers .2 and .8, and from Mach numbers .2 to the greatest investigated the values of K_D will not change in ratio much greater than 4 or 5 to 1.

The parameters $\ell_2/\ell_1, \ldots, \ell_n/\ell_1$ determine the <u>shape</u> of the body, but not its size. They may be called the shape-parameters. Herein lies the answer to the possible objection that the roughness of the surface might influence the drag. Roughness, for example depth and spacing of lathe-marks, is actually shape, and there is room among the arguments $\ell_2/\ell_1, \ldots$ for numbers specifying these roughnesses in the ratio to the master-dimension ℓ_1.

16. The Stieltjes integral.

Before beginning the study of the Stieltjes integral, we wish to warn the reader that this section and the next six can be omitted without destroying the continuity of the presentation in later chapters. A knowledge of the properties of the Stieltjes integral is very useful in the study of probability theory, touched on in the next sections. Also, in the discussion of weighting factors in Chapter VIII, a process is used which is essentially the definition of a Stieltjes integral. The authors feel that the reader will understand the processes involved with greater clarity if he has a comprehension of the meaning and properties of the Stieltjes integral. Nevertheless, it is possible to avoid naming them in Chapter VIII, and it is also possible to omit the rest of this chapter, and the parts of the later chapters which make use of it, without irremediable loss.

In order to have a concrete example to guide us, we shall consider (non-rigorously) a mass-distribution lying entirely interior to some interval $a \leq x \leq b$ of the positive x-axis. Pictorially, we can think of this part of the axis as replaced by a wire, possibly of variable density and possibly loaded with several point-masses. We wish to find the moment of inertia of this mass-distribution about the origin. Let $g(x)$ be the mass of that part of the distribution lying between the origin and the point with abscissa x (not inclusive). Then for any two points $x_1 > 0$ and $x_2 > x_1$ the mass lying between these points, and including x_1 but not x_2, will be $g(x_2) - g(x_1)$; for $g(x_2)$ is the mass of the part to the left of x_2, and from it we have subtracted the mass $g(x_1)$ of the part to the left of x_1. The moment of inertia of this part of the mass cannot be less than x_1^2 times its mass, and cannot exceed x_2^2 times its mass. Hence if the interval from x_1 to x_2 is short, and ξ is any point between x_1 and x_2, the quantity $\xi^2[g(x_2) - g(x_1)]$ cannot differ greatly from the moment of inertia of this part of the distribution. If we choose points $x_1 < x_2 < \dots < x_n$ with $x_1 = a$ and $x_n = b$, and in each interval we choose an intermediate point (the chosen point between x_{i-1} and x_i being named ξ_i), it is plausible that when the intervals are all small the sum

$$(1) \qquad \sum_{i=1}^{n} \xi_i^2[g(x_i) - g(x_{i-1})]$$

cannot differ much from the moment of inertia of the distribution. Hence we may expect that if the number of subintervals is increased without bound, in such a way that the length of the longest subinterval tends to zero, the sum (1) will approach a limit, and this limit will be the moment of inertia of the distribution. This sort of discussion will be found in many elementary texts on physics, and is usually followed by another step, in which the difference $g(x_i) - g(x_{i-1})$ is replaced by $g'(x_i^*)[x_i - x_{i-1}]$, where x_i^* is some value between x_{i-1} and x_i. After this replacement the sum (1) takes the form familiar in elementary calculus, and its limit is an integral of the familiar type. However, this reduction is impossible when $g(x)$ lacks a derivative, as for example it does whenever the distribution contains point-masses. So instead of trying to devise some substitute for the reduction, we shall study limits of sums such as (1) in their own right.

Suppose then that $g(x)$ is a function defined on some interval $a \leq x \leq b$ and monotonically increasing on that interval, so that $g(x_2) \geq g(x_1)$ whenever $x_2 > x_1$. Suppose that $f(x)$ is defined and finite on the interval $a \leq x \leq b$. As in our example, we subdivide the interval from a to b into subintervals by means of points

$$(2) \qquad a = x_1 < x_2 < x_3 < \dots < x_n = b,$$

and between each pair of consecutive division-points

x_{i-1} and x_i we choose a point ξ_i. We form the sum

$$(3) \qquad \sum_{i=1}^{n} f(\xi_i)\,[g(x_i) - g(x_{i-1})\,],$$

analogous to (1). If it happens that the sum (3) approaches a limit as the number of subintervals is increased in such a way that the length of the longest subinterval tends to zero, irrespective of the manner in which the ξ_i are chosen within the subintervals, we say that $f(x)$ is Stieltjes-integrable with respect to $g(x)$ from a to b, and we denote the limit by the symbol

$$(4) \qquad \int_a^b f(x)\,dg(x).$$

Thus the moment of inertia in our example would be the Stieltjes integral

$$\int_a^b x^2\,dg(x),$$

assuming that it exists.

In any good textbook on advanced calculus there will be found a proof that if $f(x)$ is continuous on the interval from a to b, the integral

$$(5) \qquad \int_a^b f(x)\,dx$$

exists. This proof can be amended to show that if $f(x)$ is continuous the integral (4) exists; all that is needed is to replace the differences $x_i - x_{i-1}$ by the differences $g(x_i) - g(x_{i-1})$ in the proof. It is also true that if $f(x)$ is bounded and has a finite number of discontinuities the integral (4) exists, provided that no discontinuity of $f(x)$ occurs at the same place as a discontinuity of $g(x)$.

Just as in the case of the integral (5), it is easy to prove from the definition a number of theorems, some of which we now state.

(6) Theorem. <u>If</u> $f_1(x)$ <u>and</u> $f_2(x)$ <u>are both integrable with respect to</u> $g(x)$ <u>from</u> a <u>to</u> b, <u>then so is their sum</u>, <u>and</u>

$$\int_a^b [f_1(x) + f_2(x)]\, dg(x) = \int_a^b f_1(x)\, dg(x) + \int_a^b f_2(x)\, dg(x).$$

(7) Theorem. <u>If</u> $f(x)$ <u>is integrable with respect to</u> $g(x)$ <u>from</u> a <u>to</u> b, <u>and</u> k <u>is a constant</u>, <u>then</u> $kf(x)$ <u>is also integrable with respect to</u> $g(x)$ <u>from</u> a <u>to</u> b <u>and</u>

$$\int_a^b kf(x)\, dg(x) = k \int_a^b f(x)\, dg(x).$$

(8) Theorem. <u>If</u> $f(x)$ <u>is integrable with respect to</u> $g(x)$ <u>from</u> a <u>to</u> b, <u>it is also integrable with respect to</u> $g(x)$ <u>over every interval contained in the interval from</u> a <u>to</u> b. <u>Also</u>, <u>if</u> c <u>is between</u> a <u>and</u> b,

$$\int_a^b f(x)\, dg(x) = \int_a^c f(x)\, dg(x) + \int_c^b f(x)\, dg(x).$$

(9) Theorem. <u>If</u> $f(x)$ <u>is integrable with respect to</u> $g(x)$ <u>from</u> a <u>to</u> b, <u>and</u> $|f(x)| \leq M$ <u>for all</u> x <u>in the interval</u>, <u>then</u>

$$\int_a^b f(x)\, dg(x) \leq M[g(b) - g(a)].$$

To prove this we need only observe that the absolute value of the sum (3) is not decreased if we replace $f(\xi_i)$ by M in each term; and if this is done, the sum (3) changes into $M[g(b) - g(a)]$.

(10) $$\int_a^b 1\,dg(x) = g(b) - g(a).$$

This is obvious if we substitute $f(x) = 1$ in (3).

The next theorem has no analogue in the theory of the usual integral (5).

(11) <u>Theorem.</u> <u>If $g_1(x)$ and $g_2(x)$ are both monotonically increasing functions on the interval from a to b, and $g(x)$ is their sum, then every function $f(x)$ which is integrable from a to b with respect to both $g_1(x)$ and $g_2(x)$ is also integrable with respect to $g(x)$, and</u>

$$\int_a^b f(x)\,dg(x) = \int_a^b f(x)\,dg_1(x) + \int_a^b f(x)\,dg_2(x).$$

For any choice of the points x_i and ξ_i we have

$$\sum_{i=1}^{n} f(\xi_i)[g(x_i) - g(x_{i-1})]$$

$$= \sum_{i=1}^{n} f(\xi_i)[g_1(x_i) - g_1(x_{i-1})]$$

$$+ \sum_{i=1}^{n} f(\xi_i)[g_2(x_i) - g_2(x_{i-1})].$$

If we let the number of subintervals increase without limit, the length of the longest subinterval approaching zero, the two sums in the right member approach the two integrals in the right member of (11). Hence the left member approaches the same limit, which establishes (11).

If it happens that $g(x)$ has a derivative at each point of the interval from a to b, the Stieltjes integral (4) can be transformed into an integral of the type (5). For let (2) be any method of subdividing the interval from a to b. Since $g(x)$ is differentiable, it is continuous, and by the theorem of mean value we know that between each pair of consecutive division-points x_{i-1} and x_i there is a point ξ_i such that

$$g(x_i) - g(x_{i-1}) = g'(\xi_i)[x_i - x_{i-1}].$$

If we substitute this in (3), the sum takes the form

$$\sum_{i=1}^{n} f(\xi_i)\, g'(\xi_i)[x_i - x_{i-1}]. \tag{12}$$

As the number of subintervals increases, the length of the longest approaching zero, the sum (3) approaches the limit (4), while if we write the same sum in the form (12) we see that its limit is the integral of $f(x)\, g'(x)$ from a to b. Hence

$$\int_a^b f(x)\, dg(x) = \int_a^b f(x)\, g'(x)\, dx. \tag{13}$$

However, we must keep in mind that this formula holds only subject to strong restrictions on the function $g(x)$; for example, if g has a single discontinuity, (13) cannot possibly be true for all functions $f(x)$.

Suppose next that $f(x)$ and $g(x)$ are both defined for all x, and that $g(x)$ is monotonically increasing. Then we define

$$\int_a^{\infty} f(x)\, dg(x) = \lim_{b \to \infty} \int_a^b f(x)\, dg(x), \tag{14}$$

provided that the limit exists; we define

$$(15)\quad \int_{-\infty}^{b} f(x)\,dg(x) = \lim_{a\to-\infty}\int_a^b f(x)\,dg(x),$$

provided that the limit exists; and we define

$$(16)\quad \int_{-\infty}^{\infty} f(x)\,dg(x) = \lim_{\substack{a\to-\infty\\ b\to\infty}}\int_a^b f(x)\,dg(x),$$

provided that the limit exists as a tends to $-\infty$ and b to ∞ independently of each other.

In order to exhibit the versatility of the Stieltjes integral, we now look at some examples. First, the integral (5) is obviously a special case of the Stieltjes integral, with $g(x) = x$. Next, let $w_1, \ldots, w_k$ be a set of positive numbers. Define $g(x)$ to be the sum of those w_j whose subscripts do not exceed x; for example, $g(0) = 0$ and $g(3.2) = w_1 + w_2 + w_3$. This function $g(x)$ has jumps at $x = 1, 2, \ldots, k$, and is constant between jumps, above $x = k$ and below $x = 1$. The jump at $x = j$ is equal to w_j. Let $f(x)$ be any function continuous at all the jump-points of $g(x)$, and let (2) be a subdivision of the interval from 0 to j into subintervals of length less than 1. If an interval $x_{i-1} < x \leq x_i$ contains a jump-point $x = j$ in its interior or at its right end, then

$$g(x_i) - g(x_{i-1}) = w_j;$$

and if we choose $\xi_i = j$, as we may, the corresponding term in the sum (3) is $w_j f(j)$. If the interval from x_{i-1} to x_i contains no jump-point in its interior or at its right end, then $g(x_i) = g(x_{i-1})$, and the corresponding term in the sum (3) vanishes however we choose ξ_i. So the sum (3) has the value

$$w_1 f(1) + \ldots + w_k f(k).$$

This is true for all sufficiently fine subdivisions.

so the limit of the sum is also this same fixed number, and thus we have shown that

$$(17) \qquad \int_0^k f(x)\, dg(x) = \sum_{j=1}^{k} w_j f(j).$$

In particular, if each w_j is 1, the Stieltjes integral is the finite sum $f(1) + \ldots + f(k)$. If the sum of the w_j is 1, the right member of (17) is the weighted mean of the $f(j)$. If each w_j is $1/k$, the right member of (17) is the arithmetic mean of the numbers $f(j)$.

Given an infinite sequence of non-negative numbers w_j, we can define $g(x)$ as in the preceding paragraph. Equation (17) will still hold. But now we can let the upper limit increase without bound, and thus find that

$$(18) \qquad \int_0^\infty f(x)\, dg(x) = \sum_{j=1}^{\infty} w_j f(j),$$

provided that this limit exists. In particular, if all the w_j are equal to 1 the right member of (18) is the infinite series $\Sigma f(j)$. We also observe that the series $\Sigma f(j)$ is absolutely convergent if and only if $|f(x)|$ is Stieltjes-integrable with respect to this particular $g(x)$ from 0 to ∞.

Suppose next that $w(x)$ is non-negative and (for the sake of simplicity) continuous for $a \leq x \leq b$, and that

$$(19) \qquad \int_a^b w(x)\, dx = 1.$$

Then if the integral

$$(20) \qquad \int_a^b f(x)\, w(x)\, dx$$

exists it is called the integral mean of $f(x)$ with weight $w(x)$. This, too, can be written as a Stieltjes integral. We define

$$(21) \qquad W(x) = \int_a^x w(x)\, dx \qquad (a \leq x \leq b).$$

Since $w(x)$ is non-negative, this is monotonically increasing. Also $W'(x) = w(x)$. So by (13),

$$(22) \qquad \int_a^b f(x)\, w(x)\, dx = \int_a^b f(x)\, dW(x).$$

If the Stieltjes integral had no other virtue, its ability to cover finite sums, infinite series, weighted means, integral means and ordinary (Riemann) integrals in a single formula would make it worth knowing.

The definition of the Stieltjes integral can be extended without difficulty to functions which are the difference of two monotonically increasing functions. Suppose that $g(x)$ can be represented in the form

$$(23) \qquad g(x) = p(x) - n(x),$$

where $p(x)$ and $n(x)$ are monotonically increasing functions (that is, if $x' < x''$ then $p(x') \leq p(x'')$ and $n(x') \leq n(x'')$). If $f(x)$ is defined on the interval from a to b, the sum (3) takes the form

$$\sum_{i=1}^{n} f(\xi_i)\,[g(x_i) - g(x_{i-1})]$$

$$(24) \qquad = \sum_{i=1}^{n} f(\xi_i)\,[p(x_i) - p(x_{i-1})]$$

$$- \sum_{i=1}^{n} f(\xi_i)\,[n(x_i) - n(x_{i-1})].$$

Now if f(x) is integrable both with respect to p(x) and with respect to n(x) on the interval from a to b, the two sums on the right approach the two integrals with respect to p(x) and to n(x) respectively. So the left member must also approach a limit, which we call the Stieltjes integral of f(x) with respect to g(x), and we denote it by the symbol (4). Thus

$$(25)\qquad \int_a^b f(x)\,dg(x) = \int_a^b f(x)\,dp(x) - \int_a^b f(x)\,dn(x).$$

Although we do not need the utmost generality, the reader who is familiar with the concept of functions of bounded variation will recognize that it is exactly this class of functions which can be represented in the form (23). Somewhat pictorially, the functions of bounded variation can be thus described. Let x vary from a to b. The point g(x) moves up and down, perhaps with jumps (the amount of the jump is counted in as distance travelled). The total vertical distance travelled by g(x) is the total variation of the function; if it is finite, we say that g(x) is of bounded variation. For simplicity, we shall restrict our attention to a simpler (though still extensive) class of functions, namely those which have the property that the interval from a to b can be subdivided into a finite number of subintervals on each of which g(x) is either monotonically increasing or monotonically decreasing. Suppose that we can find a finite number of points

$$(26)\qquad a = X_o < X_1 < \dots < X_m = b$$

such that from each X_{i-1} to the next point X_i the function g(x) is monotonic. Take any x between a and b, and let X_k be the last one of the points X_i to the left of x. Then

$$(27)\qquad \begin{aligned} g(x) - g(a) &= [g(X_1) - g(X_o)] + \dots + [g(X_k) - g(X_{k-1})] \\ &\quad - [g(x) - g(X_k)]. \end{aligned}$$

In the right member select all the non-negative terms and denote their sum by $P(x)$; select all the negative terms and denote their sum by $-N(x)$. It is not difficult to see that as x increases, neither $P(x)$ nor $N(x)$ can decrease; and by the definition

$$(28) \qquad g(x) - g(a) = P(x) - N(x).$$

This is not quite the same as (23), but we easily obtain (23) by defining

$$(29) \quad p(x) = P(x) + g(a)/2, \quad n(x) = N(x) - g(a)/2.$$

Furthermore, the sum of $P(x)$ and $N(x)$ (which by (29) is equal to the sum of $p(x)$ and $n(x)$) is the same as the sum of the absolute values of all the terms in the right member of (27), since it is the sum of all the non-negative terms and the negatives of all the negative terms. Hence $p(x) + n(x)$ is the same as the total vertical distance travelled by $g(x)$ as x varies from a to x. In particular, $p(b) + n(b)$ is the total vertical distance travelled by $g(x)$ as x travels from a to b. This is the total variation of $g(x)$.

By making use of (25), we can easily show that statements (6), (7), (8) and (11) hold for the more general form of the Stieltjes integral. However, (9) does not hold. It can be replaced by

(30) Theorem. *If $f(x)$ is integrable with respect to $g(x)$ from a to b, and $|f(x)| \leq M$ for $a \leq x \leq b$, then*

$$\int_a^b f(x)\,dg(x) \leq M\,[\text{total variation of } g(x)].$$

We can apply (9) to the integrals with respect to $p(x)$ and $n(x)$ separately, since these are monotonically increasing functions. Then, by (25),

$$\left|\int_a^b f(x)\,dg(x)\right| \leq \left|\int_a^b f(x)\,dp(x)\right| + \left|\int_a^b f(x)\,dn(x)\right|$$
$$\leq M[p(b) - p(a)] + M[n(b) - n(a)].$$

But $p(b) + n(b)$ is the total variation of $g(x)$ from a to b, while by their definitions $P(a)$ and $N(a)$ both vanish, so that by (29), $p(a) + n(a) = 0$. Hence the preceding inequality implies (30).

If $g(x)$ is continuous at each of the points X_i, we can show by means of an example that the estimate in (30) cannot be improved. On each interval from X_{i-1} to X_i the function $g(x)$ is either increasing or decreasing. In the former case we define the function $f(x)$ to be $+ M$ from X_{i-1} to X_i; in the latter case we define it to be $- M$. In either case the absolute value of $f(x)$ is always M. Let the integral in the left member of (30) be represented as the sum of the integrals from X_0 to X_1, from X_1 to X_2, and so on. Then if $g(x)$ is increasing from X_{i-1} to X_i we have

$$\int_{X_{i-1}}^{X_i} f(x)\, dg(x) = M\int_{X_{i-1}}^{X_i} 1\, dg(x)$$

$$= M[\, g(X_i) - g(X_{i-1})\,] = M|g(X_i) - g(X_{i-1})|,$$

while if $g(x)$ is decreasing on this interval we have

$$\int_{X_{i-1}}^{X_i} f(x)\, dg(x) = (\, - M)\int_{X_{i-1}}^{X_i} 1\, dg(x)$$

$$= (\, - M)\,[\, g(X_i) - g(X_{i-1})\,] = M|g(X_i) - g(X_{i-1})|.$$

Thus, on adding these integrals, we find

$$\int_b^a f(x)\, dg(x) = \sum_{i=1}^{m} M|g(X_i) - g(X_{i-1})|.$$

But except for the factor M, the right member represents the total vertical distance travelled by $g(x)$ as x goes from a to b, so the right member is M times the total variation of $g(x)$. Thus equality holds in (30), showing that (30) cannot be sharpened by replacing the right member by any smaller number.

The definition of the Stieltjes integral can be extended to functions of any finite number of variables. We shall content ourselves with extending it to two dimensions; extensions to higher dimensions involve no essentially different ideas, but make the notation more complicated. Furthermore, for the sake of simplicity we shall consider only the integrals of continuous functions. Finally, we restrict our attention to functions $g(x, y)$ which satisfy the condition

$$(31)\quad g(x_2, y_2) - g(x_1, y_2) - g(x_2, y_1) + g(x_1, y_1) \geq 0$$

whenever $x_1 < x_2$ and $y_1 < y_2$. Such functions we shall call "positively monotonic." Condition (31) may seem unnatural, but we shall find that it is useful and relates closely to monotonic functions of one variable. To simplify the notation, if J is the interval defined by the inequalities

$$(32)\qquad x_1 \leq x \leq x_2,\ y_1 \leq y \leq y_2,$$

we define

$$(33)\qquad \Delta_g J = g(x_2, y_2) - g(x_1, y_2) - g(x_2, y_1) + g(x_1, y_1).$$

Suppose then that $g(x, y)$ is positively monotonic on an interval J defined by the inequalities

$$(34)\qquad a \leq x \leq A,\ b \leq y \leq B,$$

and that $f(x, y)$ is continuous on this same interval. We subdivide the one-dimensional intervals from a to A and from b to B by points

$$(35)\qquad \begin{aligned} a = x_0 < x_1 < \dots < x_m = A, \\ b = y_0 < y_1 < \dots < y_n = B. \end{aligned}$$

From these we form mn intervals

$$(36)\qquad J_{hk};\ x_{h-1} \le x \le x_h,\ y_{k-1} \le y \le y_k \qquad (h = 1, \ldots, m;\ k = 1, \ldots, n).$$

In each interval J_{hk} we select a point (ξ_{hk}, η_{hk}), and we form the sum

$$(37)\qquad \sum_{h=1}^{m} \sum_{k=1}^{n} f(\xi_{hk}, \eta_{hk})\ \Delta_g J_{hk}.$$

It can be shown, much as in the case of the usual double integrals, that this sum necessarily approaches a limit as the length of the longest side of the intervals J_{hk} tends to zero. The limit is the Stieltjes double integral

$$(38)\qquad \iint_J f(x, y)\ dg(x, y).$$

From the definition we immediately deduce that (6) and (7) apply to the double integral also. Instead of (8) we have

(39) Theorem. If $f(x, y)$ is continuous on J, and the interval J is subdivided into two intervals J_1 and J_2 either by inserting a point of division on the x-axis between a and A or by inserting a point of division on the y-axis between b and B, then

$$\iint_J f(x, y)\ dg(x, y)$$

$$= \iint_{J_1} f(x, y)\ dg(x, y) + \iint_{J_2} f(x, y)\ dg(x, y).$$

Instead of (9) we have

(40) <u>Theorem</u>. <u>If</u> $f(x, y)$ <u>is</u> <u>continuous</u> <u>on</u> J, <u>and</u> $|f(x)| \le M$ <u>on</u> J, <u>then</u>

$$\left| \iint_J f(x, y)\ dg(x, y) \right| \le M \Delta_g J.$$

In general, for the double Stieltjes integral the process of iterated integration, first with respect to one variable and then with respect to the other, is completely without meaning. However, there is one important case in which such iterated integration is possible. This is the special case in which the function $g(x, y)$ can be factored into the product of two monotonically increasing functions, one depending on x alone and the other depending on y alone; say

(41) $$g(x, y) = g_1(x)\, g_2(y).$$

In this case, for each fixed x in the interval from a to A, $f(x, y)$ is a continuous function of y alone; hence the integral

(42) $$\int_b^B f(x, y)\, dg_2(y)$$

has a meaning. Moreover, if ε is a positive number, because of the continuity of $f(x, y)$ there is a positive number δ such that $|f(x', y) - f(x'', y)|$ remains less than ε whenever x' and x'' differ by less than δ. If we compute the integral (42) first with $x = x'$ and then with $x = x''$, by (9) the two integrals differ by at most $\varepsilon[g_2(B) - g_2(b)]$, so the integral (42) is a continuous function of x. This in turn implies that the integral (42), regarded as a function of x, can be integrated with respect to $g_1(x)$ from a to A. So the left member of the equation

(43) $$\int_a^A \left\{ \int_b^B f(x, y)\, dg_2(y) \right\} dg_1(x)$$

$$= \int \int_J f(x, y)\, dg(x, y)$$

has a meaning. The proof that the equation (43) is in fact correct is only trivially different from the proof for the special case $g_1(x) = x$, $g_2(y) = y$ to be found in any calculus text.

17. Probability measures.

The purpose of the next six sections is to collect some of the elements of the theory of probability, listing some theorems and definitions which will be convenient for use in later chapters. Since these sections are not very easy reading, the reader should know that the only places in which we make use of probability theory are Sections 7 and 10 of Chapter VI, in which probability theory is used to estimate errors in numerical computations, and Sections 4 and 5 of Chapter XIII, in which the best coefficients for calculating drag from firing-range data are investigated. Both these topics can be omitted without destroying the continuity of the subject matter, so the reader may omit the rest of this chapter if he prefers.

It is not to be expected that these few sections will contain a complete, concise and readable account of everything of importance in probability theory. For a detailed presentation, including proofs of theorems that we shall merely state without proof, we refer the reader to J. V. Uspensky's Introduction to Mathematical Probability (New York: McGraw-Hill Book Company, Inc., 1937), to S. S. Wilks' Mathematical Statistics (Princeton, N. J.: Princeton University Press, 1943), or to H. Cramér's Mathematical Methods of Statistics (Princeton, N.J.: Princeton University Press, 1946). The account in this section has been influenced by conversations of the authors with Professors A. P. Morse and H. E. Federer; but all details are according to the authors' tastes and judgment of suitability for the present needs, and Professors Morse and Federer may justly disavow paternity if they wish.

Like all branches of applied mathematics, probability theory consists of two essential parts. There is an aggregate of mathematical theorems, having the typical mathematical form "if P is true, then Q is true"; and there is a dictionary by which the mathematical terms are translated into the language of material objects.

This dictionary contains an expression "equally likely" which has often been difficult to translate. We shall not attempt any general discussion, but shall look at a few examples instead. Consider first, a die, which is a cube cut from homogeneous material, the faces being marked 1 to 6. If this die is thrown sufficiently often (the precaution being taken that the person rolling the die is required to make it bounce off a wall) it will be found that nearly a sixth of the numbers turned up are ones, nearly a sixth are twos, and so on. We then say that the six numbers are "equally likely" to appear, and we assign each one the probability 1/6, which is a way of summarizing the statement that in a long sequence of tosses we may expect that each of the numbers will appear about a sixth of the time.

Consider next, a point moving about a horizontal circular track of circumference C, being retarded by air friction and friction of the track. We assume that the initial velocity of the moving particle is imparted by a human thumb, incapable of reproducing the imparted velocity from trial to trial with any degree of accuracy. We assume also that the coefficient of friction of the track is the same at all points. This is an idealization of the honest roulette wheel. We may reasonably anticipate that if we mark off two arcs of the track with equal lengths, in a large number of trials the particle will stop in one arc about as often as in the other; that is, the ratio of the number of times the point stops in arc 1 to the number of times it stops in arc 2 will approach unity as a limit when the number of trials increases without bound. More generally, if two arcs have the respective lengths L_1 and L_2, we may expect that the ratio of the number of times the point stops in arc 1 to the number of times it stops in arc 2 will approach L_1/L_2 as the number of trials increases without bound. In particular, if the second arc is the whole circumference, its length is $L_2 = C$, and the number of times the moving point stops in it is the same as the number of trials. So the ratio of the number of times the point stops in the arc of length L_1 to the total number of trials may be expected to approach the limit L_1/C as the number of trials increases without bound. In symbols, if A_1 is the arc and L_1 its length, we assign the arc A_1 the "probability measure" $p(A_1) = L_1/C$.

The reader may have been struck by the number of times we have used vague expressions such as "we may reasonably expect that ...." It would be beyond the scope of this study to try to investigate more deeply. This is the point at which we encounter the slightly hazy connection between the mathematical side of the dictionary and the world of concrete events. Nevertheless, the reader will almost surely find that the assignment of "probability measure" in the preceding examples is the only reasonable one. For instance, nothing in nature precludes the possibility that in twenty successive trials on the roulette wheel the moving point will each time stop in a short arc of the circumference, colored green and marked with a 0. Nevertheless, if this actually occurred the players would very probably be astonished, and would doubt that the hypotheses concerning uniform coefficient of friction, etc., were satisfied. It is in this somewhat vague sense that we understand the limiting processes of the preceding paragraphs. Once we have passed this point, we have a precise numerical measure of probability to which precise mathematical processes can be applied.

Next consider the example of throwing two homogeneous dice (still bouncing them off the wall). We suppose these dice distinguished in some visible way, for example by color, so that we can speak of "the first die" and "the second die" without confusion. Each throw of the pair provides a pair of numbers (a, b) each of which is one of the digits from 1 to 6. In a large number of trials, about one sixth will have a = 1, about one sixth will have a = 2, and so on.

But the dice are not mechanically interrelated, so that if we select from all the throws the particular ones (roughly a sixth of all) such that a = 1, for about a sixth of these (a thirty-sixth of the total) b will be 1, for about another sixth b will be 2, and so on. There are, in all, thirty-six pairs (a, b) of the kind described; and the ratio of the number of times any particular one of these pairs turns up to the total number of trials should approach 1/36 as a limit when the number of trials is increased without bound. The two dice have been assumed independent in a physical sense, the motion of one producing no forces or torques which affect the motion of the other. The result is that the numbers a and b are independent in a probability sense, that is, a given value of b is just as likely to appear when a = 1 as when a = 2 or any other particular one of the digits 1 to 6.

Finally, let us consider a system of n circular tracks each of length C, with a point moving about each circumference. In order to make it easier to record results, we suppose that a point on each circumference is chosen as the starting point and labelled 0, each other point being labelled with the number expressing the arc length measured from 0 to that point in a counterclockwise direction. Thus each point bears a label not less than 0 but less than C; the point which this process would label C is the same as the starting point, already labelled 0, so C itself is not the label of any point. If all n points are started off, and the first one stops at the place x_1, the second at the place x_2, etc., we can record the result of the trial by means of the n-tuple

$$(x_1, x_2, \ldots, x_n),$$

wherein each x_i is in the interval $0 \leq x_i < C$. To phrase it more geometrically, each trial corresponds to a point of the "cube" $0 \leq x_i < C$ $(i = 1, \ldots, n)$ in n-dimensional space. Let A_1 be an arc of length L_1 on the first circle, A_2 an arc of length L_2 on the second circle, and so on. We shall assume that the various circles (roulette wheels) are independent in the physical sense, so that the motion of the point on the first circle gives rise to no forces that could affect the motion of the point on the second circle. In a large number N of trials we may anticipate that the number of times the point on the first wheel stops in arc A_1 is nearly $(L_1/C)N$, or with the notation already introduced, $p(A_1)N$. More precisely, the ratio of the number of times the point stops in A_1 to the total number N of trials should approach $p(A_1)$ as a limit when N increases without bound. Let us now segregate those trials in which the point on the first circle stops in the arc A_1 (as just remarked, there are approximately $Np(A_1)$ of these) and investigate the number of these segregated trials in which it is true that the point on the second circle stops in arc A_2. Since the motion of the point on the first wheel does not affect that on the second, we may anticipate that the number of these is approximately $p(A_2)$ times the number of trials being investigated, which is itself approximately $Np(A_1)$. Hence the number of trials in which the point rolling on the first circle stops in arc A_1 and that rolling on the second circle stops in arc A_2 should be nearly $Np(A_1)p(A_2)$. Continuing the reasoning, the number of times that the point on the i-th circle stops in arc A_i for each of the n circles $(i = 1, \ldots, n)$ should be approximately

$$Np(A_1)p(A_2)\ldots p(A_n).$$

In order to rephrase this result in terms of the geometrical representation already mentioned, let us first observe that if an arc A_i does not contain the starting point $x = 0$ of the i-th circle, and if we denote by a_i the label of the beginning point of A_i when traversed in a counterclockwise direction and by b_i its end point, then the arc A_i consists of all the points with labels x between a_i and b_i inclusive. If the arc A_i contains the origin, and we denote by a_i its beginning point and by b_i its end point when we traverse the arc in a counterclockwise direction, then as the moving point traverses the arc in a counterclockwise direction its label x starts at a_i, increases up to C when the point reaches the starting point of the circle, at which point it drops abruptly to 0 and then starts increasing again until it rises to the value b_i. Thus if A_i does not contain the starting point $x = 0$ it consists of all points with labels x such that $a_i \leq x \leq b_i$; while if A_i contains the origin it consists of all points with labels x such that $a_i \leq x < C$ and also all points with labels such that $0 \leq x \leq b_i$. In either case, the arc A_i corresponds to one or two segments of the x_i-axis with total length $L_i = Cp(A_i)$. The trials in which the point on the i-th circle stops in arc A_i are represented by points in n-dimensional space for which the first coordinate lies in an interval or pair of intervals of total length L_1, the second in an interval or pair of intervals of total length L_2, and so on. If we understand an "n-dimensional interval" to be a set of points in n-space satisfying a system of inequalities of the form $h_i \leq x_i \leq k_i$ (or similar inequalities with some or all of the signs $\leq$ replaced by $<$), we see that the points representing the trials in which the i-th point stops in A_i are represented geometrically by points in n-dimensional space which lie in a finite number of n-dimensional intervals, not more than 2^n of them, having total volume $L_1 L_2 \ldots L_n$. The "cube" in n-dimensional space representing all possible trials has all its edges of length C, so its volume is C^n. The ratio of the number of trials in which the point on the i-th circle stops in A_i to the total number N of trials is nearly equal to the product $p(A_1)p(A_2) \ldots p(A_n)$, which is the same as $L_1 L_2 \ldots L_n/C^n$.

Next we consider a slight extension of the concept of probability measure. In the example of the single die, the probability measure of each single digit 1, 2, etc., is 1/6; in a large number N of trials, the number of times any one of these digits will turn up is approximately (1/6)N. So if S is a set consisting of several of these digits — for example, S might consist of the digits 1, 3 and 5 — and m stands for the number of digits in S, each of these will turn up approximately N/6 times, and the number of times that the throw will give us a number in S is approximately m times N/6. Therefore the probability measure to be assigned to S is m/6. Thus each subset of the aggregate 1, 2, 3, 4, 5, 6 has a probability measure equal to 1/6 of the number of elements in the subset. In exactly the same way, any subset S of the thirty-six pairs (1, 1), ..., (6, 6) in the second example is assigned probability measure equal to (1/36) times the number of pairs in S. It continues to be true that if the number N of trials is large, the number of times that a pair belonging to S is thrown will be approximately N times the probability measure of S.

In discussing the other examples, it is convenient to make use of intervals in n-dimensional space defined by inequalities of the type

$$a_i \leq x_i < b_i \qquad (i = 1, \ldots, n),$$

where the a_i and b_i are numbers such that

$$0 \leq a_i < b_i \leq C.$$

Such intervals are usually called "half-open," because they contain their lower boundaries but not their upper boundaries. The volume of the interval defined by the inequalities just written is the same as if the inequalities $a_i \leq x_i < b_i$ were changed to $a_i \leq x_i \leq b_i$, namely $(b_1 - a_1) \ldots (b_n - a_n)$. Let J_1 and J_2 be two half-open intervals in n-dimensional space, having no points in common. The probability that a trial will have a result in J_1 is $p(J_1) = (\text{volume } J_1)/C^n$, and similarly for J_2. Let S be the set consisting of the two intervals J_1 and J_2. In a large number N of trials, approximately $Np(J_1)$ will have results in J_1 and approximately $Np(J_2)$ will have results in J_2. None of these are counted twice, since no trial can have its result in both J_1 and J_2. Hence the number of trials with results in S will be the sum of the number with results in J_1 and the number with results in J_2, which is approximately

$$N [p(J_1) + p(J_2)].$$

Accordingly, the probability measure of S should be taken to be $p(J_1) + p(J_2)$. A similar reasoning applies to any finite sum of intervals $J_1 + \ldots + J_2$. If the reader happens to be familiar with the theory of the Lebesgue integral, he will realize that we are by no means limited to the consideration of finite sums. We can define p(S) for sets S consisting of the sum of an infinite sequence of intervals without common points, and then use these in turn to extend the range of meaning of the symbol p(S) to a vast class of sets S, namely the class of all Lebesgue-measurable sets. But if the reader happens not to have studied the Lebesgue integral, he is not hopelessly handicapped as far as the needs of this book are concerned.

In order to unify the discussion, we observe that in each of our four examples (and in any other example we later encounter) we have as a starting point an aggregate P of things. In the first example, P consists of the numbers 1, 2, 3, 4, 5, 6. In the second example, it consists of all the points on the circumference of the circular track, and these can be symbolized by the real numbers $0 \leq x < C$. In the third example, P consists of all the pairs (1, 1), ..., (6, 6). In the fourth example, it consists of all the n-tuples of the form (point of first circle, ..., point on n-th circle) and can be more conveniently symbolized by the n-tuple $(x_1, \ldots, x_n)$ representing a point in the "cube" $0 \leq x_i < C$, $i = 1, \ldots, n$. To certain subsets of the population we have assigned numbers called their probability measures. This is done for all subsets of P in the first and third examples, in which there are only a finite number of such subsets. In the second and third examples, the sets S to which probability measure has been assigned include all sets consisting of a finite number of intervals; and this class could conceivably (in fact, actually) be enlarged still further. Let us say that the sets for which a definite probability measure has been assigned are "measurable sets." Then to each measurable set S there corresponds a probability measure p(S), and the sets and their measures satisfy the following conditions.

(1a) *If S is measurable, then* $0 \leq p(S) \leq 1$.

(1b) *The set P consisting of the whole population is measurable, and its probability measure is* 1.

(1c) *The set consisting of no points at all is measurable, and its probability measure is* 0.

(1d) *If* S_1 *and* S_2 *are measurable, so is the set* $S_1 + S_2$ *consisting of all points belonging to one or both of the sets. Moreover, if* S_1 *and* S_2 *have no common points, then*

$$p(S_1 + S_2) = p(S_1) + p(S_2).$$

(1e) *If* S_1 *and* S_2 *are measurable, so is the set of points which belong both to* S_1 *and to* S_2.

(1f) *If* S_1 *is measurable, so is the set* S_2 *consisting of all points of P which do not belong to* S_1*; and* $p(S_1) + p(S_2) = 1$.

All four of our examples satisfy these conditions, and henceforth we shall assume them satisfied for every example in probabilities which we shall investigate.

18. *Expected values; measurable functions.*

Suppose next that the population has been subdivided into a finite number of measurable sets $S_1, \ldots, S_k$, having no common points, and that a player is playing a game in which he is to receive a certain sum f_1 if the outcome of the game is in S_1, a sum f_2 if the outcome is in S_2, and so on. If a large number of games is played, say N of them, in approximately $Np(S_1)$ of these games the outcome is in S_1, and his receipts from these games add up to approximately $f_1Np(S_1)$; in approximately $Np(S_2)$ of the games the outcome is in S_2, and his receipts from these games add up to approximately $f_2Np(S_2)$; and so on. His total receipts from all games will be approximately

$$N[f_1p(S_1) + \dots + f_kp(S_k)].$$

So his average receipts per game are given by the coefficient of N in this expression. This average is called his "mathematical expectation." To represent it somewhat more conveniently, we define a function $f(q)$ for all elements q of the whole population P, by setting $f(q) = f_i$ if q is in S_i. Then if the outcome of a game is expressed by the symbol q, the player receives an amount $f(q)$. His "mathematical expectation" is denoted by the symbol $E[f]$; and according to the foregoing discussion, it is given by the expression

(1) $$E[f] = \sum_{i=1}^{k} f_i\, p(S_i).$$

The mathematical expectation of $f(q)$ is also called its "expected value."

This definition may seem ambiguous, for the choice of the S_i is somewhat arbitrary; for example, if $f_1 = f_2$ we can combine S_1 and S_2 into a single set on which f is constant, and then there will be only k - 1 sets. Given any function f of the type described, we can combine all sets on which f has each one of its values. This corresponds to bracketing terms in (1), and does not change the value of the sum. So the freedom of choice of the S_i does not affect the value of the sum in (1).

A simple but important property of the expected value as defined in (1) is

(2) _If_ $f(q) \le g(q)$ _for each_ q _in_ P _then_ $E[f] \le E[g]$.

Let $f_1, \ldots, f_m$ be the distinct values f can assume, and let S_i be the set on which $f = f_i$. Let $g_1, \ldots, g_n$ be the distinct values g can assume, and let S_j' be the set on which $g = g_j$. We are assuming these to be measurable. Let $S_{i,j}$ be the set of all points which belong to S_i and also belong to S_j'. These sets $S_{i,j}$ have no points in common; every point of P belongs to one of them; and they are measurable by (17.1e). On each of them f and g are both constant. In $S_{i,j}$ select a point $q_{i,j}$ unless $S_{i,j}$ is empty (in which latter case discard it). Clearly

$$E[f] = \Sigma f(q_{i,j})\, p(S_{i,j}),\quad E[g] = \Sigma\, g(q_{i,j})\, p(S_{i,j}).$$

Since p is never negative, this leads at once to (2).

In the first and third of our examples this completes the definition of expected value; for when the population contains only a finite number of points every function f(q) must have only a finite number of values. But the situation is different when the population is infinite, as in the second and fourth examples. In such cases it may be possible to extend the definition of expected value so as to cover many instances of functions taking on infinitely many different values. This is done by a process essentially the same as that used in defining an integral. Suppose that f(q) is a function defined and bounded over the population. There will always exist functions G(q) which take on a finite number of values, each on a measurable set, and which are nowhere less than f(q). Since we wish to preserve the important property (2), any reasonable way of assigning an expected value to f(q) must be such that $E[f] \leq E[G]$ for every such function G(q). Consequently, if we form the mathematical expectation of each such function G(q), we obtain a collection of overestimates for E[f]. In a like manner, we can take all the functions g(q) each of which takes on only a finite number of values, each on a measurable set, and find the mathematical expectation of each such g(q). We thus obtain a collection of underestimates for E[f]. In favorable cases (for instance, in the second and fourth examples, whenever f(q) is continuous) it will happen that there is exactly one number J which separates the overestimates from the underestimates, each overestimate being $\geq J$ and each underestimate being $\leq J$. In such a case, we define the expected value of f to be this number J. Functions f(q) which are assigned mathematical expectations by this process will be called "integrable," since the process is essentially an integration. Moreover, it is easy to show that the elementary properties of the Riemann integral are also enjoyed by the function E[f]. For instance, the sum of two integrable functions is integrable, and the expected value of the sum is the sum of the expected values. If f(q) is integrable and k is a constant, then kf(q) is integrable and its expected value is kE[f]. The product of bounded integrable functions is integrable. Furthermore, from the definition we see without trouble that (2) continues to be true for integrable functions f and g.

As an instance of the foregoing, in the fourth example the finite collections of intervals of n-space could be used as the measurable sets, and in this case the definition of expected value is analogous to that of the Riemann integral. A function is integrable in the sense just defined if, and only if, it is Riemann-integrable, and in that case E[f] is C^{-n} times the Riemann integral of f(q) over the cube. If the class of measurable sets had already been enlarged as in the theory of the Lebesgue integral so as to contain all Lebesgue-measurable sets, the functions integrable in the sense of the preceding paragraph would be the same as the bounded Lebesgue-integrable functions, the expected value being C^{-n} times the Lebesgue integral.

The definition of expected value can also be extended to certain classes of unbounded functions. If f(q) is unbounded as q varies over the population P, we first form the auxiliary functions $f_{mn}(q)$ defined as follows. Whenever f(q) is between $-m$ and n inclusive,

$f_{mn}(q)$ is the same as $f(q)$. Where $f(q)$ exceeds n, we define $f_{mn}(q)$ to be equal to n; and where $f(q)$ is below $-m$, we define $f_{mn}(q)$ to be $-m$. Then for every m and n the function f_{mn} is bounded. Also, as m and n tend to ∞, $f_{mn}(q)$ approaches $f(q)$ at each q; for whenever m and n are both greater than $|f(q)|$ the definition makes $f_{mn}(q)$ equal to $f(q)$. It may happen that for each m and n the function $f_{mn}(q)$ is integrable (i.e., has an expected value) and that as m and n tend independently to ∞ this expected value of f_{mn} approaches a definite limit. In this case we say that $f(q)$ is an (unbounded) integrable function, and we define its expected value to be the limit of the expected value of f_{mn} as m and n both increase without bound.

It is easy to show that if f and g are both integrable and $f(q) \leq g(q)$ for all q, then $E[f] \leq E[g]$; this follows by a passage to the limit from (2). Moreover, if $f(q)$ is integrable and k is constant, then $kf(q)$ is integrable, and $E[kf] = kE[f]$. It is rather more difficult, but still possible, to show that if f and g are both integrable so is their sum, and $E[f + g] = E[f] + E[g]$. But it is not necessarily true that the product of unbounded integrable functions is integrable.

Given any set S, we can define the characteristic function of the set S to be the function $K(q|\text{q in S})$ which has the value 1 if q is in S and has the value 0 otherwise. We may think of $K(q|\text{q in S})$ as the "truth-value" of the statement "q is in S"; if the statement is true the "truth value" is 1, if the statement is false the "truth-value" is 0. We shall resist the temptation to abbreviate the phrase "characteristic function of the set" to "characteristic function." Although we shall not use it, there is a concept well-known in statistics under the name of the "characteristic function of a distribution," which is entirely different from the concept "characteristic function of a set," and we avoid the abbreviation in order to reduce the chance of confusion in the mind of a reader who has already encountered the concept "characteristic function of a distribution."

The function $K(q|q \text{ in } S)$ has the value 1 on the set S, and on the remainder P - S of the population it has the value 0. So if S happens to be measurable, by (1) the expected value of $K(q|q \text{ in } S)$ is

$$(3) \qquad E[K(q|q \text{ in } S)] = 1\,p(S) + 0p(P - S).$$

Thus if S is a measurable set,

$$(4) \qquad E[K(q|q \text{ in } S)] = p(S).$$

But it is possible that the left member of (4) may exist even though S was not one of the sets originally listed as measurable. In this case we can enlarge the class of measurable sets, including in the enlarged class all sets S for which $K(q|q \text{ in } S)$ is integrable and defining the probability measure by (4). For instance, in the fourth example we started with the intervals as the measurable sets. Then E[f] is defined, as we have already seen, whenever f(q) is Riemann-integrable. But there are many sets besides the intervals whose characteristic functions are integrable; in fact, all the elementary geometric figures have this property. Thus with the help of (4) the class of measurable sets is enlarged so as to contain a multitude of new sets, in particular all the elementary geometric figures. If the class of measurable sets had already been enlarged so as to include all Lebesgue-measurable sets, using (4) would have given us nothing new.

From now on we shall restrict our attention to a particular sub-class of the integrable functions, which we shall call the "measurable" functions. These are the integrable functions such that for every real number y the set $S(f < y)$ of all points q at which $f(q) < y$ is a measurable set. If the concept of measure has been extended as in the Lebesgue theory, this is not really a restriction, since every integrable function is then measurable. But if the Riemann type of definition is used for the expected value, the restriction to measurable functions may be a real restriction, which, nevertheless, does not exclude any of the functions which we shall encounter later in this book.

19. Cumulative distribution functions.

If $f(q)$ is a measurable function and y is a real number, the statement "$f(q)$ is less than y" is equivalent to the statement "q is in the set $S(f < y)$ of points q at which $f(q)$ is less than y." But the probability that q is in the set $S(f < y)$ is the same as the probability measure of the set $S(f < y)$, by the very meaning of probability measure. In turn, the probability measure of the set $S(f < y)$ is the same as the mathematical expectation of the characteristic function $K(q|f < y)$ of the set $S(f < y)$, as we saw in (18.4). Hence the probability that $f(q)$ is less than y is the same as the mathematical expectation of the function $K(q|f < y)$. For simplicity of notation, we shall denote by $k(y)$ the probability that $f(q)$ is less than y. Then $k(y)$ satisfies the equation

$$k(y) = E\left[K(q|f < y)\right]. \tag{1}$$

The function $k(y)$ is called the "cumulative distribution function" of the function $f(q)$.

In many instances all that we wish to know about $f(q)$ is summarized in the knowledge of the cumulative distribution function $k(y)$. For example, suppose that we wish to form a life insurance company, and as a basis for fixing premiums we first want to know what would have been fair premiums to charge the members of a large group of people whose life-spans have been recorded. The population P will consist of the people in this list, and for each person q in the population, $f(q)$ is the span of his life. For each y, $k(y)$ is the measure of the part of the population whose life-spans are less than y. A table of $k(y)$ is then essentially the same as a mortality table (the usual mortality tables are tables giving values of 100000(1 - k(y))). Such a table is all we need in order to determine proper premiums for any assigned rate of interest.

It is worth remarking, however, that the cumulative distribution function of any one f(q) may fail to summarize all the information about the population needed for all purposes. For instance, in order to determine the proper premiums for a special policy to be issued only to lawyers we cannot rely on k(y) alone, but must go back to the original population with its probability measure, select the lawyers from it, and construct a new cumulative distribution function for this selected subpopulation.

Since k(y) is the probability measure of a set, its values must lie between 0 and 1 inclusive. Moreover, if $y < z$ then $k(y) \leq k(z)$, for then the set on which $f(q) < y$ is included in the set on which $f(q) < z$, and so the probability measure k(y) of the former set cannot be greater than the probability measure k(z) of the latter. If f(q) is bounded, k(y) is 0 for all y less than or equal to the lower bound of f(q), for then the set on which $f < y$ contains no points at all and has probability measure 0. Also k(y) is 1 for all y greater than the upper bound of f(q), for then the set on which $f(q) < y$ consists of the entire population P and has probability measure 1.

If it happens that k(y) is the indefinite integral of some function $k^*(y)$, so that

$$(2) \qquad k(y) = \int_{-\infty}^{y} k^*(y)\,dy,$$

the function $k^*(y)$ is called the "probability density" of the distribution of f(q).

The probability that f(q) is less than c is k(c); the probability that it is less than d is k(d). So if $c < d$ the probability that $c \leq f(q) < d$ is the difference $k(d) - k(c)$ between these probabilities. Thus we are permitted to regard the values assumed by the function $f(q)$ as a new population lying on the real axis. Each interval of the form $c \leq y < d$ has probability measure $k(d) - k(c)$, which is non-negative and not greater than 1. The whole real axis has probability measure 1; for as c tends to $-\infty$ and d tends to ∞ the probability that $c \leq f(q) < d$ tends to 1. In fact, we can verify that the probability measure $k(d) - k(c)$ satisfies all the requirements (17.1). The important feature of this point of view is that the new population, consisting of real numbers, may be essentially simpler to handle than the original population, which was of a highly unrestricted nature. Of course the new population of real numbers, with probability measure $k(d) - k(c)$, can only be of use in studying functions which are determined by the values of $f(q)$. For instance, the cumulative distribution function of the function $f(q)$ which is the sum of the numbers turned up on a pair of dice will give us all the information we need to discuss probabilities in the game of craps. But it would be inadequate to determine the probability that a pair consisting of a one and a three will appear before a pair whose sum is seven.

The theorem that relates the new population of real numbers to the original population, and allows us to investigate the properties of functions determined by the value of $f(q)$ without going back to the original population, is the following:

(3) Theorem. *If $h(y)$ is continuous for all y, and $f(q)$ is an integrable function with cumulative distribution function $k(y)$, then the function $h(f(q))$ is integrable (has an expected value) if and only if $h(y)$ is Stieltjes-integrable with respect to $k(y)$ from $-\infty$ to ∞; and in that case*

$$E\left[h(f)\right] = \int_{-\infty}^{\infty} h(y)\, dk(y).$$

We shall not try to prove this in general; instead, we shall assume that $f(q)$ is bounded. Let a be a number below the lower bound of f, and b a number above its upper bound. Then the integral in (3) is the same as the integral from a to b; for $k(y)$ is identically 0 from $-\infty$ to a and identically 1 from b to ∞, and by (16.9) and (16.14) the integral over each of these infinite intervals is zero. We subdivide the interval from a to b by points

$$a = y_0 < y_1 < \dots < y_n = b,$$

and we denote by m_i and M_i respectively the least and greatest values of $h(y)$ on the subinterval

$$y_{i-1} \leq y \leq y_i.$$

If ϵ is a positive number, we may suppose that the intervals are all small enough so that $h(y)$ changes by less than ϵ on each one; then $M_i - m_i < \epsilon$ for each i. Now we define a function $M(y)$ on the interval $a \leq y < b$ as follows: if y is in the interval $y_{i-1} \leq y < y_i$, then $M(y)$ is defined to be equal to M_i. We define $m(y)$ analogously, using m_i instead of M_i. It is clear that

$$m(y) \leq h(y) \leq M(y) \tag{4}$$

for all y in the interval $a \leq y < b$. Another way of writing the functions $M(y)$ and $m(y)$ is as follows. First, let $F_i(y)$ denote the function which is 1 if $y_{i-1} \leq y < y_i$ and is 0 elsewhere. (This is the same as the characteristic function $K(y|y_{i-1} \leq y < y_i)$ of the interval $y_{i-1} \leq y < y_i$, but we wish to avoid the more complicated notation.) Then we can show that

$$m(y) = \sum_{i=1}^{n} m_i F_i(y), \quad M(y) = \sum_{i=1}^{n} M_i F_i(y). \tag{5}$$

For if y is in an interval $y_{i-1} \leq y < y_i$, only the single term with subscript i is different from zero in each sum; and this term is m_i in the first sum and M_i in the second.

Since F_i is the characteristic function of the interval $y_{i-1} \leq y < y_i$, the expected value of $F_i(f(q))$ is the same as the probability that $f(q)$ is in the interval, which in turn is the probability that $f(q) < y_i$ minus the probability that $f(q) < y_{i-1}$, which is the same as $k(y_i) - k(y_{i-1})$. Hence

$$E[F_i(f)] = k(y_i) - k(y_{i-1}).$$

We now multiply by M_i and sum for $i = 1, \ldots, n$, obtaining with the help of (5)

$$\begin{aligned} \sum_{i=1}^{n} M_i [k(y_i) - k(y_{i-1})] &= \sum_{i=1}^{n} M_i E[F_i(f)] \\ &= E[\sum_{i=1}^{n} M_i F_i(f)] \qquad (6) \\ &= E[M(f)]. \end{aligned}$$

In the same way, if we multiply by m_i and sum we find that the corresponding left member is equal to the expected value of the function $m(f(q))$:

$$\sum_{i=1}^{n} m_i [k(y_i) - k(y_{i-1})] = E[m(f)]. \qquad (7)$$

Since m_i and M_i differ by less than ε, it is easy to show that the left members of (6) and (7) also differ by less than ε. Also, by the definition of the Stieltjes integral, if the partition of the interval from a to b is fine enough, the left members of both equations (6) and (7) will be arbitrarily close to the Stieltjes integral

$$\int_a^b h(y)\, dk(y). \qquad (8)$$

Because of (4), we see that $E[M(f)]$ is an overestimate for $E[h(f)]$, while $E[m(f)]$ is an underestimate for $E[h(f)]$. Since the left members of (6) and (7) differ by less than ε, so do the right members, and thus we have an underestimate and an overestimate for $E[h(f)]$ which differ by less than ε. Thus there cannot be two distinct numbers separating all overestimates from all underestimates, which by definition means that $h(f(q))$ has an expected value ("is integrable"). Moreover, there are both underestimates and overestimates as close as desired to the Stieltjes integral (8), which is possible only if this Stieltjes integral is itself the number which separates overestimates from underestimates. That is, we have proved (3) for bounded functions $f(q)$.

To suggest the type of use to which this theorem can be put, let us imagine a game played with n roulette wheels in which a player is to receive a sum $g(y)$ determined by the sum of the numbers $x_1, \ldots, x_n$ which appear on the various wheels. Let

$$f(q) = x_1 + \ldots + x_n.$$

The player wishes to know the expected value of the sum he will receive. This sum is $g(f(q))$, and its expected value is $E[g(f)]$. By (3), we can compute this without further reference to the population P if we only know the cumulative distribution function $k(y)$ of the sum $f(q)$.

If the function $f(q)$ takes on only a finite number of values, which we may suppose arranged in increasing order and denoted by $f_1, f_2, \ldots, f_n$, and if we denote by S_i the set on which $f(q) = f_i$, then for each number y such that $y_{i-1} < y \leq y_i$ the inequality $f(q) < y$ is satisfied on the set $S_1 + \ldots + S_{i-1}$ and nowhere else. So by its definition $k(y)$ is the probability measure of the set:

(9) $k(y) = p(S_1) + \ldots + p(S_{i-1})$ for $y_{i-1} < y \leq y_i$.

For example, consider the population in the second example, consisting of the thirty-six pairs

$$(1, 1), \ldots, (6, 6).$$

For $q = (a, b)$ we define $f(q) = a + b$. The values of $f(q)$ are the integers 2 to 12 inclusive. For $y \leq 2$ there is no q with $f(q) < y$, so $k(y) = 0$. For $y > 12$ the inequality $f(q) < y$ holds for all q, so $k(y) = 1$. The set S_1, on which $f(q) = 2$, contains the single member (1, 1), so $p(S_1) = 1/36$. The set S_2, on which $f(q) = 3$, has two members, (1, 2) and (2, 1), and so $p(S_2) = 2/36$. Proceeding thus, we find that $k(y)$ has jumps at 2, 3, ..., 12, being constant between these values; and on the intervals $2 < y \leq 3$, $3 < y \leq 4$, ..., $11 < y \leq 12$ it has values 1/36, 3/36, 6/36, 10/36, 15/36, 21/36, 26/36, 30/36, 33/36, 35/36.

For the function $k(y)$ in (9), the integral of the continuous function $g(y)$ with respect to $k(y)$ is

$$(10) \quad \int_{-\infty}^{\infty} g(y)\, dk(y) = g(f_1)p(S_1) + \ldots + g(f_n)p(S_n).$$

In the example of the two dice, if $f(q)$ is the sum of the numbers on the two dice, the mathematical expectation of any function $g(f)$ is

$$\begin{aligned} &g(2)/36 + 2g(3)/36 + 3g(4)/36 + 4g(5)/36 \\ &\quad + 5g(6)/36 + 6g(7)/36 + 5g(8)/36 + 4g(9)/36 \\ &\quad + 3g(10)/36 + 2g(11)/36 + g(12)/36. \end{aligned}$$

Thus if a player is to receive one unit of money if he throws a natural (meaning a 7 or an 11 on the first roll) the function $g(f)$ has value 1 for $f = 7$ and for $f = 11$, and has value 0 elsewhere. Hence the expected value is $6g(7)/36 + 2g(11)/36 = 2/9$.

20. Variance; the normal distribution.

Let $f(q)$ be a function having a mathematical expectation $\bar{f}$. For each q, the difference $f(q) - \bar{f}$ will be called the deviation of $f(q)$, or its deviation from the mean. The mathematical expectation of the deviation is

$$E[f - \bar{f}] = E[f] - E[\bar{f}]$$
$$= \bar{f} - \bar{f} = 0.$$

If $f(q)$ has a cumulative distribution function $k(y)$, and also is bounded, by (19.3) with $h(y) = y$ we find

$$E[f] = \int_{-\infty}^{\infty} y \, dk(y). \tag{1}$$

The expected value of the square of $f(q)$, if it exists, is called the second moment of the distribution about 0. The expected value of the square of the deviation, $E[(f - \bar{f})^2]$, is called the variance (or dispersion) of the distribution. If $f(q)$ is bounded and has a cumulative distribution function $k(y)$, we need only take $h(y) = y^2$ in (19.3) to see that the variance is necessarily defined and satisfies

$$E[(f - \bar{f})^2] = \int_{-\infty}^{\infty} y^2 \, dk(y). \tag{2}$$

The variance is the difference between the second moment about 0 and the square of the mathematical expectation, for

$$\begin{aligned} E[(f - \bar{f})^2] &= E[f^2] - E[2f\bar{f}] + E[\bar{f}^2] \\ &= E[f^2] - \bar{f}^2. \end{aligned} \tag{3}$$

The "standard deviation" of $f(q)$ (or of its distribution) is the square root of the variance. It is customarily denoted by the letter σ.

A particularly important class of functions consists of those with "normal distribution," which means that there is a number σ such that the cumulative distribution function is

$$(4) \quad k(y) = (1/\sigma\sqrt{2\pi}) \int_{-\infty}^{y} e^{-(y^2/2\sigma^2)}\, dy.$$

It does not follow at once from the proof of (19.3) that for every continuous function g(y) the expected value of g(f(q)) is

$$(5)\ E[g(f)] = (1/\sigma\sqrt{2\pi}) \int_{-\infty}^{\infty} g(y)\, e^{-(y^2/2\sigma^2)}\, dy,$$

because (19.3) applies only to bounded distributions, and the normal distribution corresponds to unbounded functions. However, if g(y) is a polynomial (5) can be obtained from (19.3) by a fairly simple limiting process, which we shall not exhibit here.

It is obvious from (4) that k(y) tends to 0 as y tends to $-\infty$. It is not at all obvious that k(y) tends to 1 as y tends to ∞, as is required of every cumulative distribution function. But the proof that this is in fact the case can be found in most advanced calculus texts.

Taking g(y) = y in (5) shows that E[f] = 0, since the integrand in the right member is an odd function of y. The variance is found by setting $g(y) = y^2$ in (5). By integration by parts, with the help of the relations $k(\infty) = 1$ and $k(-\infty) = 0$, we can show that the variance is σ^2. Thus the σ in (4) and (5) is actually the standard deviation of the distribution, as the notation suggests.

There are many published tables of the function (4) as a function of y/σ, or what amounts to the same thing, of the function (4) as a function of y for $\sigma = 1$. From any such table we find that

$$(6) \qquad k(2.585\sigma) - k(-2.585\sigma) = 0.99.$$

So the probability that f(q) lies between -2.585σ and $+2.585\sigma$ is 0.99.

A function f(q) over a finite population P cannot be normally distributed; for it has only a finite number of values and, as (19.9) shows, its cumulative distribution function has to be discontinuous. However, if ε is a positive number, the number of elements q in the finite population P may be large enough to allow the possibility of functions f(q) whose cumulative distribution functions are everywhere within ε of being equal to a normal distribution function. If f(q) is such a function, we see by (6) that the fraction of the population having f(q) less than 2.585 standard deviations different from 0 is somewhere between $0.99 - 2\varepsilon$ and $0.99 + 2\varepsilon$. Thus the tables of the normal distribution function can serve conveniently in the study of functions over a population containing a finite but large number of elements.

A concept that is often referred to, especially in connection with symmetric distributions (in which $k(y) + k(-y) = 1$), is the "probable error." This is the number y such that

$$k(y) - k(-y) = 0.5.$$

In other words, there is a probability of 1/2 that f(q) is between $-y$ and y. If the distribution is normal, we find from the tables of the probability integral that the probable error is 0.67449σ, so that

$$k(0.67449\sigma) - k(-0.67449\sigma) = 0.5.$$

The concept of probable error is at its most useful when the distribution is known to be normal, for then it determines the standard deviation, being equal to 0.67449σ, and this in turn determines the distribution completely. Thus when we know the probable error we can find the value of y for which there is any desired probability that f(q) is between - y and y. But if the distribution is not normal we can draw no such conclusions. We must state separately, from other information, what values of y correspond to different probabilities. For instance, the distribution of the ranges of rockets fired from a tube at a fixed small elevation is far from normal. There is presumably zero probability of negative range, and the distribution is markedly unsymmetrical. Hence the probable error does not tell us very much about the distribution. Because of this defect, the use of the probable error has become less frequent in the last few years.

21. Independent distributions.

When we need to discuss two functions $f_1(q)$ and $f_2(q)$ simultaneously, we often profit by using the "bivariate cumulative distribution function" or "joint distribution function" k(y, z) defined by the equation

$$(1)\quad k(y, z) = E\left[K(q \mid f_1(q) < y \text{ and } f_2(q) < z)\right].$$

That is, k(y, z) is the probability that f_1 be less than y and f_2 less than z at the same time. Given any four numbers, a, A, b and B such that $a < A$ and $b < B$, the probability that $f_1 < A$ and $f_2 < B$ is k(A, B), while the probability that $f_1 < A$ and $f_2 < b$ is k(A, b). Hence the probability that $f_1 < A$ while $b \leq f_2 < B$ is the difference k(A, B) - k(A, b). In the same way we see that the probability that $f_1 < a$ while $b \leq f_2 < B$ is k(a, B) - k(a, b). By subtraction, the probability that $a \leq f_1 < A$ and $b \leq f_2 < B$ at the same time is

$$(2)\quad \Delta_k J = k(A, B) - k(A, b) - k(a, B) + k(a, b),$$

J denoting the interval $a \leq y < A$, $b \leq z < B$. (The symbol in the left member was already defined in (16.33)). In particular, the left member of (2) is non-negative, so that the integration theory in the latter part of Section 16 can be applied. Also, we can follow the proof of (19.3) for bounded f_1 and f_2, making only notational changes, and thus prove

(3) Theorem. _If $f_1(q)$ and $f_2(q)$ are bounded measurable functions, with joint cumulative distribution function $k(y, z)$, and if $h(y, z)$ is any continuous function, then_

$$E[h(f_1, f_2)] = \int\int h(y, z)\, dk(y, z),$$

the double integral being taken over the whole (y, z)-plane.

In the example of the two dice, we remarked that the two were independent in the sense that the motion of the first produced no forces or torques affecting the motion of the second; and that, as a consequence, the probability that the number on the first die would have a value a_1 and simultaneously the number on the second die would have a value b_1 was equal to the product of the probability (1/6) that the number on the first die would be a_1, and the probability (also 1/6) that the number on the second die would be b_1. From this it follows that for any two subsets S_1, S_2 of the population [1, 2, 3, 4, 5, 6], the probability that the number pair (a, b) is such that a is in S_1 and b in S_2 is equal to the product of the probability that a is in S_1 and the probability that b is in S_2. This can be conveniently formulated with the help of the mathematical expectation symbol. Let $k(q \mid a \text{ in } S_1)$ be, as before, the function which is 1 when the point q, or (a, b), has its first component a in the set S_1 and is 0 when a is not in S_1. The functions $K(q \mid b \text{ in } S_2)$ and $K(q \mid a \text{ in } S_1 \text{ and } b \text{ in } S_2)$ are analogously defined. Then the probability that a is in S_1 is the same as the mathematical expectation of $K(q \mid a \text{ in } S_1)$, and similarly for the other two functions. The independence, in a physical sense, of the two dice produces the mathematical consequence

$$(4)\quad E[K(q|a \text{ in } S_1 \text{ and } b \text{ in } S_2)] = E[K(q|a \text{ in } S_1)]\ E[K(q|b \text{ in } S_2)].$$

In the example of the n roulette wheels, a similar situation was observed. If S_1, S_2, ..., S_n were arcs on the first, second, ..., n-th circular tracks, the probability that the point moving on the first circle would stop in arc S_1, that on the second circle in arc S_2, and so on, was the same as the product of the probabilities of the separate events that the first point would stop in S_1, that the second would stop in S_2, and so on. In the notation of mathematical expectation

$$(5)\quad E[K(q|x_1 \text{ in } S_1 \text{ and } x_2 \text{ in } S_2 \text{ and } \dots \text{ and } x_n \text{ in } S_n)] = E[K(q|x_1 \text{ in } S_1)]\ E[K(q|x_2 \text{ in } S_2)] \dots E[K(q|x_n \text{ in } S_n)].$$

These mathematical consequences of the physical unrelatedness of the two dice, or of the n roulette wheels, lead us to a definition of the (mathematical, or statistical) independence of two or more functions over a population P. Let $f_1(q)$, ..., $f_n(q)$ be measurable functions defined over a population P. Let $k_i(y)$ be the cumulative distribution function of $f_i(q)$ (that is, the probability that f_i is less than y), and let $k(y_1, \dots, y_n)$ be their joint cumulative distribution function (that is, the probability that all the conditions $f_1(q) < y_1$, ..., $f_n(q) < y_n$ shall be true). Then the functions $f_i(q)$ are defined to be independent if for every set of real numbers y_1, ..., y_n it is true that the probability that all the conditions $f_i(q) < y_i$ hold simultaneously is the same as the product of the probability that the first one is true, the probability that the second is true, and so on.

In symbols, the f_i are independent if and only if

(6) $k(y_1, \ldots, y_n) = k_1(y_1)\, k_2(y_2) \ldots k_n(y_n)$.

This may seem to ask a little less than we asked in the several examples, since these examples dealt with intervals. But there is in fact no difference. Let J_i be the interval $a_i \leq y < b_i$ for each separate i $(i = 1, \ldots, n)$, and let J be the interval in n-dimensional space defined by the whole set of inequalities $a_i \leq y_i < b_i$. If $n = 2$, by substitution of (6) in (2) we find that

(7) $\Delta_k J = [\, k_1(b_1) - k_1(a_1)\,]\ [\, k_2(b_2) - k_2(a_2)\,]$.

Analogous results hold in space of n dimensions. In words, the probability that the point with coordinates $(f_1(q), \ldots, f_n(q))$ is in J is equal to the product of the probabilities that $f_1(q)$ is in J_1, that $f_2(q)$ is in J_2, and so on. We have already met functions satisfying (6) in Section 16; it was for just such functions that the double integral could be computed as an iterated integral.

One of the many important properties of sets of independent functions is the following.

(8) Theorem. If $f(q)$ and $g(q)$ are independent bounded functions, and $F(y)$ is a continuous function of the real variable y on some interval containing all the values of $f(q)$, and $G(z)$ is a continuous function of the real variable z on some interval containing all the values of $g(q)$, then

$$E[\, F(f)\, G(g)\,] = E[\, F(f)\,]\ E[\, G(g)\,].$$

Let $a \leq y \leq A$ be an interval containing all the values of $f(q)$, and let $F(y)$ be continuous on this interval. Let the interval $b \leq z \leq B$ contain all the values of $g(q)$, and let $G(z)$ be continuous on this interval. By (3), (6), (16.43) and (19.3),

$$E[F(f)\,G(g)] = \int_a^A \int_b^B F(y)\,G(z)\,dk(y, z)$$

$$= \int_a^A F(y) \left\{ \int_b^B G(z)\,dk_2(z) \right\} dk_1(y)$$

$$= E[G(g)] \int_a^A F(y)\,dk_1(y)$$

$$= E[G(g)]\;E[F(f)].$$

In particular, let $\bar{f}$ be the mathematical expectation of f, and $\bar{g}$ the mathematical expectation of g. Taking $F(y) = y$ and $G(z) = z$ in (8), we find that the mathematical expectation of f(q) g(q) is the product $\bar{f}\,\bar{g}$ of their mathematical expectations. Still more particularly, if the expected value of either function is zero, so is the expected value of the product, provided that the functions are independent.

Next consider two independent functions each with expected value 0. Applying (8) with $F(y) = y^2$ and $G(z) = z^2$ yields

$$(9) \qquad E[f^2 g^2] = E[f^2]\,E[g^2],$$

so that the variance of the product is equal to the product of their variances.

Again, let f and g be independent (bounded) functions, having the respective mathematical expectations $\bar{f}$ and $\bar{g}$. The mathematical expectation of the sum is $\bar{f} + \bar{g}$, and by definition the variance of the sum is

$$(10) \qquad \begin{aligned} &E[(f + g - \bar{f} - \bar{g})^2] \\ &= E[(f - \bar{f})^2] + 2E[(f - \bar{f})(g - \bar{g})] + E[(g - \bar{g})^2]. \end{aligned}$$

Applying (8) with $F(y) = y - \bar{f}$ and $G(z) = z - \bar{g}$, we find that the second term in the right member has value zero. Thus the variance of the sum of two independent (bounded) functions is the sum of their variances.

From this it follows at once that the standard deviation of the sum of two independent bounded functions is the square root of the sum of their squares, and similarly for the sum of any finite number of independent functions. This does not apply at once to normal distributions, which are unbounded. But the limiting process involved is a legitimate one, and it can be shown that the relation holds for normal distributions too. Since the probable error of a normal distribution is a constant multiple (0.6745) of its standard deviation, it is also true that the probable error of the sum of a finite number of independent normally distributed variables is the square root of the sum of the squares of their several probable errors.

22. Distributions with different domains of definition.

Suppose that a player engages in a sequence of n games, independent of each other in the physical sense (the result of a game having no influence on the outcome of any other game) and not necessarily all alike. In the first game the outcome is represented by a point q_1 in a population P_1, and the player's gain is $f_1(q_1)$; in the second game the outcome is represented by a point q_2 in a population P_2 (which may or may not be the same as P_1), and the player's gain is a function $f_2(q_2)$; and so on for all n games. The outcome of the whole sequence of games can be represented by a "point" $(q_1, \ldots, q_n)$, in which q_1 is the outcome of the first game, q_2 the outcome of the second game, and so on. The aggregate of all such n-tuples is itself a new population, which we designate by the symbol P. If the outcome of the sequence of games is represented by the element $(q_1, \ldots, q_n)$ of P, the player won $f_1(q_1)$ in the first game, $f_2(q_2)$ in the second, and so on. His total gain is

$$(1) \quad \begin{aligned} f(q) &= f(q_1, \ldots, q_n) \\ &= f_1(q_1) + \ldots + f_n(q_n). \end{aligned}$$

The function $f_1(q_1)$ can be regarded as a function on the population P. For if $q = (q_1, \ldots, q_n)$ is in P, the first element q_1 in the n-tuple q determines the value of f_1. In the same way each of the other functions f_i can be regarded as functions on the whole population P. It does not yet make sense to say that these are independent functions, because the definition in the paragraph containing equation (21.6) presupposes a probability measure on the population P, and we have not yet defined one. However, there is an obvious way to introduce such a probability measure. Let $S_1, S_2, \ldots, S_n$ be measurable sets in the respective populations $P_1, P_2, \ldots, P_n$. We then say that the set of all elements $q = (q_1, \ldots, q_n)$ in P such that q_1 is in S_1, q_2 in S_2, etc., is a measurable set, and we may assign it the probability measure $p(S_1)\, p(S_2) \ldots p(S_n)$. This accords with the idea of the physical independence of the games, since it amounts to saying that the probability that all the events q_1 in S_1, q_2 in S_2, etc., is the product of their individual probabilities. In particular, let $y_1, \ldots, y_n$ be real numbers, and let S_1 be the part of P_1 on which $f_1(q_1) < y_1$, S_2 the part of P_2 on which $f_2(q_2) < y_2$, and so on. Then if $k_i(y)$ is the cumulative distribution function of f_i, we have

(2) $$k_1(y_1) = p(S_1), \ldots, k_n(y_n) = p(S_n).$$

The joint cumulative distribution function of the f_i is the function $k(y_1, \ldots, y_n)$ which is the probability that all the conditions $f_1 < y_1, \ldots, f_n < y_n$ hold simultaneously. This last is equivalent to saying that all the conditions q_1 in $S_1, \ldots, q_n$ in S_n hold simultaneously. But by our definition of probability measure, the probability measure of the part of P on which all conditions q_1 in S_1, etc., hold is equal to the product of the separate probability measures $p(S_1) \ldots p(S_n)$. From this and (2) we see that (21.6) is satisfied, and the functions f_i are independent.

Some discussion such as the foregoing is needed to give a precise meaning to the idea of independence of the results of several games. If many games are played on one (honest) roulette wheel, the outcome of each single game can be represented by a single point of the interval $0 \leq x < C$, where C is the circumference of the wheel. But to obtain a satisfactory method of expressing the idea that the outcomes of the games are independent, and to be able, for example, to give a precise meaning to the idea of the cumulative distribution function of the sum of several functions, representing the gains in the individual games, we need to construct a new population, each of whose points can represent the outcomes of a whole sequence of games. On the other hand, once the concept has been clearly grasped, it is often possible and desirable to avoid specific mention of this population in the statements of our theorems. For example, by recalling that the expected value of the sum of functions is the sum of their expected values, we find from (1) that

$$(3) \qquad E[f] = E[f_1] + \dots + E[f_n].$$

In order to comprehend the meaning of this formula, we must have thought through the construction of the population P and the measure function on it, or some equivalent mental process. This is essential in order to understand what the left member means. In the right member we can think of each term as determined by the particular function f_i on the particular population P_i. But in the statement of the formula, no visible reference to the population P occurs, which contributes to the simplicity of the statement. Equation (3) is merely a simple example of a large class of formulas. One highly important formula permits computing the cumulative distribution function of the sum $f_1 + \dots + f_n$ from the cumulative distributions of the several functions f_i, without going back to the original populations. We do not need this formula, so we shall not derive it. But the possibility of finding the cumulative distribution function of the sum without mentioning the populations P, P_1, etc. shows that we can, for example, find the variance of the sum without exhibiting the details of the construction of the population P. This is especially convenient when we wish to consider an unending sequence of games and to find some property of the sum (for instance, the expected value, or the cumulative distribution function) at the end of one game, at the end of two games, at the end of three games, and so on without end. For then each step in the process requires construction of a new population P, first P_1, then pairs of elements (q_1, q_2) with q_1 in P_1 and q_2 in P_2, and so on. But if the properties being studied can be expressed in terms of the cumulative distribution functions of the sums, then we can avoid the detailed exposition of these populations, and state the results more compactly, often with a gain in intelligibility.

This gain in compactness of statement will now be exhibited in a special case of the highly important "central limiting theorem of probability theory." This we shall state without proof; for the proof, the reader may refer to the books by Uspensky and Cramér cited in Section 17.

(4) Theorem. Let $f_1, f_2, \ldots$ be an infinite sequence of functions all having the same bound, each of which has expected value zero, and every finite set of which is independent. Let V_n be the variance of f_n, and assume that the series $V_1 + V_2 + \ldots$ is divergent. Define

(5) $$s_n = (f_1 + \ldots + f_n)/\sqrt{V_1 + \ldots + V_n}\,.$$

Then the cumulative distribution function of s_n tends uniformly to the cumulative distribution function of a normal distribution with standard deviation $\sigma = 1$.

Let $k_n(y)$ be the cumulative distribution function of s_n; by the theorem,

(6) $$\lim_{n \to \infty} k_n(y) = (1/\sqrt{2\pi}) \int_{-\infty}^{y} e^{-y^2/2}\, dy.$$

For example, let an unbiased coin be tossed; if it falls heads (H) a player wins a dollar, if it falls tails (T) he loses a dollar. The set consisting of the single point H has measure 1/2, as has the set consisting of the single point T. The function defining the player's winnings is $f(H) = 1$, $f(T) = -1$. The mathematical expectation is $(1)(1/2) + (-1)(1/2) = 0$, the variance is $(1)^2(1/2) + (-1)^2(1/2) = 1$. The radical in the denominator of (5) is $\sqrt{n}$. The sum of the player's winnings after n games is the numerator in the right member of (5), which is the same as $\sqrt{n}\, s_n$. If n is large, this will have a distribution function which is nearly equal to the right member of (6). In particular, the right member of (6) is 0.005 for $y = -2.585$. So for large n the value of $k_n(-2.585)$ will also be nearly 0.005. That is, in a large number n of games the player has one chance in two hundred of losing more than $2.585\sqrt{n}$ dollars.

For another example, closely related to an application made in a later chapter, let us suppose that we are adding n numbers, each having been rounded off to the nearest multiple of some number α. (If for instance we were carrying one decimal place, α would be 0.1.) We wish to find the distribution of the resulting rounding error in the sum of the numbers. Clearly we need not bother about the integral multiples of α; all we have to watch is the discarded part of each summand. This discarded part (rounding error) is equally likely to be any number between $-\alpha/2$ and $\alpha/2$. Its cumulative distribution function is 0 for $y < -\alpha/2$, 1 for $y \geq \alpha/2$, and is linear between $-\alpha/2$ and $\alpha/2$. It is in fact the integral of the function $k'(y)$ which is equal to $1/\alpha$ between $-\alpha/2$ and $\alpha/2$ and is zero elsewhere; so by (16.13) and (19.3), if $h(y)$ is continuous for $-\alpha/2 \leq y \leq \alpha/2$

$$E[h] = \int_{-\alpha/2}^{\alpha/2} h(y)\,[1/\alpha]\,dy.$$

By (20.1), the expected value of the error is 0. By (20.2), its variance is

$$(7) \qquad V = \int_{-\alpha/2}^{\alpha/2} y^2(1/\alpha)\, dy.$$

Thus the denominator in the right member of (5) is $\alpha\sqrt{n/12}$. As in the preceding example, there is a probability 0.005 (approximately) that the sum of n errors will be less than $-2.585\ \alpha\sqrt{n/12}$, which is about $-0.7465\ \alpha\sqrt{n}$. Likewise there is an equal probability that the sum will exceed $0.7465\ \alpha\sqrt{n}$. So the probability is 0.99 that the absolute value of the error will not exceed $0.7465\ \alpha\sqrt{n}$. For instance, if the summands are rounded to the nearest whole number, so that $\alpha = 1$, and there are 45 summands, the probability is 0.99 that the error in the sum will not exceed $0.7465\sqrt{45}$, which is very nearly 5. If we wish the sum of 45 summands to have an error which has only one chance in a hundred of being as great as 0.5, we must write each summand accurately to the nearest tenth.

Chapter II

THE FORCE SYSTEM ON A PROJECTILE

1. Gravitational and Coriolis forces on a projectile.

A body moving through the atmosphere is subject to the action of two distinct classes of forces. First, there is a force due to the flow of the air about the body. Second, there are the gravitational attractions of the earth and the other members of the solar system; and if as usual the motion is referred to axes fixed with respect to the earth, there will also be the centrifugal and Coriolis forces introduced by the fact that the axes do not form an inertial frame. These latter forces will form the subject of this section. The aerodynamic forces will be discussed in the following sections.

First, let us dispose of the effects of the attractions of members of the solar system other than the earth itself. For this purpose it is accurate enough to regard the earth and the sun as spherical, and composed of concentric homogeneous spherical shells. In any text on celestial mechanics it is shown that the gravitational field of such bodies, outside of their surfaces, is the same as though their masses were concentrated at their centers. We shall also suppose that the earth moves in a circular orbit of radius 93,000,000 miles, having its center at the center of gravity of the system consisting of earth and sun. The angular velocity of the earth's center about the sun is about $2\pi/31,557,000 = 1.99 \cdot 10^{-7}$ radians per second, since there are approximately

31,557,000 seconds in a year. The centrifugal acceleration of the earth's center is found by multiplying the square of this quantity by the radius of the orbit in feet; it is about .019 feet per second per second. Consider now a system of axes with origin at the center of the earth and with axes having fixed directions with respect to an inertial frame. All points fixed with respect to this system have the same velocity with respect to an inertial frame, and therefore have the same acceleration, which is the acceleration at the earth's center caused by the gravitational acceleration field of the sun.

Let $\mathbf{a}_P$ denote the acceleration due to the sun's gravitational field at the point P. If a particle has position vector $\mathbf{X}$ with respect to the axes just described, its acceleration relative to an inertial frame is the sum of its acceleration $\mathbf{X}^{(\cdot\cdot)}$ with respect to the axes and the acceleration of the axis system, which is the same as the gravitational acceleration $\mathbf{a}_C$ at the center C of the earth. So if $\mathbf{F}$ is the sum of all forces other than the sun's gravitational attraction acting on the particle, its motion satisfies the equation

$$\mathbf{X}^{(\cdot\cdot)} + \mathbf{a}_C = \mathbf{F}/m + \mathbf{a}_P,$$

where P is the point at which the particle is located. Therefore if $\mathbf{a}_P$ were identically equal to $\mathbf{a}_C$ we could cancel these terms and find that the motion of the particle would be the same as though the earth's orbital motion and sun's attraction simultaneously disappeared. The amount by which the equation $\mathbf{X}^{(\cdot\cdot)} = \mathbf{F}/m$ fails to be correct is the difference between $\mathbf{a}_C$ and $\mathbf{a}_P$. It is not difficult to see that for all points on the surface of the earth, this difference has its greatest value at the point P nearest the sun. At that point the acceleration $\mathbf{a}_P$ has the same direction as $\mathbf{a}_C$, and its magnitude is greater in the ratio of $(93,000,000)^2$ to $(92,996,000)^2$, since the radius of the earth is about 4,000 miles. The ratio differs from unity by about 1/11,500, and $|\boldsymbol{a}_C|$ is about .019 feet per second per second, so $|\boldsymbol{a}_P - \boldsymbol{a}_C|$ cannot exceed .000,001,8 feet per second per second, which is entirely negligible for ballistic purposes. A similar discussion shows that the effect of the moon's attraction, though somewhat larger, is also negligible, while the effects of the other planets are far smaller.

The effects of the rotation of the earth cannot be so simply dealt with. They turn out to be large enough to be worth computing when the trajectory is a long one. Let us denote the angular velocity vector of the earth by $\boldsymbol{\omega}$. This vector is parallel to the earth's axis, in the direction from center to north pole. Since there are 86,164 seconds in a sidereal day, the magnitude of $\boldsymbol{\omega}$ is

$$(1) \qquad \omega = |\boldsymbol{\omega}| = 2\pi/86,164 = 7.29 \cdot 10^{-5} \text{ radians per second,}$$

approximately.

Let us choose a coordinate system with origin O on or outside of the surface of the earth, and having axes fixed relative to the earth. Let $\mathbf{r}$ be the vector from the center C of the earth to the point O, and let $\mathbf{x}$ be the vector from O to a moving point P. If the earth were a homogeneous sphere, the gravitational acceleration due to the earth's attraction at P would be proportional to $|\mathbf{r} + \mathbf{x}|^{-2}$ and opposite in direction to $\mathbf{r} + \mathbf{x}$, and would therefore have the form $\boldsymbol{a} = -k(\mathbf{r} + \mathbf{x})/|\mathbf{r} + \mathbf{x}|^3$, where k is a positive constant. Since the earth is not a sphere, this is in error by a small quantity $\mathbf{e}$, and we have

$$(2) \qquad \boldsymbol{a} = -k(\mathbf{r} + \mathbf{x})/|\mathbf{r} + \mathbf{x}|^3 + \mathbf{e}.$$

Both by theory and by experiment it can be shown that $|\mathbf{e}|$ is well under one per cent of $|\boldsymbol{a}|$ at all points on or near the surface of the earth.

Let $\mathbf{F}$ denote the sum of all the forces acting on the particle located at P, other than the gravitational attraction of the earth and fictitious forces arising from the fact that the coordinate frame is not inertial. These forces $\mathbf{F}$ consist of the aerodynamic forces in the case of a bomb or a shell; for rockets they also include the propulsive force. If the mass of the particle is m, by (I.13.8) the acceleration $\mathbf{x}^{(\cdot\cdot)}$ with respect to the axis system is given by

$$\mathbf{x}^{(\cdot\cdot)} = \mathbf{F}/m - k(\mathbf{r} + \mathbf{x})/|\mathbf{r} + \mathbf{x}|^3 + \mathbf{e}$$

$$(3) \qquad + [(\boldsymbol{\omega}\cdot\boldsymbol{\omega})(\mathbf{r} + \mathbf{x}) - (\boldsymbol{\omega}\cdot(\mathbf{r} + \mathbf{x})\boldsymbol{\omega})]$$

$$- 2\,\boldsymbol{\omega} \times (\mathbf{r} + \mathbf{x})^{(\cdot)}.$$

Since

$$|\mathbf{r} + \mathbf{x}|^2 = (\mathbf{r} + \mathbf{x})\cdot(\mathbf{r} + \mathbf{x})$$

$$= |\mathbf{r}|^2 + 2\,\mathbf{r}\cdot\mathbf{x} + |\mathbf{x}|^2,$$

if $|\mathbf{x}|$ remains less than 100 miles the last term may be omitted without causing error greater than one part in 1600. Using the binomial theorem,

$$|\mathbf{r} + \mathbf{x}|^{-3} = (|\mathbf{r} + \mathbf{x}|^2)^{-3/2}$$

$$(4) \qquad = |\mathbf{r}|^{-3}(1 + 2\mathbf{r}\cdot\mathbf{x}/|\mathbf{r}|^2)^{-3/2}$$

$$= |\mathbf{r}|^{-3} - 3|\mathbf{r}|^{-5}(\mathbf{r}\cdot\mathbf{x}) + \ldots,$$

the omitted terms amounting to less than one part in a thousand if $|\mathbf{x}|$ remains less than 100 miles.

In the term in square brackets in (3), the part involving $\mathbf{x}$ is $(\boldsymbol{\omega}\cdot\boldsymbol{\omega})\mathbf{x} - (\boldsymbol{\omega}\cdot\mathbf{x})\boldsymbol{\omega}$. In the discussion following (I.13.8) it was shown that the magnitude of this vector is the product of ω^2 by the length of the component of $\mathbf{x}$ perpendicular to $\boldsymbol{\omega}$. If $|\mathbf{x}|$

remains less than 100 miles, or 528,000 feet, equation (1) of this section shows that the magnitude of the vector cannot exceed .003 feet per second per second. We therefore neglect it. Furthermore, we observe that $\mathbf{r}$ is fixed in our coordinate system, so that $\mathbf{r}^{(\cdot)} = \mathbf{0}$. When we make all these substitutions in (3), it reduces to

$$\mathbf{x}^{(\cdot\cdot)} = \mathbf{F}/m - k\mathbf{r}/|\mathbf{r}|^3 + \bullet$$

(5)
$$+[(\omega \cdot \omega)\mathbf{r} - (\omega \cdot \mathbf{r})\omega]$$
$$- k\mathbf{x}/|\mathbf{r}|^3 + 3k(\mathbf{r} \cdot \mathbf{x})\mathbf{r}/|\mathbf{r}|^5$$
$$- 2\omega \times \mathbf{x}^{(\cdot)} ,$$

only small terms having been omitted. The small error term $\bullet$ should properly be computed at the point P. But it does not come up to one per cent of the principal term $k\mathbf{r}/|\mathbf{r}|^2$, and its change over a hundred-mile region will be of the order of a ten thousandth of this principal term, so we may consider it to be a constant.

If a body is held at rest with respect to the coordinate system at O and then released, and no forces except gravity act on it so that $\mathbf{F} = \mathbf{0}$, it will begin to move with an acceleration found from (5) by setting $\mathbf{F} = \mathbf{x}^{(\cdot)} = \mathbf{0}$. This acceleration we shall denote by $\mathbf{g}$, so that

(6) $$\mathbf{g} = -k\mathbf{r}/|\mathbf{r}|^3 + \bullet + (\omega \cdot \omega)\mathbf{r} - (\omega \cdot \mathbf{r})\omega .$$

It is this acceleration which is determined by any of the experiments used to determine local gravity. Its direction is the direction of the plumb line with plumb bob at O; its magnitude is determined, for example, by measuring the period of a pendulum at O, and is customarily denoted by g:

(7) $$g = |\mathbf{g}| .$$

The line through O parallel to $\mathbf{g}$ is called the <u>vertical</u> at O; "down" is by definition the direction of $\mathbf{g}$, and "up" is the opposite direction. The plane through O perpendicular to $\mathbf{g}$ is called the <u>horizontal plane</u> through O, and any line through O and lying in this plane is a horizontal line at O. A surface (necessarily curved) which is horizontal (perpendicular to $\mathbf{g}$) at each of its points is a <u>level surface</u>.

Suppose in particular that the axis system, which we have been assuming to be fixed relative to the earth and to have its origin at O, has one of its axes vertical and positive upwards; we call this the y-axis. The other two axes are horizontal at O, and have directions such that the (x, y, z)-system is right-handed. The vector $\mathbf{x}$ has components (x, y, z). In (6) the first term on the right has magnitude over a hundred times as great as the others, so in computing the small terms in (5) it is sufficiently accurate to replace (6) by

$$(8) \qquad \mathbf{g} = -k\mathbf{r}/|\mathbf{r}|^3,$$

whence

$$(9) \qquad g \doteq k/|\mathbf{r}|^2.$$

Then

$$(10) \qquad \begin{aligned} -k\mathbf{x}/|\mathbf{r}|^3 &\doteq -(g/|\mathbf{r}|)\mathbf{x} \\ &= (-gx/|\mathbf{r}|, -gy/|\mathbf{r}|, -gz/|\mathbf{r}|). \end{aligned}$$

The vector $\mathbf{r}$ has direction opposite $\mathbf{g}$, approximately, by (8), so it is vertically upward at O, and

$$\mathbf{r}/|\mathbf{r}| = (0, 1, 0).$$

Hence

$$3k(\mathbf{r} \cdot \mathbf{x})\mathbf{r}/|\mathbf{r}|^5$$

$$(11) \quad \doteq (3k/|\mathbf{r}|^3)(0x + 1y + 0z)(0, 1, 0)$$

$$= (0, 3gy/|\mathbf{r}|, 0).$$

In order to transform the last term in (5) to a form more convenient for computation, it is desirable to define two new expressions. The astronomical latitude of O is defined to be the angle between the earth's axis (or the vector $\boldsymbol{\omega}$) and the plane horizontal at O. We shall count this angle as positive in the northern hemisphere and negative in the southern, and we shall designate it by the symbol λ. Thus λ is the same as 90° minus the angle between the vector $\boldsymbol{\omega}$ and the positive y-axis, and the y-component of $\boldsymbol{\omega}$ is $\boldsymbol{\omega} \sin \lambda$.

The azimuth of a non-vertical vector at O is defined to be the angle from north to the horizontal projection of the vector, counted positive in a clockwise direction. (It is also fairly common to define the azimuth as starting from south instead of north, and confusion can result if it is not made clear which convention is used. Here we shall always use the definition just given.) If α is the azimuth of the positive x-axis, the positive z-axis, which lies 90° clockwise from the positive x-axis, will have azimuth $90^\circ + \alpha$. So a unit vector drawn horizontally northward at O will have x- and z-components which are respectively $\cos \alpha$ and $\cos (90^\circ + \alpha) = - \sin \alpha$. The horizontal projection of $\boldsymbol{\omega}$ points northward and has length $\boldsymbol{\omega} \cos \lambda$, so its x- and z-components are respectively $\omega \cos \lambda \cos \alpha$ and $- \omega \cos \lambda \sin \alpha$. Thus three components of $\boldsymbol{\omega}$ are given by

$$\boldsymbol{\omega} = (\omega \cos \lambda \cos \alpha, \ \omega \sin \lambda, \ - \omega \cos \lambda \sin \alpha).$$

The components of the position vector $\mathbf{x}$ of the particle are $(x(t), y(t), z(t))$ in our chosen coordinate system, so $\mathbf{x}^{(\cdot)} = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$. From these last two equations we see that

$$\omega \times x^{(\cdot)}$$

$$(12) \quad = \omega(\dot{z}\sin\lambda + \dot{y}\cos\lambda\sin\alpha,\ -\dot{x}\cos\lambda\sin\alpha - \dot{z}\cos\lambda\cos\alpha,\ \dot{y}\cos\lambda\cos\alpha - \dot{x}\sin\lambda).$$

Let us denote the components of F by (F_x, F_y, F_z) and substitute (6), (10), (11) and (12) in (5). We obtain the three equations

$$\ddot{x} = F_x/m - gx/|r| - 2\omega\dot{z}\sin\lambda - 2\omega\dot{y}\cos\lambda\sin\alpha,$$

$$(13) \quad \ddot{y} = F_y/m - g + 2gy/|r| + 2\omega\dot{x}\cos\lambda\sin\alpha + 2\omega\dot{z}\cos\lambda\cos\alpha,$$

$$\ddot{z} = F_z/m - gz/|r| - 2\omega\dot{y}\cos\lambda\cos\alpha + 2\omega\dot{x}\sin\lambda,$$

wherein we recall that $|r|$ is the distance from 0 to the center of the earth and may be replaced with adequate accuracy by the mean radius of the earth.

If the axis system is chosen so that the velocity vector at 0 is in the (x, y)-plane, the initial values of z and $\dot{z}$ are both zero. But the trajectory of a projectile lies nearly in a vertical plane, so in this case $\dot{z}$ will never depart greatly from 0, and the terms in (13) which contain z or $\dot{z}$ as a factor may be discarded without perceptible error.

A unit vector drawn horizontally eastward at 0 will have x- and z-components which are respectively $\sin\alpha$ and $\cos\alpha$. Thus the terms in (13) which involve ω may be regarded as the sum of three vectors. First, the terms with factor $\dot{y}$ are the components of

$-2\omega\dot{y}\cos\lambda\,(\sin\alpha, 0, \cos\alpha)$, which represents an acceleration westward if $\dot{y}$ is positive and eastward if $\dot{y}$ is negative, and with magnitude proportional to the cosine of the latitude and the vertical component of velocity. Second, the terms with factor $\sin\lambda$ are the components of $2\omega\sin\lambda\,(-\dot{z}, 0, \dot{x})$, which is directed horizontally to the right of the direction of motion and has magnitude proportional to the sine of the latitude and to the horizontal component of the velocity. Third, the remaining terms are the components of $2\omega\cos\lambda\,(0, \dot{x}\sin\alpha + \dot{z}\cos\alpha, 0)$, which is vertical and is proportional to the cosine of the latitude and to the eastward component of the velocity.

For a simple example, let us consider the case of a particle moving in a vacuum, starting at time $t = 0$ from O with vertical velocity. Since $\dot{x}$ and $\dot{z}$ clearly remain small, equations (13) take the approximate forms

$$\ddot{x} = -2\omega\dot{y}\cos\lambda\sin\alpha,$$

$$\ddot{z} = -2\omega\dot{y}\cos\lambda\cos\alpha.$$

By integrating twice, recalling that $\dot{x} = \dot{z} = 0$ at time $t = 0$, we find that

$$x = -2\omega\cos\lambda\sin\alpha\int_0^t y\,dt,$$

$$z = -2\omega\cos\lambda\cos\alpha\int_0^t y\,dt.$$

The magnitude of this vector is $2\omega\cos\lambda\left|\int_0^t y\,dt\right|$; it is directed to the west if the integral is positive and to the east if the integral is negative. Thus a particle fired upward in a vacuum will return to its original level at a point to the west of its point of departure, since y is positive in the intervening time. A particle dropped from rest down a mine shaft (aerodynamic forces being ignored) will fall to the east of the point of release, since y is negative for all t.

Henceforth we shall use coordinate systems attached to the earth or other systems expressed in terms of such systems in all trajectory computations. The effects of the rotation of the earth are accounted for by the last two terms in each of equations (13), and in all other respects the coordinate systems attached to the earth behave like inertial frames. Thus we are safe in treating coordinate systems attached to the earth as though they were inertial frames, provided only that we remember to include the correction terms in (13) whenever we are dealing with a motion in which these terms produce an appreciable effect. In particular, the parentheses enclosing the dots in (5) and similar equations have now served their purpose, and henceforth will be omitted; if a vector $\mathbf{x}$ has components (x, y, z) in some coordinate system fixed to the earth, the vector $(\dot{x}, \dot{y}, \dot{z})$ will be designated $\dot{\mathbf{x}}$.

2. Aerodynamic force system; preliminary discussion.

The major task of this chapter is to find some reasonable description of the aerodynamic force system on a projectile. This is a rather difficult but an exceedingly important project. Early ballisticians failed to realize the magnitude of the aerodynamic forces, in particular the drag. Since early determinations of the velocity were based on the range obtained by the projectile there were large and systematic errors in the values of velocity obtained. In fact, when the velocity of a projectile was first measured accurately by Robins there was widespread disbelief in his results. Further, when he made a rough determination of the loss in velocity due to air resistance by measuring velocity at different distances from the muzzle of the gun his results were subject to widespread criticism and disbelief by the ballisticians of his day. (We refer the reader to Section 1 of the following chapter where, in the discussion of resistance firings, this situation is further discussed.) Today we know that for some projectiles the air resistance may amount to 25 times the force of gravity! Thus the importance of analyzing the aerodynamic forces on a projectile cannot be overemphasized.

The analysis which we use today is the result of a long development. After the importance of the air resistance was recognized there was still a long period when all other forces were neglected. In explaining certain observed phenomena, errors were made even by such distinguished mathematicians as Poisson. The fundamental work on this subject, which did much to resolve the various difficulties encountered, was done by R. H. Fowler, E. G. Gallop, C. N. H. Lock and H. W. Richmond in their paper "The Aerodynamics of a Spinning Shell," which appeared in the Philosophical Transactions of the Royal Society of London in 1920. This paper, together with the application by F. R. Moulton of the numerical methods of astronomy to ballistic problems, marks the beginning of modern exterior ballistics. The aerodynamic force system hypothesized by these authors was not derived on the basis of any mathematical analysis, but on intuitive physical reasoning. It is therefore not entirely surprising that certain inconsistencies arose. These difficulties were explained by K. L. Nielsen and J. L. Synge in a paper published as a Ballistic Research Laboratory Report, and essentially the same analysis was made by M. A. Biot in a manuscript which as far as we know is unpublished. The description of the force system which is given in this chapter is precisely that of Nielsen and Synge. The analysis, we shall see, gives, in a sense, the description of the complete force system on a projectile.

We shall first make an analysis of the force system in an oversimplified case, in order to illustrate the method of procedure. Suppose that we have a shell whose surface is a surface of revolution which is moving with respect to the air with velocity $\mathbf{u}$. Suppose that the angular velocity of the shell is zero. For convenience, we shall choose a coordinate system with X- , Y- , Z-axes so that the vector **u** is along the X-axis and the axis of the shell is contained in the XY-plane. We call the angle between the vector **u** and the axis of the shell the angle of yaw. We know that we may replace the collection of all aerodynamic forces acting on the shell by a single force **R**, the resultant, acting at a point attached to the body. This point is called the center of pressure, and is attached to the body only in a mathematical sense, since for certain projectiles it may be situated in front of the nose. Since the projectile is supposed symmetric and non-rotating, and its axis lies in the XY-plane, the vector **R** must be contained in the XY-plane. The force **R**, acting at the center of pressure, may be replaced by an equal force acting at the center of mass, together with a torque about the center of mass. This torque is simply the cross product of the vector from the center of mass to the center of pressure by **R**, and hence is perpendicular to the XY-plane. The overturning moment M is the component of this torque along the Z-axis. The drag D and the lift L are the components of **R** along the X-axis (parallel to the vector velocity) and the Y-axis respectively.

The three (scalar) quantities D, L and M depend on the angle of yaw, δ. It is clear that if the direction of yaw is reversed, L and M change signs while D remains unchanged — that is, L and M are odd functions of δ and D is even. We are not interested in these functions for all possible values of the yaw. The aim of the design of a projectile is, among other things, to insure that it will fly with small yaw. For small yaw, it is possible to approximate the variation of D, L and M by assuming that D is independent of yaw, and that L and M depend linearly upon the yaw. We define ℓ and m by the equations

$$(1) \qquad \begin{aligned} L &= \ell \sin \delta, \\ M &= m \sin \delta \cos \delta, \end{aligned}$$

and presume that ℓ, m and D are independent of the yaw. It is rather strange to use in one case sin δ and in the other, sin δ cos δ, but to first-order terms these are the same. The form for L is actually the conventional ballistic one. The form for M is motivated by the fact that we shall use axial component of velocity rather than total velocity in defining aerodynamic coefficients in the later sections, and m is defined to be consistent with later usage.

The factors D, $\mathcal{I}$ and m depend on many variables: notably on the shape of the projectile, its size and velocity, and on a number of constants of the air, its elasticity, coefficient of viscosity, etc. This sort of dependence has been discussed in Section I.15, and we shall assume that D, $\mathcal{I}$ and m are functions of the shape of the projectile, the speed u, the density of air ρ, the diameter of the projectile d, the coefficient of viscosity μ, and the velocity of sound a. (The last is equivalent to the statement that D, $\mathcal{I}$ and m depend on the temperature, since the velocity of sound in air is proportional to the square root of the absolute temperature.) According to (I.15.42), D can be written in the form $D = \rho u^2 d^2 K_D$, where the dimensionless function K_D depends on the dimensionless power-products u/a, $\rho ud/\mu$, $\mathcal{I}_2/d, \ldots, \mathcal{I}_n/d$. For simplicity, we suppress the shape parameters $\mathcal{I}_2/d, \ldots, \mathcal{I}_n/d$ from the notation, understanding that when we speak of the K_D we mean the function K_D corresponding to a given shape of object, which must be specified. An analogous discussion can be applied to $\mathcal{I}$ and m, and we find as in Section I.15 that D, $\mathcal{I}$ and m can be expressed with the help of <u>aerodynamic</u> <u>coefficients</u> — that is, dimensionless functions K_D, K_L and K_M, called the drag, lift and moment coefficients respectively. These are defined by

$$D = \rho d^2 u^2 K_D,$$

$$(2)\quad L = \rho d^2 u^2 K_L \sin\delta \text{ or } \mathcal{I} = \rho d^2 u^2 K_L,$$

$$M = \rho d^3 u^2 K_M \sin\delta\cos\delta \text{ or } m = \rho d^3 u^2 K_M.$$

In view of the Buckingham theorem and the discussion

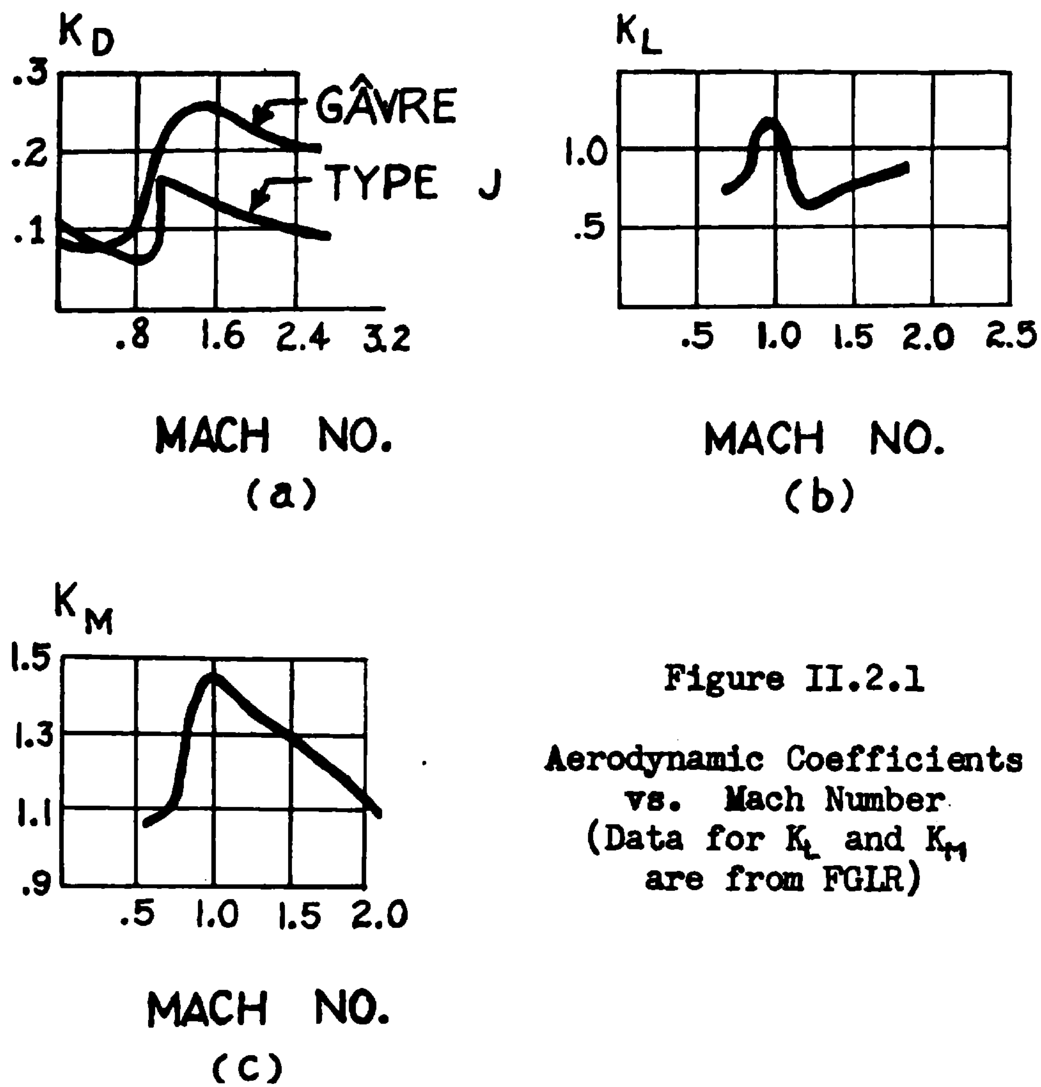


Figure II.2.1

Aerodynamic Coefficients vs. Mach Number (Data for K_L and K_M are from FGLR) at the end of Section I.15, for a given shape of projectile these aerodynamic coefficients depend on the dimensionless quantities u/a and $\rho du/\mu$. (In the case of K_M, the specification of shape must also include the specification of location of the center of mass in the projectile.) Hence, summing up, the aerodynamic force and moment acting on a projectile whose surface is a surface of revolution which has angular velocity zero can be described completely by means of the three dimensionless functions K_D, K_L, K_M of u/a and $\rho du/\mu$. It has usually been found that for militarily useful projectiles the dependence on Reynolds number $\rho du/\mu$ is slight, and K_D, K_L, K_M are commonly regarded as functions of u/a alone. However, this is a simplification which we must be prepared to abandon in those cases where experiments indicate that it is an oversimplification.

Graphs of these functions for a particular shell are shown in Figure 1. The ratio

$$u/a = (\text{speed of a projectile})/(\text{speed of sound})$$

is called the Mach number. It is clear from the graphs that the aerodynamic coefficients remain almost constant for Mach numbers well below one (i.e., speeds well below the speed of sound). In the neighborhood of sound (near Mach number one) all the coefficients change markedly. An intuitive reason for this behavior is not difficult to find. Well below the speed of sound the pattern of the air flow is largely independent of the speed of the projectile. The moving projectile will affect the pressure and density of the air both in front and behind it. However, above the speed of sound, the projectile cannot affect the air in front of it, since the speed of propogation of a disturbance is the speed of sound. There is, then, an entirely different character to the air flow about the projectile above and below the speed of sound. This is strikingly illustrated in Figure 2 by a series of shadowgraphs of spheres, taken at various speeds.

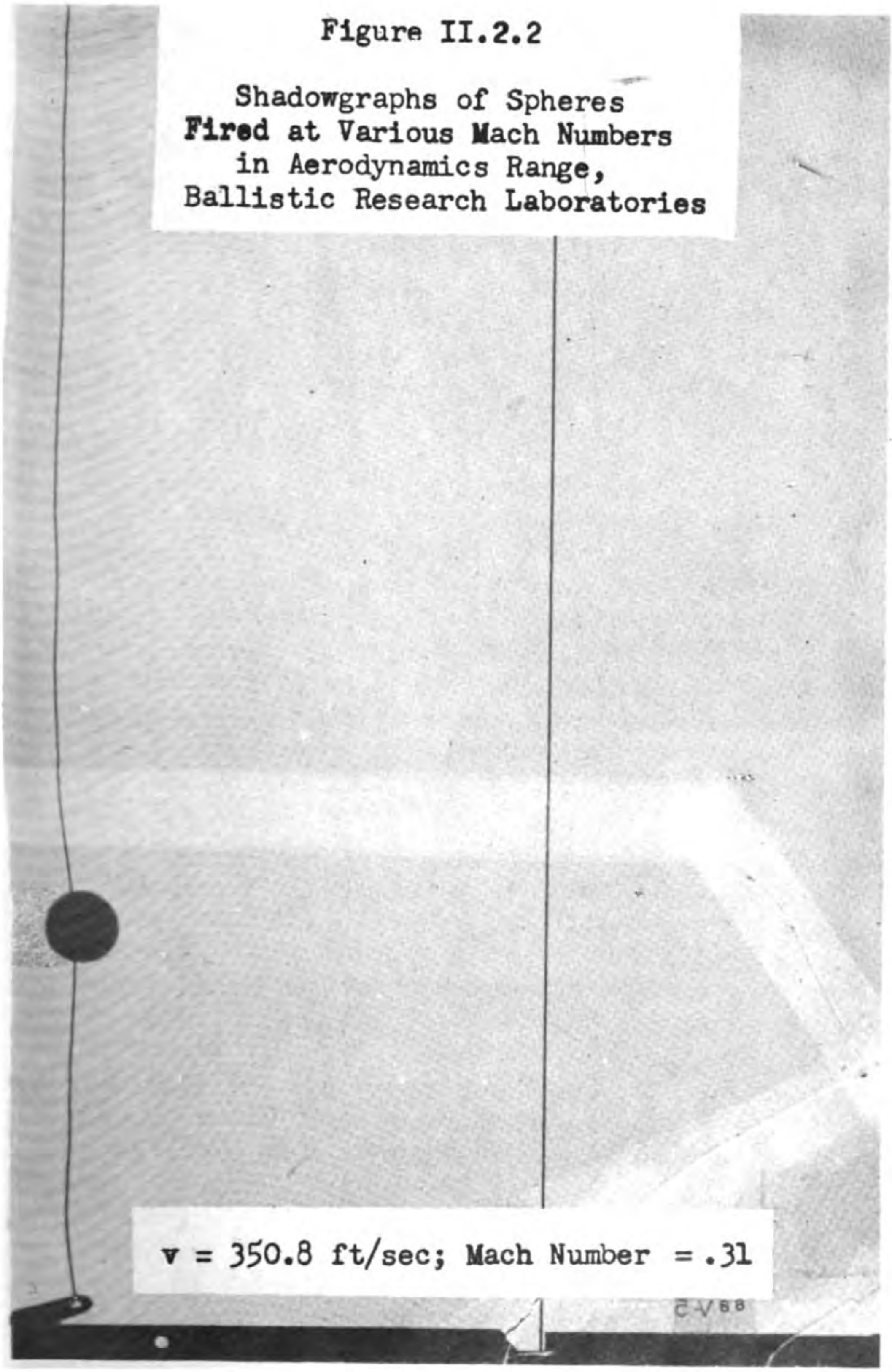
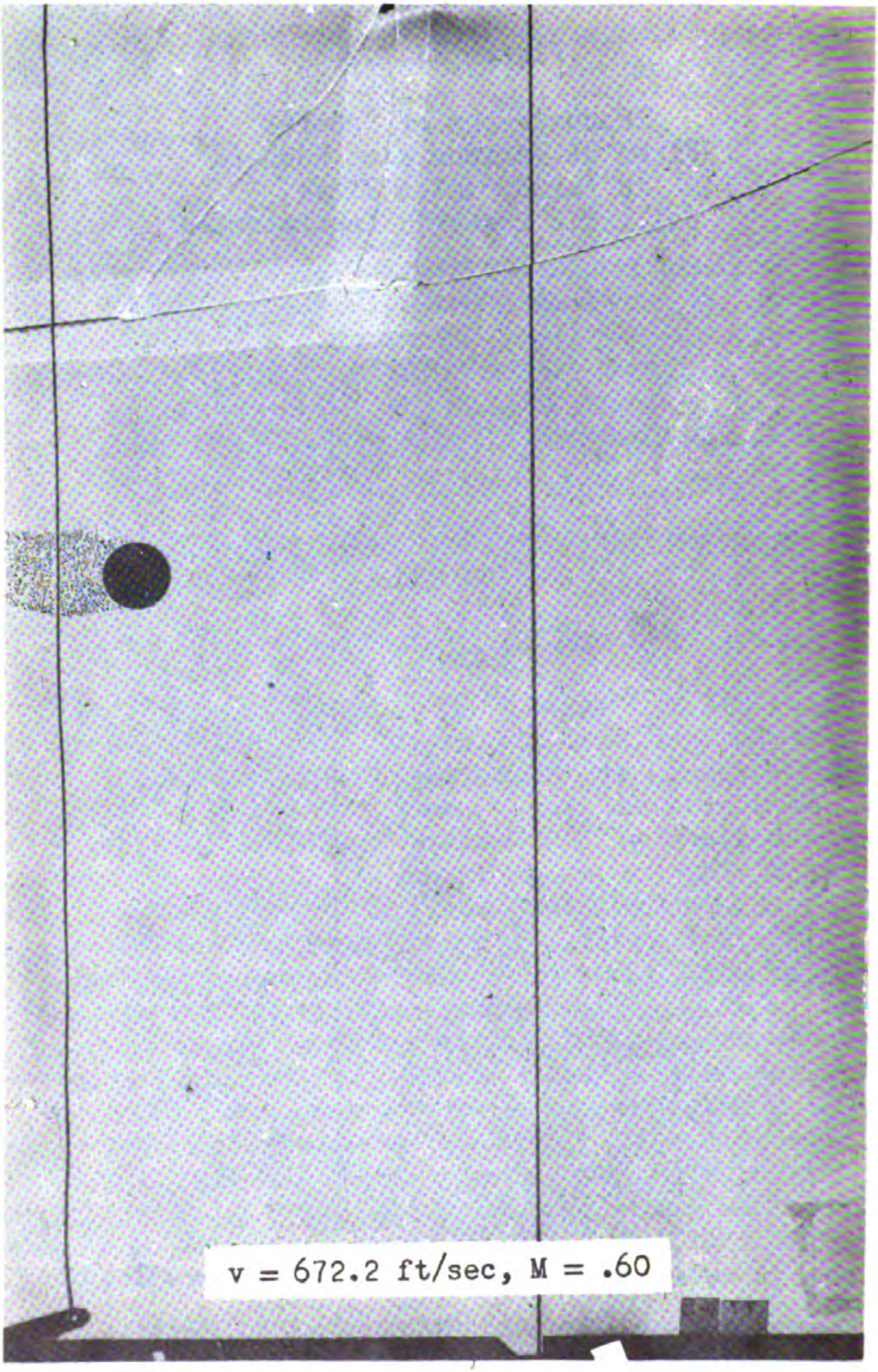
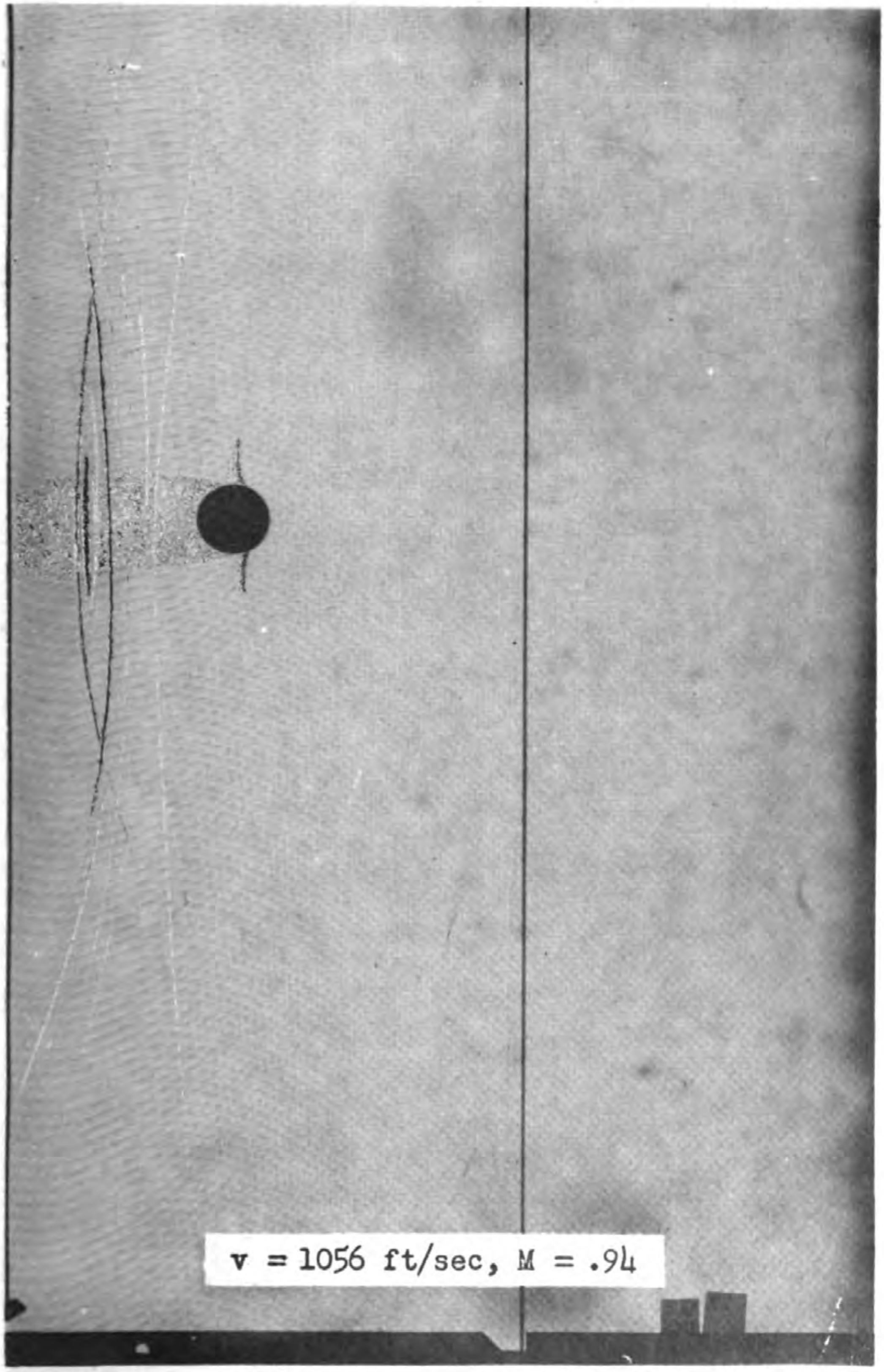


Figure II.2.2

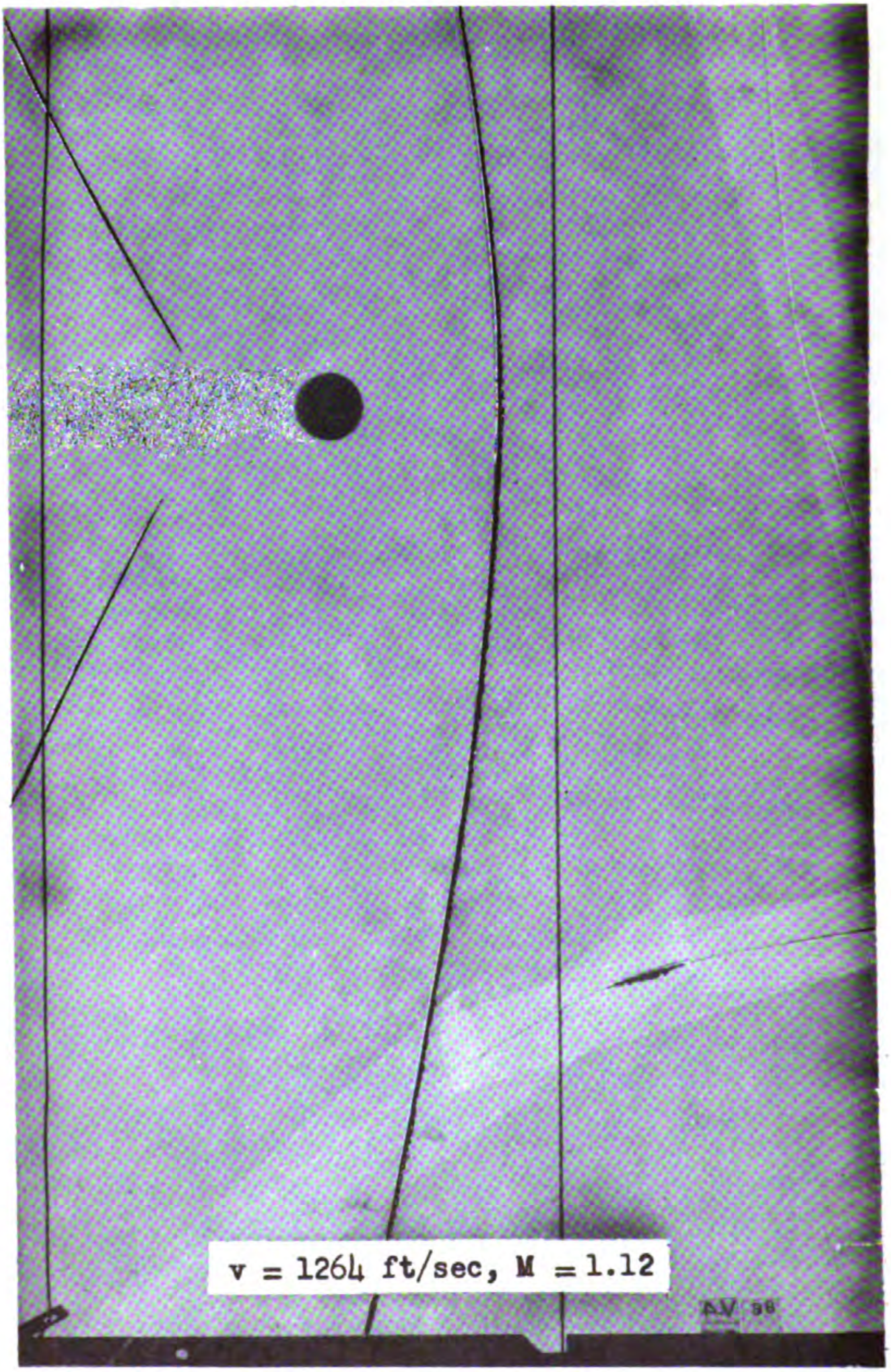
Shadowgraphs of Spheres
Fired at Various Mach Numbers
in Aerodynamics Range,
Ballistic Research Laboratories

v = 350.8 ft/sec; Mach Number = .31

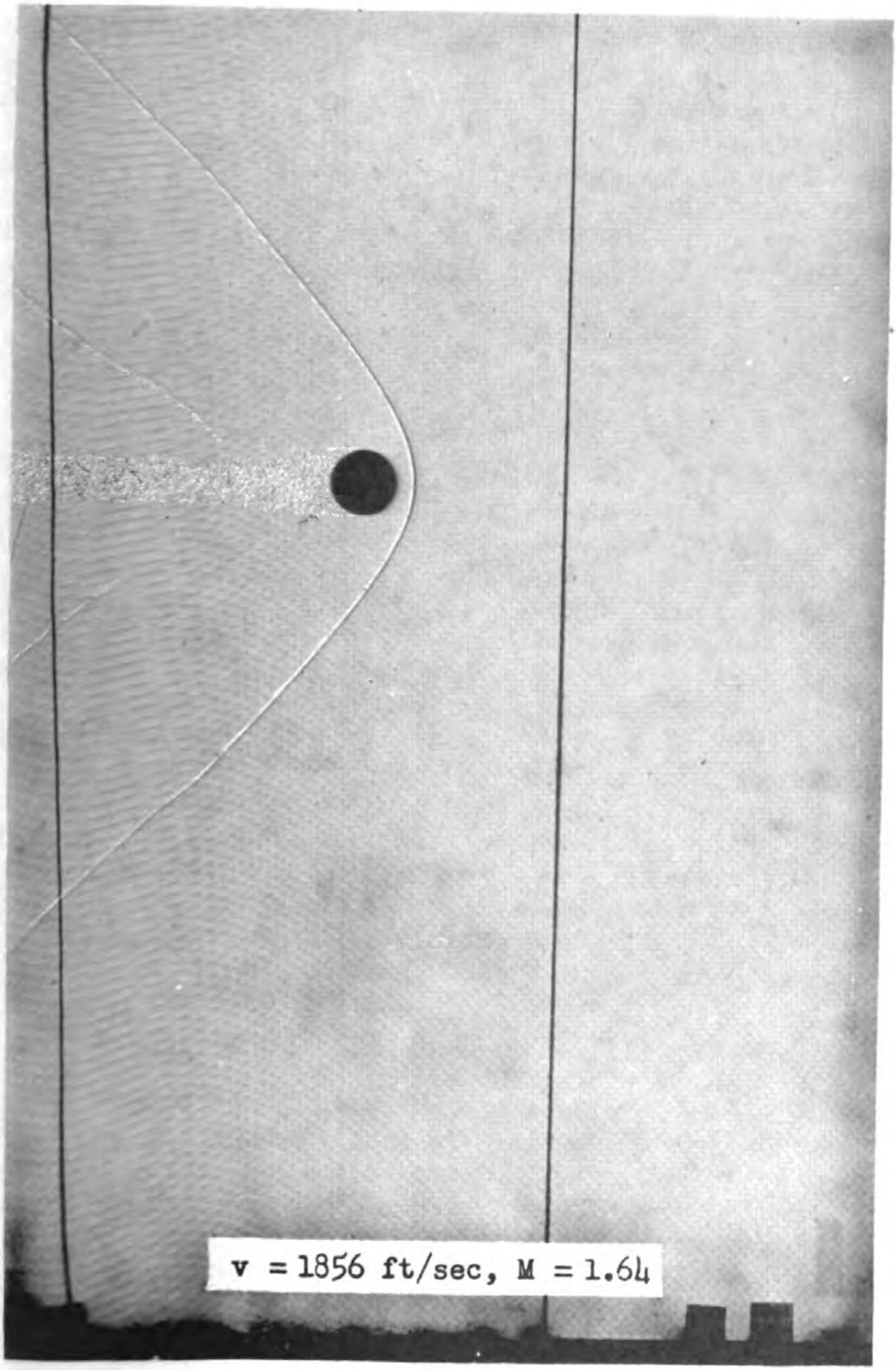

v = 672.2 ft/sec, M = .60

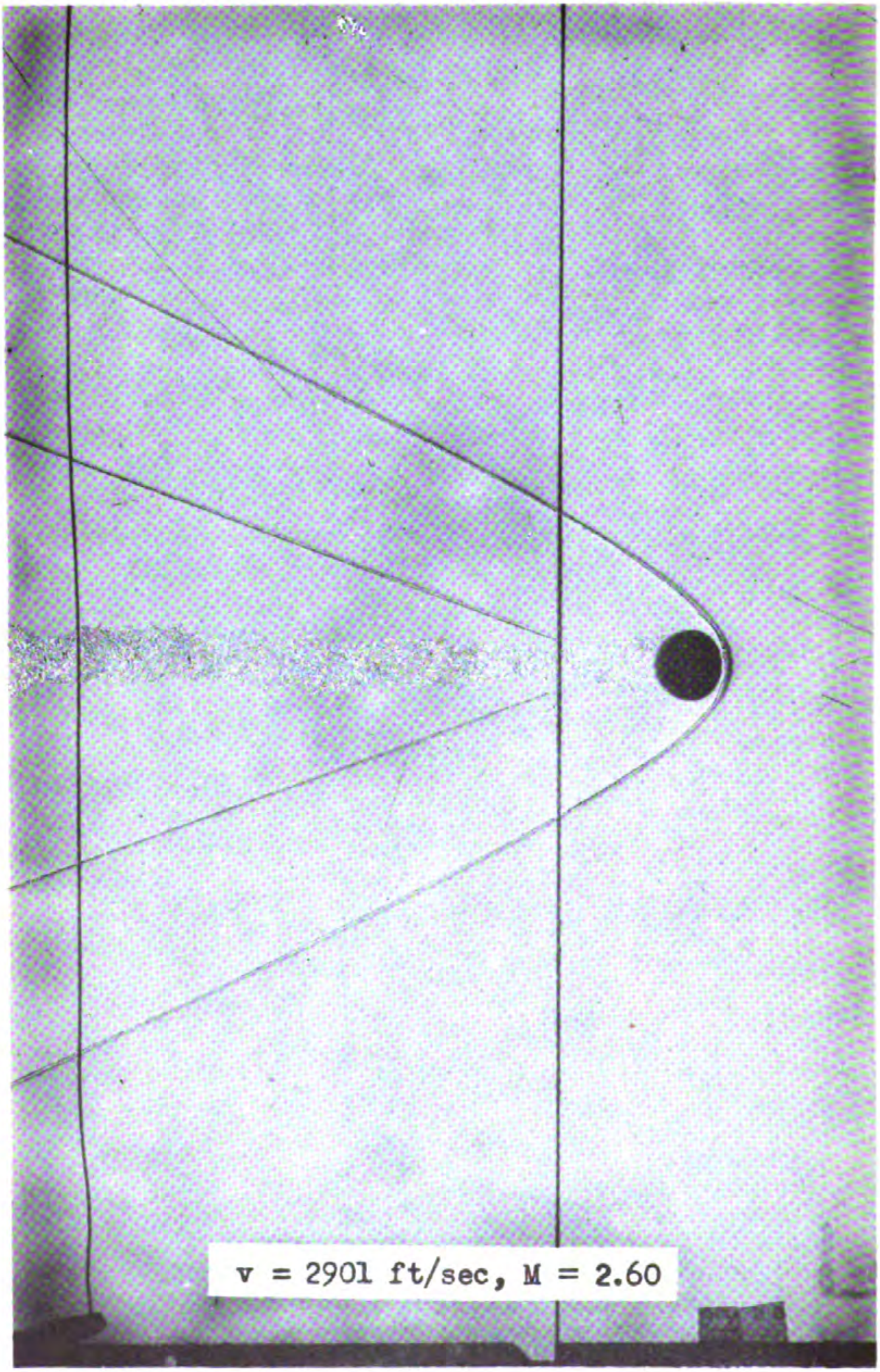


v = 1056 ft/sec, M = .94



v = 1264 ft/sec, M = 1.12


v = 1856 ft/sec, M = 1.64


v = 2901 ft/sec, M = 2.60

These shadowgraphs will be discussed further in a following chapter.

It is noteworthy that the aerodynamic coefficients depend on the shape of the projectile. Nothing we have said would give any indication of the comparative drag of a sphere and a conical-headed shell. This fact was not clearly realized at one time, and the first drag coefficient to be tabulated, by a commission at Gâvre, France, in 1888, actually depended on experimental data obtained on many different shell types. The effect of shape is clearly shown by Figure 1, showing K_D versus Mach number for two different shapes.

It should be emphasized that the aerodynamic coefficients are not all of equal importance. If a shell were to pass along on an ideal trajectory, its angle of yaw would be always zero, and lift and moment would have no effect. The drag coefficient is by far the most important from a computational point of view, and it will be seen that trajectories computed on the basis of drag alone give very good first approximations to the motion of the shell. Of course, a shell will not travel with zero yaw. The problem of arranging the shell-gun design so that the yaw will be small requires a knowledge of K_M and several other coefficients. Further, the computation of the difference between the actual trajectory and a "drag-only," or "particle," trajectory requires knowledge of K_L, K_M and certain other coefficients, although to a degree of accuracy rather less than that required in K_D.

3. Aerodynamic forces on an arbitrary projectile.

The discussion of the preceding section actually furnishes an extremely inadequate basis on which to discuss the motion of a projectile. As we have said before, it is desirable for a projectile to travel with a small yaw. It is clear since a trajectory will in general be curved, that this requires that the projectile have some angular velocity. Furthermore, shell are usually stabilized by imparting to them a high spin. It would surely seem dangerous to try to analyze the motion of a shell on the basis of an aerodynamic system which postulated zero angular velocity. Our next task is, then, to find a reasonable description of the complete force on a projectile. The methods used in accomplishing this are similar to those of the preceding section. We reduce the determination of aerodynamic forces to the determination of a (finite) number of aerodynamic coefficients. These are functions of Mach number and projectile shape and can be evaluated experimentally, at least to a degree of accuracy corresponding to their importance in predicting the motion of the projectile. The approach is essentially empirical, and dimensional analysis again plays a rather important role.

Let us then suppose that a projectile is moving with velocity $\mathbf{u}$ and angular velocity $\boldsymbol{\omega}$ through air which is at rest. The force and torque about the center of mass exerted by the air on the projectile are denoted by $\mathbf{F}$ and $\mathbf{G}$ respectively. As in the previous analysis, it is supposed that $\mathbf{F}$ and $\mathbf{G}$ are functions of $\mathbf{u}$, $\boldsymbol{\omega}$, the density of air ρ, the size and shape of the projectile and the speed of sound a. We are not interested in the values of $\mathbf{F}$ and $\mathbf{G}$ for all values of $\mathbf{u}$ and $\boldsymbol{\omega}$. The normal position of a shell, for example, is that in which both $\mathbf{u}$ and $\boldsymbol{\omega}$ are parallel to the axis of the shell, and only small deviations from this configuration are expected. In view of this it will be appropriate to approximate $\mathbf{F}$ and $\mathbf{G}$ by a Taylor expansion about "normal" values of $\mathbf{u}$ and $\boldsymbol{\omega}$. It is therefore assumed that the projectile has an axis, which is distinguished by the fact that in flight the velocity $\mathbf{u}$ and the angular velocity $\boldsymbol{\omega}$ are nearly parallel to this axis. We choose three unit vectors, $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$, which are fixed in the projectile and form a right-handed orthogonal system; that is,

$$\mathbf{x}_1 \times \mathbf{x}_2 = \mathbf{x}_3,$$

$$\mathbf{x}_2 \times \mathbf{x}_3 = \mathbf{x}_1,$$

and

$$\mathbf{x}_3 \times \mathbf{x}_1 = \mathbf{x}_2.$$

Further, $\mathbf{x}_1$ is supposed to coincide with the axis of the projectile. The vectors $\mathbf{F}$, $\mathbf{G}$, $\mathbf{u}$ and $\boldsymbol{\omega}$ can be written as linear combinations of these unit vectors. The quantities, F_i, G_i, u_i and ω_i, for $i = 1, 2, 3$, are defined by

$$(1) \quad \begin{aligned} \mathbf{F} &= F_1 \mathbf{x}_1 + F_2 \mathbf{x}_2 + F_3 \mathbf{x}_3, \\ \mathbf{G} &= G_1 \mathbf{x}_1 + G_2 \mathbf{x}_2 + G_3 \mathbf{x}_3, \\ \mathbf{u} &= u_1 \mathbf{x}_1 + u_2 \mathbf{x}_2 + u_3 \mathbf{x}_3, \\ \boldsymbol{\omega} &= \omega_1 \mathbf{x}_1 + \omega_2 \mathbf{x}_2 + \omega_3 \mathbf{x}_3. \end{aligned}$$

The quantities F_i and G_i are now functions of ρ, d, $\mathbf{u}$, $\boldsymbol{\omega}$, a. Making a Taylor expansion in u_2, u_3, ω_2, ω_3 in accordance with the program outlined above leads to:

$$(2) \quad \begin{aligned} F_i &= A_i + B_i u_2 + C_i u_3 + D_i \omega_2 + E_i \omega_3, \\ G_i &= A_i' + B_i' u_2 + C_i' u_3 + D_i' \omega_2 + E_i' \omega_3, \end{aligned}$$

for $i = 1, 2, 3$, where, to a first-order approximation, the A_i, B_i, etc., are independent of u_2, u_3, ω_2 and ω_3. The coefficients A_i, B_i, etc., will, after certain factors involving velocity, etc., are divided out, be again called aerodynamic coefficients. If the projectile is without symmetry there are thirty such coefficients. We shall now hopefully examine the consequences of symmetry.

4. Consequences of rotational symmetry.

Let us now suppose that the projectile has an angle s of rotational symmetry, and that $0 < s < \pi$ radians. That is, it is presumed that the projectile, rotated about its axis through the angle s, exactly covers itself. In order to exploit this symmetry, it is convenient to use the mechanism of complex numbers. We define

$$(1) \qquad \xi = u_2 + iu_3, \quad \eta = \omega_2 + i\,\omega_3,$$

where $i^2 = -1$. A bar over a complex number will mean its conjugate. Thus $\bar{\xi} = u_2 - iu_3$, and $\bar{\eta} = \omega_2 - i\,\omega_3$. Since u_2 and u_3 can be obtained as linear combinations of ξ and $\bar{\xi}$, and similarly for ω_2, ω_3 and η, $\bar{\eta}$, it follows that any expression which is linear in u_2, u_3, ω_2 and ω_3 can be written as a linear function of ξ, $\bar{\xi}$, η, and $\bar{\eta}$. The first of equations (3.2) can therefore be rewritten in the form

$$F_1 = a_1 + b_1\xi + b_2\eta + \bar{b}_1\bar{\xi} + \bar{b}_2\bar{\eta},$$

$$(2) \qquad \mathcal{F} = F_2 + i\,F_3$$

$$= a_2 + c_1\xi + c_2\eta + d_1\bar{\xi} + d_2\bar{\eta}.$$

In these equations the coefficients, except a_1, are in general complex. The special form of the first equation is due to the fact that F_1 must be real for all values of ξ and η. Let us now consider an axis system which is that obtained by rotating the vectors $\mathbf{x}_2$ and $\mathbf{x}_3$ through the angle of symmetry s. The various vectors, $\mathbf{u}$, $\mathbf{F}$ and $\boldsymbol{\omega}$ will be represented by a different set of numbers in this system, which will be denoted the $(\mathbf{x}_1, \mathbf{x}_2', \mathbf{x}_3')$-system. In particular, instead of the quantities given in (1) there will be a new set which will be related as follows:

In (x_1, x_2, x_3)-system	In (x_1, x_2', x_3')-system*
ξ	$\xi \exp(-is)$
η	$\eta \exp(-is)$
$\bar{\xi}$	$\bar{\xi} \exp(is)$
$\bar{\eta}$	$\bar{\eta} \exp(is)$
F_1	F_1
$\mathcal{J}$	$\mathcal{J} \exp(-is)$

The coefficients in (2), like those in (3.2), depend on ρ, d, u_1, ω_1 and a and also on the position of the axes with respect to the body. But turning the axes about x_1 by angle s is equivalent to rotating the body through an angle - s about x_1 while holding axes fixed; and because of the symmetry, such a rotation cannot change the coefficients in (3.2) or in (2). Hence in the new axes F_1 and $\mathcal{J}$ are still given by (2), and therefore

$$F_1 = a_1 + b_1 \xi \exp(-is) + b_2 \eta \exp(-is) + \bar{b}_1 \xi \exp(is) + \bar{b}_2 \bar{\eta} \exp(is),$$

$$(3) \quad \mathcal{J} \exp(-is) = a_2 + c_1 \xi \exp(-is) + c_2 \eta \exp(-is) + d_1 \cdot \bar{\xi} \exp(is) + d_2 \bar{\eta} \exp(is).$$

---

*We use exp ( ) to mean e to the power ( ), where e is the basis for natural logarithms. It will be recalled that exp (a + ib), where a and b are real, is simply $(\cos b + i \sin b)e^a$. The fact which is used above is that in order to rotate a plane vector by the angle s it is only necessary to multiply its complex number representation by exp (is).

Comparing the first of these with the first of (2) yields

$$b_1\xi + \overline{b_1}\overline{\xi} + b_2\eta + \overline{b_2}\overline{\eta}$$
$$= b_1\xi \exp(-is) + \overline{b_1}\overline{\xi} \exp(is)$$
$$+ b_2\eta \exp(-is) + \overline{b_2}\overline{\eta} \exp(is),$$

identically in ξ and η. If we choose $\eta = 0$, $\xi = \overline{b_1}$, this yields $2b_1\overline{b_1} = 2b_1\overline{b_1} \cos s$. Since $0 < s < \pi$, $\cos s \neq 1$ and so $b_1 = 0$. Similarly $b_2 = 0$. Comparing the second of equations (3) with the second of (2), we find that for all ξ and η,

$$d_1\overline{\xi} + d_2\overline{\eta} = d_1\overline{\xi} \exp(2is) + d_2\overline{\eta} \exp(2is).$$

This implies $d_1 = d_1 \exp(2is)$, $d_2 = d_2 \exp(2is)$. But $\exp(2is) \neq 1$, because $0 < s < \pi$; so $d_1 = d_2 = 0$.

A similar argument can be applied to G_1, G_2 and G_3, and we thus arrive at the conclusion:

(4) If a projectile is symmetric under rotation about its axis through an angle s, where $0 < s < \pi$, its force and moment system can be written, to a first order of approximation, in the form

$$F_1 = a_1, \ \mathcal{F} = F_2 + iF_3 = c_1\xi + c_2\eta,$$
$$G_1 = e_1, \ \mathcal{G} = G_2 + iG_3 = c_3\xi + c_4\eta,$$

where a_1, e_1 and the c_i's are functions of ρ, d, u_1, ω_1 and a, and are independent of the orientation of the axes x_2 and x_3 with respect to the body.

The last clause still needs justification. We form a new axis system, the (x_1, x_2', x_3')-system, by rotating x_2 and x_3 through angle ϑ about x_1. The table before (3) holds if we replace s by ϑ. If a_1', ..., c_4' are the coefficients for the new system, then

$$(c_1\xi + c_2\eta) \exp(-i\vartheta) = \mathcal{F} \exp(-i\vartheta)$$
$$= \mathcal{F}' = c_1'(\xi \exp(-i\vartheta)) + c_2'(\eta \exp(-i\vartheta))$$

for all ξ, η. This implies $c_1' = c_1$, $c_2' = c_2$. Similarly the other coefficients are unaffected by the rotation.

As in the treatment of the previous section, it will be desirable to replace the c's and a_1 and e_1 by dimensionless coefficients. However, the situation is different in this respect. Before, the powers of ρ and d required to give dimensionless coefficients were uniquely determined, and the form of the aerodynamic coefficient was completely specified when we decided to use u rather than a to obtain zero dimension in time. In the present case, even if we decide not to use a to get coefficients which have dimension zero in time, there is still a choice of u_1 or ω_1. We shall be guided in making this choice by a consequence of a different kind of symmetry.

Suppose the projectile has mirror symmetry in a plane containing the axis. That is, suppose that there is a plane such that if each point of the projectile is moved to the point on the opposite side of, and equidistant from, the plane, then the projectile is carried exactly onto itself. Choose the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ so that the plane of symmetry is parallel to the plane determined by these. As in the discussion of rotational symmetry, we consider two coordinate systems. Let us compute the representation of $\mathbf{u}$, $\boldsymbol{\omega}$, $\mathbf{F}$ and $\mathbf{G}$ in a coordinate system based on $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3' = -\mathbf{x}_3$. The results which must be demonstrated are that corresponding to u_1, u_2, u_3, ω_1, ω_2, ω_3, F_1, F_2, F_3, G_1, G_2 and G_3 are u_1, u_2, $-u_3$, $-\omega_1$, $-\omega_2$, ω_3, F_1, F_2, $-F_3$, $-G_1$, $-G_2$ and G_3. It is easy to see that the forms for the u's and F's are correct, but in order to see the way the ω's transform it is necessary to return to the definition of $\boldsymbol{\omega}$. For any three perpendicular vectors $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$, fixed in a rigid body, $\boldsymbol{\omega}$ is chosen (see (I.7.25)) to satisfy

$$\omega = (\dot{y}_2 \cdot y_3)y_1 + (\dot{y}_3 \cdot y_1)y_2 + (\dot{y}_1 \cdot y_2)\, y_3.$$

Writing $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3'$ instead of the y's the correctness of the form for the new ω's is demonstrated. In order to obtain a form for the G's, recall that $\mathbf{G}$ is the vector product $\mathbf{r} \times \mathbf{F}$, where $\mathbf{r}$ is a vector joining two points. Since we know the forms $\mathbf{r}$ and $\mathbf{F}$ take in the new coordinate system, it is only necessary to write out the form for the vector product to verify the statement about the G's.

An argument such as was applied in the case of rotational symmetry can now be used. The functional relationship in the two systems must be the same. Writing this statement out mathematically and displaying the dependence of the various quantities on ω_1 explicitly leads to

$$c_1(-\omega_1)\,\bar{\xi} - c_2(-\omega_1)\bar{\eta} = \mathcal{J} = \bar{c}_1(\omega_1)\bar{\xi} + \bar{c}_2(\omega_1)\bar{\eta},$$

$$c_3(-\omega_1)\bar{\xi} - c_4(-\omega_1)\bar{\eta} = -\mathcal{G} = -\bar{c}_3(\omega_1)\bar{\xi} - \bar{c}_4(\omega_1)\bar{\eta},$$

$$F_1(-\omega_1) = F_1(\omega_1),$$

$$G_1(-\omega_1) = -G_1(\omega_1).$$

From these equations we can at once determine which are odd and which are even functions of ω_1. Thus:

(5) <u>If a projectile has rotational symmetry under the angle</u> s, $0 < s < \pi$, <u>and further has a plane of mirror symmetry, then in the equations</u> (4), F_1, <u>the real part of</u> c_1 <u>and</u> c_4, <u>and the imaginary part of</u> c_2 <u>and</u> c_3 <u>are even functions of</u> ω_1; G_1, <u>the real part of</u> c_2 <u>and</u> c_3 <u>and the imaginary part of</u> c_1 <u>and</u> c_4 <u>are odd functions of</u> ω_1.

This result will decide the definition of the dimensionless coefficients. For each of the various functions of ω_1, u_1, ρ, d and a, one removes the dimension of mass by dividing by ρ, for odd functions of ω_1 the dimension of t^{-1} is removed by dividing by ω_1, remaining dimensions of t^{-1} are removed by dividing by u_1, and remaining dimensions of length are removed by dividing by the appropriate powers of d. We are thus led to define the aerodynamic coefficients by the following:

$$
\begin{aligned}
c_1 &= - \rho d^2 u_1 K_N + i \rho d^3 \omega_1 K_F, \\
c_2 &= \rho d^4 \omega_1 K_{XF} + i \rho d^3 u_1 K_S, \\
c_3 &= - \rho d^4 \omega_1 K_T - i \rho d^3 u_1 K_M, \\
c_4 &= - \rho d^4 u_1 K_H + i \rho d^5 \omega_1 K_{XT}, \\
F_1 &= - \rho d^2 u_1^2 K_{DA}, \\
G_1 &= - \rho d^4 u_1 \omega_1 K_A.
\end{aligned}
\qquad (6)
$$

It is not hard to verify that the various coefficients are of dimension zero in mass, length and time. It should be stated again that the definitions of these coefficients are by no means uniquely determined. We have chosen them so that the K's are all even functions of ω_1, which is a reasonable procedure but not the only logically correct one. The choice embodied an attempt to get coefficients which are slowly varying functions. The K's are functions of ρ, u_1, d, ω_1 and a, and applying again the Buckingham theorem, are therefore functions of u_1/a and $\omega_1 d/u_1$. This last variable has a geometric meaning; it is the spin per caliber of travel of the projectile. Absolutely nothing is known of the variation of the coefficients with this parameter, and we shall of necessity ignore this dependence. Of course, the K's actually depend also on the Reynolds number and on other dimensionless parameters, but we shall limit our discussion to the dependence on Mach number, which is surely the most important ballistically. Of course, these also depend very strongly on the shape of the projectile.

Two more facts should be pointed out in connection with this definition. The rather odd sprinkling of minus signs is in conformity with the notation of Fowler, Gallop, Lock and Richmond. It was chosen so that for a "normal" shell all of the coefficients would be positive. As a matter of fact, for most shell, K_T is negative. The other fact which should be pointed out is that the factor u_1 is divided out of the various expressions for force and moment. Fowler et al used the total velocity $u = |\mathbf{u}|$ instead, as did the authors in their previous work on this subject. The definition (6) is precisely that of Nielsen and Synge. Its advantages will be shown in Section 5 of this chapter and in the chapter on the solution of the equations of motion.

Each of the various terms in the expression for $\mathcal{J}$ and $\mathcal{G}$ has a name, or to be more precise, one or more names. This nomenclature is listed below; the first name given is that of Fowler et al, and the second is that used by Nielsen and Synge.

(7)

	Force or Moment	Nomenclature	
a.	$\rho d^2 u_1 K_N \xi$	Normal force	Cross force due to cross velocity
b.	$i\rho d^3 \omega_1 K_F \xi$	Magnus force	Magnus cross force due to cross velocity
c.	$\rho d^4 \omega_1 K_{XF} \eta$	*	Magnus cross force due to cross spin
d.	$i\rho d^3 u_1 K_S \eta$	*	Cross force due to cross spin
e.	$\rho d^4 \omega_1 K_T \xi$	Magnus moment	Magnus cross torque due to cross velocity

f.	$i\rho d^3 u_1 K_M \xi$	Overturning (or righting) moment	Cross torque due to cross velocity
g.	$\rho d^4 u_1 K_H$	Damping moment, or yawing moment due to yawing	Cross torque due to cross spin
h.	$i\rho d^5 \omega_1 K_{XT}$	*	Magnus cross torque due to cross spin
i.	$\rho d^2 u_1{}^2 K_{DA}$	Axial drag	Axial drag
j.	$\rho d^4 u_1 \omega_1 K_A$	Spin-decelerating moment	Spin-decelerating moment

The adjective "Magnus" always refers to a force or torque which vanishes when $\omega_1 = 0$. The forces and torques marked * were missing from the force system of Fowler et al. In fact, the motivation for the work of Nielsen and Synge lay in their observation that the force system used earlier was of necessity incomplete. The inconsistency which led to this conclusion will be remarked in the following section.

At first glance it would appear that the lift and drag coefficients defined in an earlier section did not occur in this analysis. Actually, the force **R** of Section 2 has simply been resolved along and perpendicular to the projectile instead of along and perpendicular to the velocity vector **u**. The connection between the two resolutions is easy to obtain. The normal force (see Table (7)) has the magnitude $\rho d^2 u_1 K_N |\xi| = \rho d^2 (u \cos\delta) K_N (u \sin\delta)$ and the axial drag has magnitude $\rho d^2 (u \cos\delta)^2 K_{DA}$, where δ is the yaw. Computing the components of these along and perpendicular to the velocity vector, (all other forces vanish in the case, investigated in Section 2, where the angular velocity $\boldsymbol{\omega} = \mathbf{0}$),

$$\rho d^2u^2K_L \sin\delta = \rho d^2u^2K_N \sin\delta\cos^2\delta - \rho d^2u^2K_{DA}\sin\delta\cos^2\delta,$$

$$\rho d^2u^2K_D = \rho d^2u^2K_N \sin^2\delta\cos\delta + \rho d^2u^2K_{DA}\cos^3\delta.$$

Hence

$$K_L = K_N\cos^2\delta - K_{DA}\cos^2\delta,$$

(8)

$$K_D = K_N\sin^2\delta\cos\delta + K_{DA}\cos^3\delta.$$

To a first order of approximation, for small yaw,

$$K_L = K_N - K_{DA},$$

(9)

$$K_D = K_{DA}.$$

This approximation will be used in a later chapter.

Finally, a simple examination will show that the definition of K_M given in (6) is identical with that given in Section 2.

5. Dependence of the aerodynamic coefficients on the position of the center of mass.

The aerodynamic force and moment on a projectile are clearly independent of the position of the center of mass. Yet in the definitions we have given, it is reasonably certain that the aerodynamic coefficients do depend on the C_m position. If we change the C_m position by the amount $\mathbf{r}$ the torque about the center of mass is surely changed by the amount $\mathbf{r} \times \mathbf{F}$. The coefficients must change accordingly. This dependence is precisely the same sort of thing as the dependence of a vector on the coordinate system used. It is the purpose of this section to find the equations showing the change of the aerodynamic coefficients with change of position of the center of mass. The principle employed is that for the same values of velocity and angular velocity, the same force and moment must result, regardless of the C_m position.

Let us suppose that the aerodynamic coefficients are known for the force at C_m and torque about this point, giving these quantities when the angular velocity of the projectile and the velocity of the point C_m are known. The problem is then to find the aerodynamic coefficients for a point C_m^*, which is distance r in front of C_m and on the axis of the projectile. These coefficients will permit the evaluation of the force and the torque about C_m^* when the angular velocity and the velocity of C_m^* are known. Let the force at C_m^* be denoted $\mathbf{F}^*$, the torque about C_m^* by $\mathbf{G}^*$, and the velocity of the point by $\mathbf{u}^*$. If the vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ are as before, it is easy to see that

$$
\begin{aligned}
&\mathbf{F}^* = \mathbf{F}, \quad \boldsymbol{\omega}^* = \boldsymbol{\omega}, \\
&\mathbf{G}^* = \mathbf{G} - r\mathbf{x}_1 \times \mathbf{F} \\
(1) \qquad &\phantom{\mathbf{G}^*} = (G_1, G_2 + rF_3, G_3 - rF_2), \\
&\mathbf{u}^* = \mathbf{u} + \boldsymbol{\omega} \times r\mathbf{x}_1 \\
&\phantom{\mathbf{u}^*} = (u_1, u_2 + r\omega_3, u_3 - r\omega_2).
\end{aligned}
$$

As far as axial components are concerned we have

$$(2) \qquad F_1 = F_1^*, \; G_1 = G_1^*.$$

In terms of complex numbers, we compute from (1)

$$
\begin{aligned}
&\eta^* = \omega_2^* + i\omega_3^* = \eta, \\
&\xi^* = u_2^* + iu_3^* = \xi - ir\eta, \\
(3) \qquad &\mathcal{F}^* = F_2^* + iF_3^* = \mathcal{F}, \\
&\mathcal{G}^* = G_2^* + iG_3^* = \mathcal{G} - ir\mathcal{F}.
\end{aligned}
$$

Let an aerodynamic coefficient with a * denote this coefficient with C_m^* as a reference point, and let c's with * denote the corresponding combinations as in (4.6). Then, the definition of the K^*'s states that

$$\mathcal{J}^* = c_1^* \xi^* + c_2^* \eta^*, \qquad (4)$$
$$\mathcal{G}^* = c_3^* \xi^* + c_4^* \eta^*.$$

The equations (3) give the values of $\mathcal{J}^*$, $\mathcal{G}^*$, ξ^* and η^* in terms of the original system. Substituting these gives

$$\begin{aligned} &c_1^* (\xi - ir\eta) + c_2^*\eta = \mathcal{J} = c_1\xi + c_2\eta, \\ (5)\quad &c_3^* (\xi - ir\eta) + c_4^*\eta = \mathcal{G}^* = \mathcal{G} - ir\,\mathcal{J} \\ &\qquad = c_3\xi + c_4\eta - ir(c_1\xi + c_2\eta). \end{aligned}$$

This must be an identity in ξ and η and the coefficients of these are therefore equal.

$$\begin{aligned} &c_1^* = c_1, \\ &c_2^* = c_2 + irc_1, \\ (6)\quad &c_3^* = c_3 - irc_1, \\ &c_4^* = c_4 - irc_2 + ir(c_3 - irc_1) \\ &\quad\ = c_4 - ir(c_2 - c_3) + r^2c_1. \end{aligned}$$

These equations give the desired relations, if the values of the c's in terms of the aerodynamic coefficients are substituted from the definition (4.6). These relations are more convenient if stated in terms of $p = r/d$, which is the distance from C_m to C_m^* measured in calibers. The same process may be applied to equations (2) on the axial components. These results may be summarized as follows.

(7) <u>If the center of mass of a projectile is moved forward by p calibers the aerodynamic coefficients change as given by the following equations, where * refers to the coefficient at the new position.</u>

$$K_{DA}^* = K_{DA},\ K_A^* = K_A,\ K_N^* = K_N,\ K_F^* = K_F,$$
$$K_{XF}^* = K_{XF} - pK_F, \quad K_S^* = K_S - pK_N,$$
$$K_T^* = K_T - pK_F, \quad K_M^* = K_M - pK_N,$$
$$K_H^* = K_H - p(K_S + K_M) + p^2K_N,$$
$$K_{XT}^* = K_{XT} - p(K_{XF} + K_T) + p^2K_F.$$

These equations may be used to point out the inconsistency in the force system used by Fowler et al. If one omits all consideration of K_S then the transformation formula for K_H must read

$$K_H^* = K_H - pK_M + p^2K_N.$$

Let us consider the value of K_H at a point $p + q$ calibers ahead of the original C_m. Then one can compute the value in two ways. First, substitution of $p + q$ for p in the preceding equation must give it, and second, the value may be determined from the value p units ahead of the original C_m. These two determinations must yield the same result. Using functional notation to show the place at which the coefficients are evaluated, the two computations follow.

$$K_H(p + q) = K_H - (p + q)K_M + (p + q)^2K_N,$$
$$K_H(p + q) = K_H(p) - qK_M(p) + q^2K_N(p)$$
$$= K_H - pK_M + p^2K_N - q(K_M - pK_N) + q^2K_N.$$

Comparison of these two formulas shows a discrepancy of pqK_N, leading to the conclusion that K_N is zero. Actually, as a matter of fact, consideration of all the transformation formulas shows that not only K_S but also the other coefficients must be adjoined to obtain a consistent system. (Assuming K_{XF} and K_{XT} zero implies K_F and K_T are zero.) The system with all Magnus coefficients zero is consistent, as was pointed out by Nielsen and Synge, but unfortunately this system is not adequate to explain the experimental facts. On the other hand, we shall see in solving the equations of motion of a spinning projectile that it will

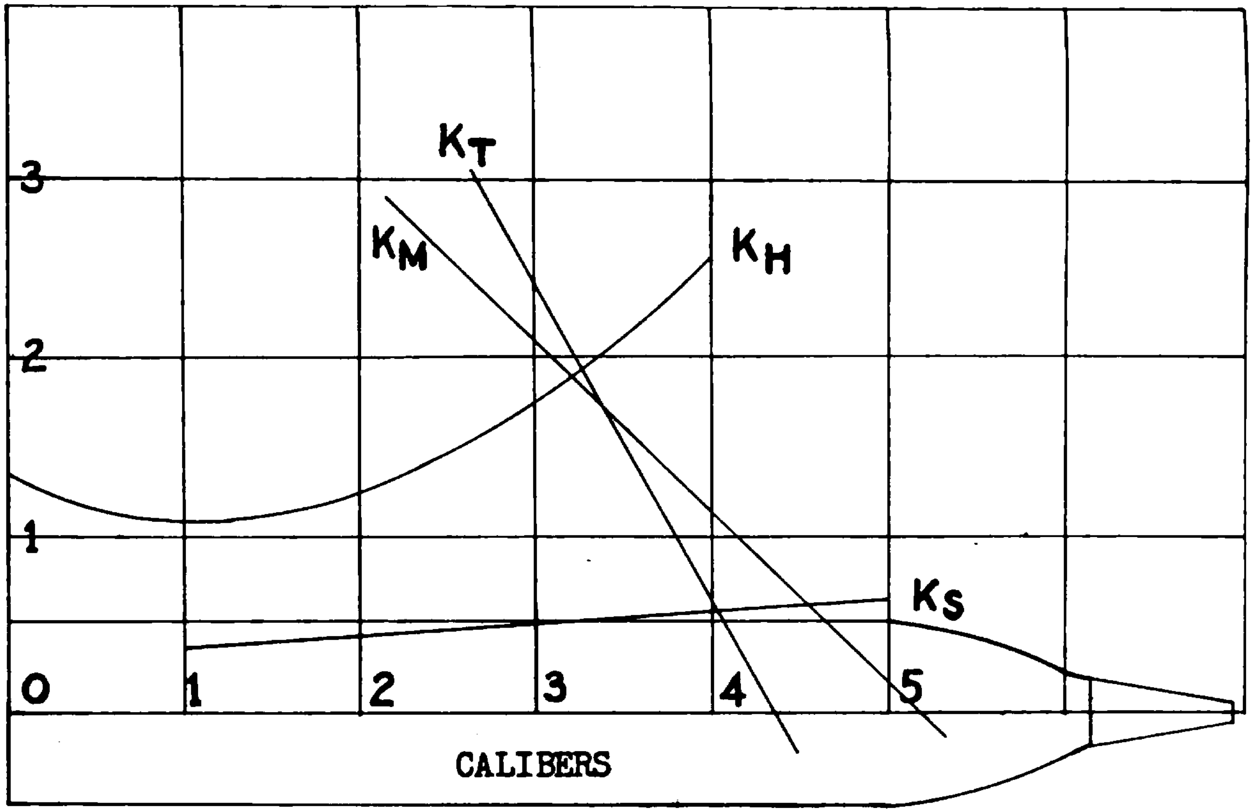


Figure II.5.1

Aerodynamic Coefficients vs. Center of Mass Position
K_M = ordinate, K_H = ordinate X 5
K_T = ordinate /10, K_s = - ordinate X 10

be impossible to deduce the values of K_{XT} and K_{XF} unless these coefficients are more than ten times as large as any of the others. The remaining system is not consistent, but has the property that, if known for one C_m position the values for any other position may be deduced. These coefficients comprise the set of all those which with present techniques can be evaluated.

Figure 1 shows these aerodynamic coefficients graphed against position of C_m. These data were obtained in the aerodynamic range at Aberdeen from firings of models of caliber 4.5-inch rockets. They constitute the most complete set of data which has, to our knowledge, been obtained for a projectile. In a later chapter we will discuss the method of deducing the aerodynamic coefficients from the motion of a projectile. For the present, the graph shows the approximate magnitudes one might expect for the various coefficients.

6. Angle of attack formula for aircraft.

In discussing the ballistics of rockets fired from aircraft a result from the aerodynamic theory of aircraft is needed. As this seems the natural place to discuss this question we will interrupt the general discussion of rotationally symmetric projectiles. The problem which must be solved is simple but quite fundamental. From the information available to the pilot of the aircraft the direction in which the aircraft is going must be deduced. To be more precise, we wish to know, relative to a frame of reference fixed in the aircraft, the direction of the vector velocity of the airplane. It will be sufficient to deduce this direction in a very special case, the so-called equilibrium condition. Suppose that the aircraft with its wings level is flying along a straight-line path at a constant speed with zero angular velocity. As in Section 2, it can be seen that the only aerodynamic force acting on the plane can be resolved into two components, the drag and the lift, directed along and perpendicular to the velocity vector. For any line fixed in an aircraft, such as the line of thrust of the propeller, the angle of attack is defined to be the angle from the velocity vector to the line. The positive direction is taken as upward. We will denote by α_T the angle of attack of the thrust line. The lift and drag coefficients for the aircraft are usually defined in aerodynamic theory by the following equations:

$$\text{(1)}\qquad \begin{aligned} \text{Lift} &= \tfrac{1}{2}\rho u^2 S\, C_L(\alpha_T), \\ \text{Drag} &= \tfrac{1}{2}\rho u^2 S\, C_D(\alpha_T). \end{aligned}$$

In these equations S is the area of the wings, ρ and u, the air density and speed. The aerodynamic coefficients C_L and C_D are functions of α_T. Suppose that the aircraft is diving at an angle θ below the horizontal. Since the aircraft is presumed to be in uniform motion in a straight line, the resultant of all the forces acting on it must be zero. If T is the thrust exerted by the propeller, this statement takes the following mathematical form:

$$\text{(2)}\qquad \begin{aligned} \tfrac{1}{2}\rho u^2 S\, C_L(\alpha_T) + T \sin \alpha_T &= mg \cos \theta, \\ \tfrac{1}{2}\rho u^2 S\, C_D(\alpha_T) - T \cos \alpha_T &= mg \sin \theta. \end{aligned}$$

The mass of the aircraft is m. The thrust, T, can be eliminated from these equations giving the following equation which determines the angle of attack.

$$\text{(3)}\qquad \begin{aligned} &\tfrac{1}{2}\rho u^2 S\, [C_L \cos \alpha_T + C_D \sin \alpha_T] \\ &\qquad\qquad = mg \cos(\theta - \alpha_T). \end{aligned}$$

This equation is exact, but a simpler approximate form is accurate enough for the use we intend to make of it. As in Section 2, it is convenient to approximate the variation of the lift coefficient with angle by a linear function. There is an angular position of the aircraft for which there is no lift, i.e., a zero of C_L. It is possible, therefore, to approximate C_L by the following form:

(4) $$C_L(\alpha_T) = (dC_L/d\alpha_T)(\alpha_T - \alpha_o).$$

The derivative should be taken as a mean value, and α_o is the angle between the line of thrust and the line of zero lift. Again, setting up the equation stating that the resultant force is zero, but ignoring the contribution of thrust, leads to the equation:

(5) $$\tfrac{1}{2}\rho u^2 S\,(dC_L/d\alpha)(\alpha_T - \alpha_o) = mg \cos \theta.$$

This equation can be solved for α_T, resulting in:

(6) $$\alpha_T = \alpha_o + 2mg \cos \theta/(\rho u^2 dC_L/d\alpha).$$

This is the approximate form required.

In the preparation of rocket firing tables, the line of thrust is not used as the line of reference. It is more convenient to use a line which is marked in the aircraft by two studs. This line is called the gun-level lug line, or more simply, the level line. The angle of attack of the level line differs from α_T by a constant amount. It can be written in the form:

(7) $$\alpha_{LL} = c + km \cos \theta/\rho u^2.$$

The numbers c and k are constants depending on the particular type of aircraft. The mass m depends on the loading and is known to the pilot. This formula is used in the construction of rocket tables, the constants c and k being deduced from angle of attack measurements.

It might be thought that the factor ρu^2 would be difficult to determine in the aircraft. This is not actually the case. The usual device for measuring air speed is a Pitot static tube which compares the free stream pressure with the total head. The total head is the pressure which would be measured at the front of a tube pointing directly into the air stream.

The result of this comparison, as read on the pilot's speed indicator, is $u\sqrt{\rho/\rho_0}$ where ρ_0 is a standard value of the density. This quantity is called indicated air speed, and is actually the parameter needed for angle of attack determination.

7. General equations of motion of a symmetrical projectile.

The equations of motion in full generality, including the complete system of aerodynamic forces, a variable wind, density and temperature, and the Coriolis force, would be almost impossibly complicated. We shall, in the later work, go to considerable trouble to devise a system whereby the various factors can be treated one at a time, instead of simultaneously. It will, however, still be necessary to have available the equations which determine the angular motion of a projectile, and these form a rather complicated system. It is the purpose of this section to derive these equations under the following assumptions.

(1) X, Y, Z are the coordinates of the projectile in a frame of reference fixed to the earth. This is supposed to be an inertial frame (the Coriolis force is neglected). The Y-axis is supposed to point vertically upward.

(2) The density ρ and the speed of sound a are functions of Y only.

(3) There is no wind. The atmosphere has no motion relative to the XYZ-system.

(4) The projectile has an angle s, $0 < s < \pi$, of rotational symmetry and a plane of mirror symmetry.

As in the earlier part of the chapter, $\mathbf{u}$ is the vector velocity of the shell, $\boldsymbol{\omega}$ is its angular velocity and $\mathbf{F}$ and $\mathbf{G}$ respectively are the aerodynamic force and moment. The unit vectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are fixed in the shell, the $\mathbf{x}_1$-vector lying along the axis. As before, a letter with the subscript 1, 2 or 3 means the component of the vector along $\mathbf{x}_1$, $\mathbf{x}_2$, or $\mathbf{x}_3$. As before, one real and one complex number will be used as a representation of a vector. The following shows the notation used for the various vectors.

(5)

Vector	Real and Complex Parts	
$\mathbf{u}$, velocity,	u_1,	$\xi = u_2 + iu_3$,
$\boldsymbol{\omega}$, angular velocity,	ω_1,	$\eta = \omega_2 + i\omega_3$,
$\mathbf{F}$, aerodynamic force,	F_1,	$\mathcal{F} = F_2 + iF_3$, $= c_1\xi + c_2\eta$,
$\mathbf{G}$, aerodynamic torque,	G_1,	$\mathcal{G} = G_2 + iG_3$, $= c_3\xi + c_4\eta$,
$\mathbf{y}$, unit vector parallel to Y-axis,	y_1,	$\mathcal{Y} = y + iy_3$,
$-g\,\mathbf{y}$, acceleration due to gravity,	$-gy_1$,	$-g\mathcal{Y}$.

Since the projectile is supposed to be symmetric, its principal moments of inertia about axes perpendicular to its axis of symmetry are, by Theorem (I.11.14), the same. Its moment of inertia about its axis of symmetry (its axial moment of inertia) will be denoted A, and B will be the moment of inertia about any transverse axis through the center of mass. The mass of the shell is m. According to Newton's second law, the rate of change of momentum is the sum of the exterior forces. In this case, there are two forces to consider, the aerodynamic force $\mathbf{F}$ and the gravitational force $-mg\,\mathbf{y}$. Hence,

(6) $$[m\mathbf{u}]^{\bullet} = \mathbf{F} - mg\,\mathbf{y}.$$

The angular momentum of the shell, since $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, are principal axes of inertia, is

$$A\omega_1 \mathbf{x}_1 + B\omega_2 \mathbf{x}_2 + B\omega_3 \mathbf{x}_3.$$

It will be more convenient to write this in the equivalent form

$$(A - B)\omega_1 \mathbf{x}_1 + B\,\boldsymbol{\omega}.$$

Since the only torque acting on the shell is $\mathbf{G}$, the second of the equations of motion, stating that the rate of change of angular momentum is the impressed torque, is:

(7) $$[(A - B)\omega_1 \mathbf{x}_1 + B\,\boldsymbol{\omega}]^{\cdot} = \mathbf{G}.$$

Let us take time out for a small lemma.

(8) <u>Lemma. Let $\mathbf{p}$ be an arbitrary vector, with components p_1, p_2, p_3 in the $\mathbf{x}_1$-, $\mathbf{x}_2$-, $\mathbf{x}_3$- coordinate system. If its real-complex number representation is p_1, $\mathcal{P} = p_2 + ip_3$ then the corresponding representation of $\dot{\mathbf{p}}$ is</u>

$$\dot{\mathbf{p}} \cdot \mathbf{x}_1 = \dot{p}_1 + (\mathcal{P}\bar{\eta} - \bar{\mathcal{P}}\eta)i/2,$$

$$\dot{\mathbf{p}} \cdot \mathbf{x}_2 + i\dot{\mathbf{p}} \cdot \mathbf{x}_3 = \dot{\mathcal{P}} - ip_1\eta + i\omega_1\mathcal{P}.$$

Proof: Throughout this proof it will be convenient to use $\Sigma(\ )_j$ to mean the sum of three quantities $(\ )_1$, $(\ )_2$ and $(\ )_3$. Thus $\mathbf{p} = \Sigma p_j \mathbf{x}_j$. Hence

$$\dot{\mathbf{p}} = \Sigma \dot{p}_j \mathbf{x}_j + \Sigma p_j \dot{\mathbf{x}}_j.$$

Since for each value of j, $\mathbf{x}_j$ is a vector fixed in the projectile,

$$\dot{\mathbf{x}}_j = \boldsymbol{\omega} \times \mathbf{x}_j.$$

This vector equation can be reduced to scalar equations by taking components in the directions $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$. The component in the $\mathbf{x}_k$-direction, $k = 1, 2, 3$, is

$$\dot{\mathbf{p}} \cdot \mathbf{x}_k = \dot{p}_k + \Sigma p_j \boldsymbol{\omega} \times \mathbf{x}_j \cdot \mathbf{x}_k.$$

The last term can be rewritten:

$$\dot{\mathbf{p}} \cdot \mathbf{x}_k = \dot{p}_k - \quad \Sigma p_j \, \omega \cdot (\mathbf{x}_k \times \mathbf{x}_j), \text{ for } k = 1, 2, 3.$$

This form is convenient for computing since $\mathbf{x}_k \times \mathbf{x}_j$ is either zero, if $k = j$, or, except for sign, the third of the three unit vectors. Writing out the three formulas obtained by setting $k = 1, 2, 3$ gives equations

$$\dot{\mathbf{p}} \cdot \mathbf{x}_1 = \dot{p}_1 - p_2\omega_3 + p_3\omega_2,$$
$$\dot{\mathbf{p}} \cdot \mathbf{x}_2 = \dot{p}_2 - p_3\omega_1 + p_1\omega_3,$$
$$\dot{\mathbf{p}} \cdot \mathbf{x}_3 = \dot{p}_3 - p_1\omega_2 + p_2\omega_1.$$

The first of these can be rewritten:

$$\dot{\mathbf{p}} \cdot \mathbf{x}_1 = \dot{p}_1 + [\, (p_2 + ip_3)\, (\omega_2 - i\omega_3)$$
$$- (p_2 - ip_3)\, (\omega_2 + i\omega_3)]\, i/2$$
$$= \dot{p}_1 + (\rho\bar{\eta} - \bar{\rho}\eta)i/2.$$

The second equation, added to the third multiplied by i, gives

$$\dot{\mathbf{p}} \cdot \mathbf{x}_2 + i\dot{\mathbf{p}} \cdot \mathbf{x}_3 = \dot{p}_2 + i\dot{p}_3 - ip_1(\omega_2 + i\omega_3)$$
$$+ i\omega_1(p_2 + ip_3)$$
$$= \dot{\rho} - ip_1\eta + i\omega_1\rho.$$

which completes the proof of the lemma.

Returning now to the main argument, the lemma applied to the case $\mathbf{p} = \mathbf{u}$ enables us to replace the vector equation (6) by one equation in reals and one equation in complex numbers. These are obtained by taking components along $\mathbf{x}_1$, and taking

(component along $\mathbf{x}_2$) + i(component along $\mathbf{x}_3$).

Equation (6) then becomes:

$$m[\dot{u}_1 - (\xi\bar{\eta} - \bar{\xi}\eta)i/2] = F_1 - mg_1,$$

$$(9) \quad m[\dot{\xi} - iu_1\eta + i\omega_1\xi] = \mathcal{F} - mg\mathcal{Y}$$
$$= c_1\xi + c_2\eta - mg\mathcal{Y}.$$

The same procedure will now be applied to the vector equation (7). By the lemma, the vector

$$[\omega_1 \cdot \mathbf{x}_1 + 0 \cdot \mathbf{x}_2 + 0 \cdot \mathbf{x}_3]^\cdot$$

has $\dot{\omega}_1$, $-i\omega_1\eta$ for its real and complex number representation. Similarly, $\dot{\boldsymbol{\omega}}$ has

$$\dot{\omega}_1 + (\eta\bar{\eta} - \eta\bar{\eta})i/2 = \dot{\omega}_1$$

and

$$\dot{\eta} - i\omega_1\eta + i\omega_1\eta = \dot{\eta}$$

for its representation. Using these facts, the equation obtained by taking the components along $\mathbf{x}_1$ in (7) is

$$(A - B)\dot{\omega}_1 + B\dot{\omega}_1 = G_1,$$

or

$$(10) \qquad A\dot{\omega}_1 = G_1.$$

The equation obtained from the components along $\mathbf{x}_2$ and $\mathbf{x}_3$ is:

$$(A - B)(-i\omega_1\eta) + B\dot{\eta} = \mathcal{G},$$

or

$$(11) \quad B\dot{\eta} + (B - A)i\omega_1\eta = \mathcal{G}$$
$$= c_3\xi + c_4\eta.$$

It is now necessary to derive the differential equation governing $\mathbf{y}$, for although $\mathbf{y}$ is a constant vector, its components in the moving frame of reference will surely change. The differential equation is obtained very simply. Since $\mathbf{y}$ is constant, $\dot{\mathbf{y}}$

is zero. Replacing $\mathbf{p}$ by $\mathbf{y}$ in the lemma gives the equation:

$$
(12) \qquad \begin{aligned} \dot{y}_1 &= (\mathcal{Y}\bar{\eta} - \bar{\mathcal{Y}}\eta)i/2, \\ \dot{\mathcal{Y}} &= iy_1\eta - i\omega_1\mathcal{Y}. \end{aligned}
$$

If the density and the velocity of sound did not depend on the altitude Y, the equations (9), (10), (11), (12) would be solvable. That is, they could be solved and any other information desired about the motion of a shell could be obtained after completing the solution. Since ρ and a depend on Y, it is necessary to adjoin one more equation to this collection in order to solve the system. This equation is

$$\dot{Y} = \mathbf{u} \cdot \mathbf{y},$$

or

$$
(13) \qquad \begin{aligned} \dot{Y} &= u_1 y_1 + u_2 y_2 + u_3 y_3 \\ &= u_1 y_1 + (\xi\bar{\mathcal{Y}} + \bar{\xi}\mathcal{Y})/2. \end{aligned}
$$

The equations (9) to (13) consist of four real and three complex equations involving the four real and three complex numbers u_1, ξ, ω_1, η, y_1, $\mathcal{Y}$, Y. Their solution would completely determine the yaw of the shell at every point along its trajectory. This is a tenth-order system of differential equations, but it is not difficult to see that it can be reduced to a ninth-order system. The vector $\mathbf{y}$ is a unit vector, and this fact can be used to eliminate one of the unknowns y_1, y_2, y_3. Or, instead of the variables $\mathbf{y}$, Y the product $Y\mathbf{y}$ could be used. Thus a ninth-order system of equations determines the yawing motion of a shell. However, the solution of this system does not completely determine all of the facts about the trajectory of the projectile. Two of the three coordinates of the center of mass of the shell, namely X and Z, are still undetermined. Since the prediction of the position of the shell is the principal problem of exterior ballistics, one can hardly say the problem is solved. Actually, both X and Z are solutions of the same set of equations, (12) and (13), which Y satisfies, but the initial conditions differ in each case. Adjoining X and Z to the set of variables in the equations would, on the surface, appear to increase the order of the system by eight, since (12) and (13) form a fourth-order system. Again, there are identities and these reduce the total order of the complete system to twelve. The reader will undoubtedly be considerably relieved to know that we do not intend to solve this twelfth-order system. It will be quite sufficient for future calculations if we retain the facts which are collected together in the following theorem.

(14) Theorem. The yawing motion of a rotationally and mirror symmetric projectile which has small yaw is determined by the solution of the equations:

a. $m(\dot{\xi} - iu_1\eta + i\omega_1\xi) = c_1\xi + c_2\eta - mg\mathcal{Y},$

b. $B\dot{\eta} + (B - A)i\omega_1\eta = c_3\xi + c_4\eta,$

c. $m[\dot{u}_1 - (\xi\bar{\eta} - \bar{\xi}\eta)i/2] = F_1 - mgy_1,$

d. $A\dot{\omega}_1 = G_1,$

e. $\dot{y}_1 = (\mathcal{Y}\bar{\eta} - \bar{\mathcal{Y}}\eta)i/2,$

f. $\dot{\mathcal{Y}} = iy_1\eta - i\omega_1\mathcal{Y},$

g. $\dot{Y} = u_1y_1 + (\xi\bar{\mathcal{Y}} + \bar{\xi}\mathcal{Y})/2.$

In these equations c_1, c_2, c_3, c_4, F_1 and G_1 represent polynomials in the aerodynamic coefficients, in ρ, d, u_1 and ω_1 as given in (II.4.6).

8. Normal equations of motion.

For computational work it will be seen that the solution of a relatively simple system of equations gives an excellent approximation to the actual motion of a projectile. The deviations of an actual trajectory from this approximate trajectory can be computed by a theory of small corrections, or perturbations. This approximate trajectory is obtained by making the following simplifying assumptions: the projectile moves with its axis tangent to its trajectory, and the only forces acting on the projectile are the drag and the force due to gravity. The density and the velocity of sound are functions of the height of the projectile above the earth's surface and there is no wind. The coordinate frame is inertial.

In this case the projectile will remain within the vertical plane containing its initial velocity vector. We choose a coordinate system with the Y-axis pointing vertically upward and the X-axis horizontal, so that the initial velocity vector lies in the XY-plane. The velocity has components $\dot{X}$, $\dot{Y}$ and the drag is directed oppositely. The drag has magnitude $\rho d^2u^2K_D$, where

$$u^2 = \dot{X}^2 + \dot{Y}^2.$$

Hence the vector drag is

$$\rho\, d^2\, u^2\, K_D(\dot{X},\, \dot{Y})/u.$$

If m is the mass of the projectile, the force due to gravity is $-(0,\, 1)mg$. Setting the rate of change of momentum equal to the impressed force gives

$$(1) \qquad m(\ddot{X},\, \ddot{Y}) = -\rho d^2uK_D(\dot{X},\, \dot{Y}) - (0,\, 1)mg.$$

This vector equation can be resolved into components. If this is done and the dependencies written out explicitly, the results are the <u>normal</u> or <u>particle</u> equations of motion:

$$(2) \qquad \begin{aligned} m\ddot{X} &= -\rho d^2uK_D(u/a)\dot{X}, \\ m\ddot{Y} &= -\rho d^2uK_D(u/a)\dot{Y} - mg, \\ u^2 &= \dot{X}^2 + \dot{Y}^2,\ \rho = \rho(Y),\ a = a(Y). \end{aligned}$$

Most of the problems in exterior ballistics will require a highly precise solution of the equations (2) by numerical integration. This solution will usually be carried out for a standard density function $\rho(Y)$ and temperature function $a(Y)$ and an experimental drag coefficient $K_D(u/a)$. In many cases the solution so obtained will be modified by corrections obtained from an approximate solution of equations (7.14) and in almost all cases the solution must be modified because the density and temperature structures are non-standard. However, the solution of equations (2) will generally form a basic part of our computation.

The equations (2) have been put in a variety of forms, convenient for various applications. A very simple variant, which will presently be useful, is to write the equations in terms of the inclination of the tangent to the trajectory and the velocity. We leave as an exercise the derivation of the following system:

$$(3)\qquad \begin{aligned} &\dot{X} = u\cos\theta, \quad \dot{Y} = u\sin\theta, \\ &m\dot{u} = -\rho d^2 u^2 K_D - mg\sin\theta, \\ &\dot{\theta} = -g\cos\theta/u, \\ &\rho = \rho(Y),\ a = a(Y). \end{aligned}$$

Another form which will be useful is in terms of slant coordinates. That is, instead of locating a point in the plane by means of its distances from two perpendicular axes it is located by means of its distance along a slant line, the P-axis, and its vertical drop Q from this line. The transformation required, if the P-axis passes through the XY-origin at the angle S, is

$$(4)\qquad \begin{aligned} &X = P\cos S,\ Y = P\sin S - Q, \\ &P = X\sec S,\ Q = X\tan S - Y. \end{aligned}$$

Under this transformation, the equations (2) become

$$
\begin{aligned}
m\ddot{P} &= -\rho\, d^2\, u \quad K_D\, \dot{P}, \\
m\ddot{Q} &= -\rho\, d^2\, u \quad K_D\, \dot{Q} + mg, \\
u^2 &= \dot{P}^2 + \dot{Q}^2 - 2\dot{P}\dot{Q} \sin S, \qquad (5) \\
\rho &= \rho\,(P \sin S - Q), \\
a &= a(P \sin S - Q).
\end{aligned}
$$

These equations are usually used only when some simplifying approximation is justified. The verification of (5) is left to the reader.

9. <u>Higher order terms in the expansion of the force function.</u>

In all of our discussion of the aerodynamic force on a projectile we have been content to use the leading terms in the Taylor expansion in cross velocity ζ and cross spin η. It is of considerable importance to know under what conditions this approximation will be adequate. Some information can be obtained from further mathematical investigation, but to a very large extent this question must be answered on the basis of experimental work. Each of the functions F_1, G_1, $\mathcal{F}$ and $\mathcal{G}$, can be expanded in quadratic terms in ζ and η. In each case, terms containing the factors

$$\zeta^2, \eta^2, \bar{\zeta}^2, \bar{\eta}^2, \zeta\eta, \ \zeta\bar{\eta}, \ \dot{\zeta}\,\bar{\zeta}, \ \eta\bar{\eta}, \ \bar{\zeta}\bar{\eta}, \ \bar{\zeta}\eta,$$

will be obtained. As in considering the linear terms, the consequences of rotational symmetry can be investigated. Precisely the same argument used before will show that all of the quadratic terms vanish in the expansion of $\mathcal{F}$ and $\mathcal{G}$. (The details of the argument have been published by C. G. Maple and J. L. Synge, in the <u>Quarterly Journal of Applied Mathematics</u>, Vol. VI (1949), pp. 345-366.) There are, however, possible non-vanishing quadratic terms in the expansion of F_1 and G_1. When the projectile has symmetry under rotation through angle s $(0 < s < \pi)$, the surviving terms in the expansion of F_1 are

$$k_1 \xi \bar{\xi} + k_2 \eta \bar{\eta} + k_3 \xi \bar{\eta} + \bar{k}_3 \bar{\xi} \eta \qquad (k_1,\ k_2 \text{ real}),$$

and there are similar terms in the expansion of G_1. Expressed otherwise, the quadratic terms are multiples of the square of the magnitude of the cross velocity $|\xi|^2$, of the magnitude of cross spin $|\eta|^2$, of the product of these magnitudes by the cosine of the angle between cross velocity and cross spin, and the product of these magnitudes by the sine of that angle.

Of these higher order terms only one has received any attention. The spin-decelerating moment G_1, is itself small and its dependence on cross spin and cross velocity is, with present experimental equipment, impossible to determine. Considerable experimentation has been conducted on the dependence of drag on yaw. In the usual form, one considers not the axial drag, $\rho d^2 u^2 K_{DA}$ but the component $\rho d^2 u^2 K_D$ along the trajectory. We write:

(1) $$\text{Drag} = \rho d^2 u^2 K_D(\xi \bar{\xi},\ u,\ a),$$

neglecting the dependence on $\eta \bar{\eta}$ and $\xi \bar{\eta} + \bar{\xi} \eta$. These three variables, $\xi \bar{\xi}$, u, a have u/a and $\xi \bar{\xi}/u^2$ as the only dimensionless products. One usually denotes $\xi \bar{\xi}/u^2$ by δ^2, the square of the yaw (actually the square of the sine of the yaw). Both K_D' and $K_{D\delta}$ have been used to describe this dependence, their definitions being

(2) $$K_D = K_{DO} + K_D'\delta^2,$$
$$K_D = K_{DO}(1 + K_{D\delta}\,\delta^2).$$

For normal shell one might expect the drag to double at about $\delta = 12^\circ$. The coefficient $K_{D\delta}$ is the <u>yaw drag</u> coefficient and its value would accordingly be around .04. It has been found by Dr. A. C. Charters that the coefficient $K_D' = K_{DO} K_{D\delta}$ is more nearly constant from shell to shell, and it is presumed that this coefficient will be used more generally in the future.

There is some evidence that K_M and K_L vary with yaw. The amount of this variation depends quite strongly on the shape of the shell. This topic will be discussed further in the following chapter where experimental determination of the aerodynamic coefficients is considered.

Chapter III

METHODS OF MEASUREMENT OF AERODYNAMIC COEFFICIENTS

1. Devices currently used for velocity and retardation measurement.

We propose to discuss briefly, from the point of view of method, some of the equipment now in use at the Aberdeen Proving Ground for measuring velocity of projectiles. Since, in principle at least, any velocity-measuring device can be used to measure deceleration, and hence drag, a discussion of the measurement of velocity can be considered as preliminary to a discussion of drag measurement.

In any velocity measurement a basic requirement is a time-measuring device. During the greater part of the last war the commonly used measuring device was some model of drum chronograph. In principle this instrument was as follows. A drum, revolving at high speed, carries a strip of tape. This tape is marked by a recording device in one of three ways. The simplest scheme, but one of the slowest in response, consists merely of a mechanical pen-and-ink writing on the tape. A more satisfactory apparatus uses a waxed tape which is punctured by an electric spark. The most satisfactory chronographs use tape of light sensitive paper, the marking being done by a cathode ray. In any case, a sequence of "timing lines" is marked upon the tape. The timing source originally used was a tuning fork, but most modern equipment uses as standard the fundamental frequency of a piece of quartz crystal. The velocity-measuring experiment requires that marks, corresponding to the time that the projectile passed two measured points, be put on the tape, as well as the timing lines. The timing lines then permit, by linear measurement of the tape, an estimation of the time interval required by the projectile to travel the measured distance. The accuracy obtained in the measurement of the time interval depends on the particular model of chronograph used. On the most accurate machines—machines for which a measuring engine is used for the mensuration of the tape—the probable error of a single time reading is of the order of one millionth of a second. This accuracy is sufficient for all present ballistic experiments, and indeed, is so much better than is required for velocity measurements alone, that the extra time required to use the device makes it highly desirable to use a simpler machine. Nevertheless, this time-measuring instrument is now being superseded, at least for experiments of the most exacting character, by a simpler and more accurate machine.

This new machine is called a counter chronograph—its basic unit is called a cycle counter. A cycle counter consists of a sequence of tubes, each with a possible simple on-or-off response. The first of the sequence is actuated by the fundamental frequency of a timing standard. (A timing frequency of 100,000 cycles per second is commonly used, and the finest counters operate at 1,600,000 cycles per second.) The first tube then simply turns on and off at the frequency of the timing standard. By means of an electronic circuit, the second tube is arranged so as to be activated by every <u>second</u> impulse of the preceding tube. The device is then used as follows. The counter is started by a tripping mechanism as the projectile passes the first of the two measured positions, and is stopped as the projectile passes the second. It is then recorded which tubes in the sequence are on and which off. The time interval can now easily be calculated as follows. Let us suppose that the unit of time is the period of one cycle of the timing standard. If only the first tube of the sequence is on, the time interval is 1 unit. If the second is on, the interval is 2 units; the third, 4 units. If the first, third, sixth, eighth, ninth and twelfth are on the time interval is $2^0 + 2^2 + 2^5 + 2^7 + 2^8 + 2^{11}$ units. In fact, the counter records units of time in the number system of base two! The extraordinary ease of reading this sort of chronograph, together with its accuracy, makes it extremely valuable for many uses.

The second important component of any velocity-measuring apparatus is the device used to record the passage of the projectile. An instrument, commonly used at one time, consisted of two foil screens, set parallel to each other and very near together. The pair of screens were set at a measured position in front of the gun, and upon being ruptured by the passage of the projectile, shorted a circuit and sent a signal to the timing device. A single thin wire was also used on occasion, the breaking of the wire giving a time signal. Both of these schemes suffer from the defect that they affect, more or less seriously, the phenomenon which is being observed. The most commonly used instrument of this last war was a solenoid. This device is quite usable—the time of reversal of electromotive force from the solenoid actually depends on the particular area through which the shell passes, but the dependence is not extreme. For high-angle fire, a certain amount of scaffolding is necessary to support the solenoids, and the equipment is in general not easily portable. It is therefore not convenient for field calibration of artillery pieces.

A scheme of replacing the solenoids was developed in England during the Second World War. In principle, the apparatus consisted of a photoelectric cell, mounted so that it is exposed to a narrow band of the sky. When the shell passes this band of sky there is a rapid change of the total light falling on the cell, and this change of intensity is used to provide a signal to a chronograph. An adaption of this device was worked out at the Ballistic Research Laboratories during the war, and proved to be well adapted to field use, and in fact also turned out to be convenient for many laboratory tests as well. These "sky screens," though simple in concept, require rather delicate design and intelligent handling.

Finally, we mention briefly the device used on the aerodynamic range. Here, an electric charge on the projectile is used to trigger a spark gap. The spark photographs the shell on a plate, and the discharge of the spark also transmits a signal to the chronograph. This method is discussed quite fully in Chapter XIII.

For any of the equipment discussed above the basic data resulting from a velocity measurement consist of the times at which the projectile passes two known space positions. To measure acceleration, we require the times at which the projectile passes three known space positions, all reasonably near the muzzle. We may then compute the mean velocity, and from the velocity loss, the force on the projectile. In practice, the computing procedure is somewhat more sophisticated than this. The form of the time as a function of distance from the muzzle is deduced from the equations of motion of the projectile, and the observed data are fitted (by means of a least squares process if there are enough observations to overdetermine the drag) to a function of this type. The fitted function then determines the drag of the shell. We shall not go into the details of this process here, since it is treated quite completely in the discussion of the spark range measurements in Chapter XIII.

A variant of this process has also been used to good purpose. Again, the time at which the projectile passes three known points is measured, but the space distribution of the three points is quite different.

Two of the points are taken near the muzzle, so that, effectively, a measurement of muzzle velocity results. The third point is at the end of a short, flat trajectory. It is quite feasible to deduce the drag of a projectile from such data as these, consisting of a muzzle velocity and a time of flight for a short, flat trajectory. The details of this procedure are discussed in Chapter V.

2. Determination of drag by means of direct observation of position.

A number of experiments have been conducted which differ from the resistance firings discussed in the previous section in details of execution but not in fundamental method. The first of these experiments (as far as we know) was conducted by Cranz. This work will be discussed rather briefly here since a full account is given in K.J. Cranz's Lehrbuch der Ballistik, Vol. III, Experimentelle Ballistik, 2nd ed. (Berlin: Julius Springer, 1927). A mechanism was designed and constructed which would fit inside a shell, the purpose of the mechanism being to emit a flash of light at intervals after the firing of the shell. The shell was then fired at night, and the position of the flashes was recorded by means of cameras. Simultaneously a record was made of the time at which the flashes occurred, so that a complete time-position record of the flight of the projectile was obtained. The acceleration and hence the force on the projectile can be deduced from these data by methods which will be discussed below. In Cranz's experiment, the flashing light was constructed by pyrotechnic methods, the mechanism being essentially a powder train, setting off periodically a charge of magnesium. The same method was used to determine experimentally the trajectory of a rocket by a group at the California Institute of Technology under Dr. W.R. Smythe in the summer of 1945. In this case, the observations were made by means of motion picture cameras, the cameras being equipped with an optical device which permitted simultaneous photography of the phenomenon and a clock. Before discussing the mathematical difficulties associated with reduction of these data it might be advisable to point out an objection to the method which is of purely experimental character. A rocket or a shell which is periodically emitting puffs of incandescent gas may or may not move in a manner closely resembling the motion of an ordinary rocket or shell. That is, measurement is made of a phenomenon which differs from the one in which we are interested by a more or less indeterminable amount. The validity of this objection depends of course on the accuracy which is desired and upon the degree to which the shell constructed differs from its prototype.

A mechanism was constructed by L.A. Delsasso and L.G. de Bey of the Ballistic Research Laboratory which is not subject to this objection. In their experiment, the projectile under study was a 500 pound General Purpose bomb, so that a great deal more room for equipment was available. (The same type of experiment had been performed on a bomb at the time of the First World War by D.L. Webster.) The equipment constructed by Delsasso and de Bey consisted of an Edgerton type lamp mounted in the nose of the bomb with batteries placed within the bomb. A condenser was charged by the batteries and at intervals of about one second discharged through the lamp, giving an extremely bright light of short duration. A transmitter in the bomb gave out a radio signal at the time of the flash and this signal was recorded on a chronograph. Plate cameras of the same type used in routine range bombing recorded the position of each flash. The particular features of the method were that the bomb used matched the prototype in external contour and in weight distribution, and the accuracy of space position and time recording was superior to any previously obtained.

Superficially, the mathematical analysis of such an experiment is extremely simple. The available data are the X-, Y-, Z-coordinates of the projectile as functions of time, the density ρ, the components of wind W_X and W_Z and the temperature as functions of altitude Y. Assuming that the yaw of the bomb is zero and that the only forces are drag and gravity, the equations governing the motion of the bomb are easy to derive. The XYZ-coordinate system is fixed to the ground, but the drag of the bomb depends on the velocity with respect to the air. Let u denote the speed with respect to the air. That is,

$$u^2 = (\dot{X} - W_X)^2 + \dot{Y}^2 + (\dot{Z} - W_Z)^2.$$

The drag has magnitude $\rho d^2 u^2 K_D$ and has the direction of $-(\dot{X} - W_X, \dot{Y}, \dot{Z} - W_Z)$. Hence the vector equation of motion is

$$(1) \quad m(\ddot{X}, \ddot{Y}, \ddot{Z}) = -\rho d^2 u K_D(\dot{X} - W_X, \dot{Y}, \dot{Z} - W_Z) - mg(0, 1, 0).$$

The quantities $\dot{X}$, $\dot{Y}$, $\dot{Z}$ and $\ddot{X}$, $\ddot{Y}$, $\ddot{Z}$ can be obtained from the experimental data by numerical differentiation and then any one of the three scalar components of the equation (1) can be used to evaluate K_D. The temperature determines the velocity of sound and hence the value of Mach number to which this value of K_D corresponds.

The difficulties arising in the execution of the above program can be illustrated by a very simple example. Suppose that the position of the bomb is determined at intervals of one second, that the timing is exact, and that there is no wind. If the bomb is dropped from medium altitude, say about ten or twelve thousand feet, its velocity at the end of its trajectory would be of the order of 800 feet per second. If the bomb is the 500 pound General Purpose mentioned earlier, the deceleration due to drag is about fifteen feet per second per second. Suppose that the errors in the determinations of the coordinates are approximately normally distributed, with standard deviations of about nine inches, or probable errors of about six inches. The y-component of the velocity at the mid-point of a one second interval would be determined as the difference of the y-coordinates at the beginning and the end of that second, and the acceleration would be the difference of two successive velocity determinations. If y_1, y_2 and y_3 are successive determinations of the altitude, the y-component of acceleration would thus be determined by the expression $y_3 - 2y_2 + y_1$. But by Section 21 of Chapter I, the probable error of this expression is $\sqrt{1 + 4 + 1}$ times the probable error of each y-coordinate, or roughly 1.25 feet per second per second. We are then determining a quantity of the order of 15 feet per second per second with a probable error of about 1.25 feet per second per second, or roughly 8 per cent. The situation is actually rather worse than this in practice, since the probable error of a coordinate is likely to exceed six inches, and furthermore the wind and density are not known perfectly and there are also errors in timing. In order to use effectively experimental data of this type, the most refined methods of numerical analysis are needed. Since these methods are not peculiar to ballistics, a detailed discussion will not be given. The same sort of problem will be discussed in some detail in connection with the reduction of data from the aerodynamic range. Due to the extreme accuracy required in the direct determination of the trajectory, very few data of this sort have been successfully reduced. If the experimental difficulties are overcome, extremely valuable data result, since the experiment is performed under conditions identical with those under which the weapon is used.

There is only one objection of a theoretical nature to the methods described. It is assumed that the yaw of the projectile is small and that drag is the only force other than gravity acting on the projectile. The experiment, since it involves no measurement of yaw, will not in itself determine whether this assumption is justified. It must be known, by means of other experiments, that this assumption is justifiable.

3. Bomb drops with accelerometers.

An experimental method was used at Aberdeen to find the drag of bombs which avoids certain of the difficulties mentioned in the previous section. An accelerometer was devised by the National Defense Research Committee to fit inside a bomb. In essence, this was simply a small weight attached to a coil spring; the spring was attached to the bomb. The axis of the spring was coincident with the axis of the bomb, so that deceleration of the bomb caused a displacement of the weight from its equilibrum position. The position of the weight is then a measure of the axial component of force on the bomb. Furthermore, if the bomb is in free flight, gravity has no effect on the position of the weight relative to the bomb since gravity acts both on the bomb and on the weight. To be more precise, in flight there are two axial components of force acting on the weight: first, the axial component of gravity and second, the force due to tension of the spring. There are also two axial components of force on the bomb, gravity and the axial component of drag. If the relative motion of the weight with respect to the bomb can be neglected, the total axial acceleration of both must be the same, and the acceleration of the weight due to displacement must be the same as the acceleration of the bomb due to drag.

In the accelerometer used in the experiments the position of the weight was telemetered to a receiving station on the ground. A small transmitter in the bomb transmitted a frequency which was a function of the position of the weight. The basic data from the experiment were the axial drag on the bomb as a function of time, the initial position and velocity of the bomb at release and the wind, density and temperature as a function of altitude. From these data it was necessary to deduce the position and velocity of the bomb throughout its trajectory, so that the speed, air density and Mach number to which the observed value of drag corresponded would be known. In order to perform the calculation the equations (2.1) may be used. Let a(t) be the acceleration measured by the accelerometer. Then, $ma(t) = \rho d^2 u^2 K_D$, and the equations (2.1) may be written in the following form:

$$(1) \quad (\ddot{X}, \ddot{Y}, \ddot{Z}) = - a(t) (\dot{X} - W_X, \dot{Y}, \dot{Z} - W_Z)/u - g (0, 1, 0).$$

These differential equations may be solved by the methods of numerical integration which are discussed in Chapter VI. The numerical solution, though tedious, does not lead to any surprising loss of accuracy as does the numerical differentiation discussed before. The results of the integration are the coordinates X, Y, Z as functions of t. The drag may then be divided by the correct values of ρ, d^2 and u^2 to give the drag coefficient K_D. The results of one such analysis, as made by W. Mills, are shown in Figure 1. It is noticeable that the record shows an oscillatory drag at about 20 seconds after release. The period of this oscillation is approximately the natural period of yaw of the bomb. This shows that the fundamental requirement that the bomb have small yaw was not met in this practical experiment.

A good overall check on this experiment is furnished by the numerical integration of the trajectory. The bomb was dropped over the regular range bombing instrumentation. The position of the airplane at release and its initial vector velocity were measured as well as the time of flight and the position of impact. The numerical integration, carried from the beginning of the acceleration record to the end, gives the total horizontal distance the bomb travelled during its flight and its total drop. These computed values may be checked against the measurements obtained in the range bombing. It was found for the bomb whose record is shown that the computed horizon-

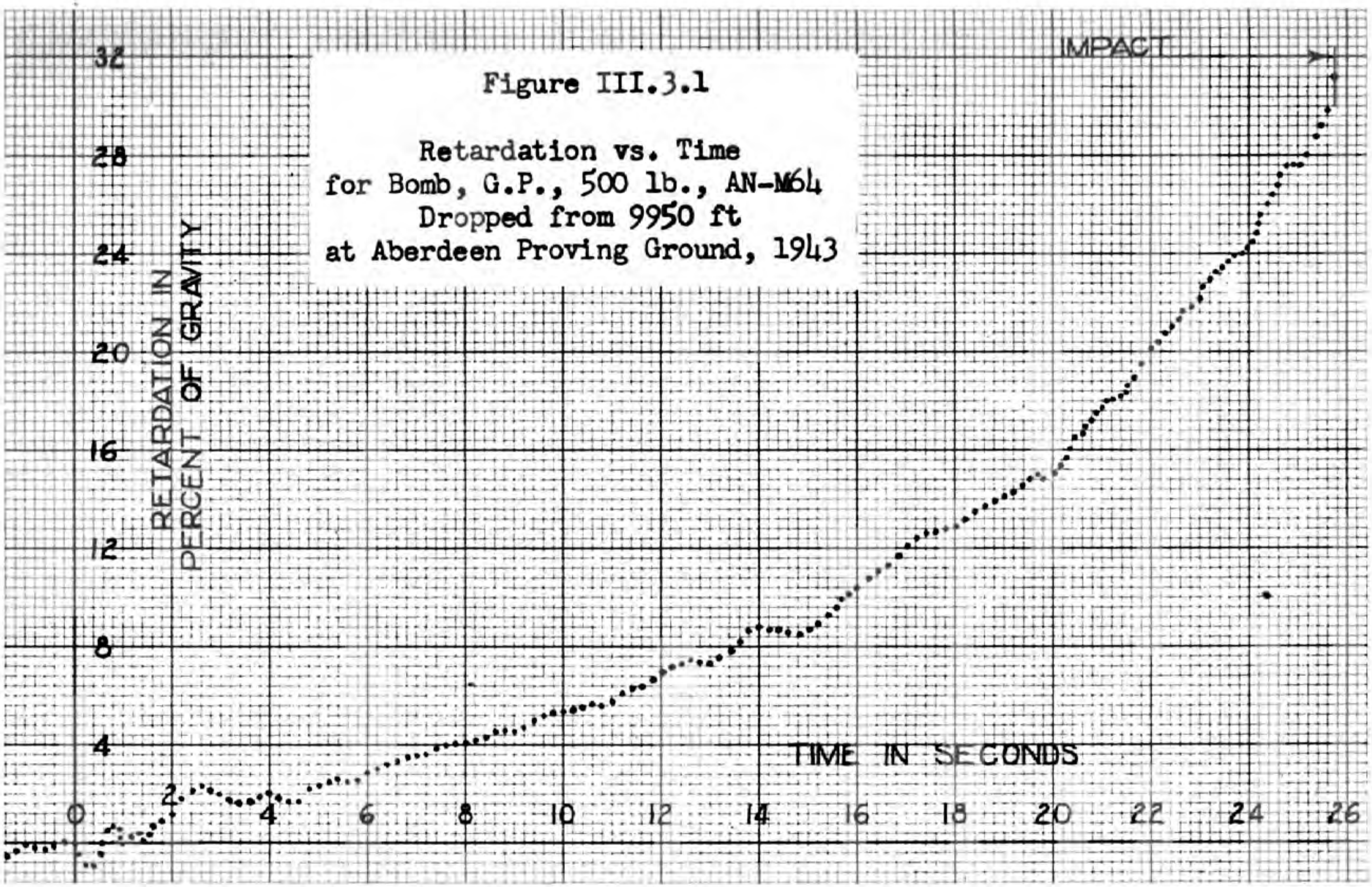


Figure III.3.1
Retardation vs. Time
for Bomb, G.P., 500 lb., AN-M64
Dropped from 9950 ft
at Aberdeen Proving Ground, 1943
RETARDATION IN
PERCENT OF GRAVITY
TIME IN SECONDS
IMPACT
32
28
24
20
16
12
8
4
0
2
4
6
8
10
12
14
16
18
20
22
24
26

tal travel was over a hundred feet less than the measured travel. Furthermore, according to the computation the bomb was two hundred feet under ground at the end of the record. A complete solution of the equations of motion as described in Chapter XI shows that this sort of result might well be expected if the computation is done on the basis of drag only. However, the discrepancies should be smaller.

The accelerometer unit used was not the final model. There is considerable reason to believe that this accelerometer did not measure the axial component of acceleration only. The later models were known to be insensitive to acceleration perpendicular to the axis of the bomb.

The accuracy of the measurement of acceleration is estimated to be 0.3 feet per second per second. The estimate of drag coefficient obtainable is accurate then to, at best, one per cent. That is, when the bomb has reached its maximum velocity the drag is of the same order of magnitude as the acceleration due to gravity and 0.3 feet per second per second is one per cent of the total drag. A rather comprehensive program was under way at Aberdeen utilizing this method and the method discussed in the previous section to evaluate the drag coefficient for several bombs. The program was temporarily suspended due to redesign of certain bombs, which was necessitated because of evidence that the yaw of the bomb during its flight was not small. This fact makes both methods ineffective, and as a matter of fact, the bombs too.

4. Wind tunnel tests; static tests.

One of the most useful experimental methods of measuring the aerodynamic forces on a projectile is essentially a reversal of the method so far discussed. Instead of firing or dropping the projectile through the atmosphere the projectile is held fixed and air is moved past it. The measurements are not limited to drag alone, and this makes the method particularly useful in the design of finned projectiles. These projectiles, bombs, mortar shell and rockets, are supposed to have a restoring moment. That is, the torque due to yaw is supposed to tend to decrease the yaw, and is supposed to be negative. Measurement of drag alone is useless in the design of such projectiles.

Most of the work of this sort in this country has been done at the National Bureau of Standards by H. L. Dryden and R. H. Heald. During the war a rather large body of data was also obtained at the Hydraulic Machinery Laboratory, California Institute of Technology, under the direction of R. T. Knapp. Although this later work was performed in a water tunnel, the results compare very well with the most reliable subsonic wind tunnel work.

Good descriptions of wind tunnel method are available in W. F. Durand, Aerodynamic Theory (Berlin: Julius Springer, 1934). Therefore we will be content to discuss briefly the general principles. A wind tunnel (or a water tunnel) is designed to send air (or water) through a test section in such a manner that the vector velocity of the fluid at every point in the section is the same. For subsonic flow, immediately preceding the test section there is a larger section where the velocity is much lower. The density of air, its velocity and temperature in the test sections are computed from various measurements. In an idealized case, the computation proceeds on the following principles. The three basic measurements are measurement by means of orifices in the tunnel walls of pressure P_t in the test section, and of the pressure P_o and the temperature T_o in the section preceding the test section.

The temperature is difficult to measure if the velocity is high, so that a direct measurement of T_t is inaccurate. The gas law, $P/\rho = kT$, k a constant depending on the gas, gives a relation between the pressure, the temperature and the density, so the density ρ_0 in the pre-test section can be computed. The two velocities V_t, V_0 and the density ρ_t in the test section must be computed. This may be done on the basis of three physical laws. These are the laws of conservation of matter, of momentum and of energy. Each of these laws gives one equation relating P_t, ρ_t, V_t and P_0, ρ_0, V_0, and these equations can be solved for V_t, ρ_t and V_0. The temperature T_t is then obtained from P_t, ρ_t by means of the gas law. In actual practice this procedure is modified considerably, and corrections must be made, based on comprehensive calibration tests.

The model to be tested is suspended in the test section on a strut. The strut is rigidly connected to a balance system which consists essentially of a table which is free to make small movements. The force and torque which are necessary to prevent movement of the table are measured and give directly the force and torque on the model and the strut. In order to get the force on the model alone we must subtract from the measured force the force on the strut. A way of measuring the force on the strut alone is to remove the model and repeat the same series of tests. It is, of course, desirable to keep this correction as small as possible. For this reason, a windshield usually fits around the strut. The windshield is connected to the tunnel wall, or to some framework which is independent of the balance system. The correction, then, is for the part of the strut that is exposed to the air flow. This is called a tare correction. It is also desirable to make a so-called interference correction. The air flow around the bomb is not the same as if the strut were not there. The change of the force on the bomb due to the presence of the strut (not the force on the strut) is called an interference effect. It can be estimated by means of an image test. A dummy strut is fitted into the opposite side of the model and the same series of tests is run. Two sets of data are then available. One gives the total force on the model with two struts and one gives the force with one strut. An extrapolation then gives the force on the model with no strut. This procedure would be perfectly correct if there were no air flow across a plane through the axis of the model and perpendicular to the strut.

For subsonic work probably the best suspension is not by strut, but by a net-work of fine wires. The force on the wires is then computed by removing various ones of them and measuring the resulting force. Unfortunately this sort of suspension makes the work go slowly and one of the big advantages of the wind tunnels is lost, namely, the speed of operation.

In testing a particular model a program something like the following might be laid out. A certain number of speeds would be selected, depending on the speed expected in actual flight and on the capabilities of the tunnel. At each of these speeds the position of the model would be varied through a number of angular positions. At each angular position, measurement would be made of drag D, lift L, and moment M. The resulting values can be graphed as functions of yaw δ for each speed. It is noteworthy that these graphs are not usually symmetric about $\delta = 0$. The lack of symmetry is supposed to be due to lack of symmetry of the model. However, there is usually a point of symmetry on the curve. That is, on the lift and moment curve there is a point such that the curve falls on itself if rotated about the point through 180°, and for the drag curve there is a vertical line of mirror symmetry. The best estimate of the corresponding force or moment of the idealized projectile is obtained by taking averages of points on both sides of the point of symmetry. This has a justification if we assume the asymmetry of the curve is due to a small protuberance which adds a constant amount to the force or moment measured.

It is not necessary in examining the data to make the assumption that the lift and moment are linear in the angle of yaw. Instead we may examine the data and decide for ourselves over what range the linear approximation is valid. It is still desirable to examine dimensionless coefficients and the ones usually chosen in ballistics are the following:

$$K_D = \text{drag}/\rho d^2u^2,$$

$$k_L = \text{lift}/\rho d^2u^2, \quad (1)$$

$$k_M = \text{moment}/\rho d^3u^2.$$

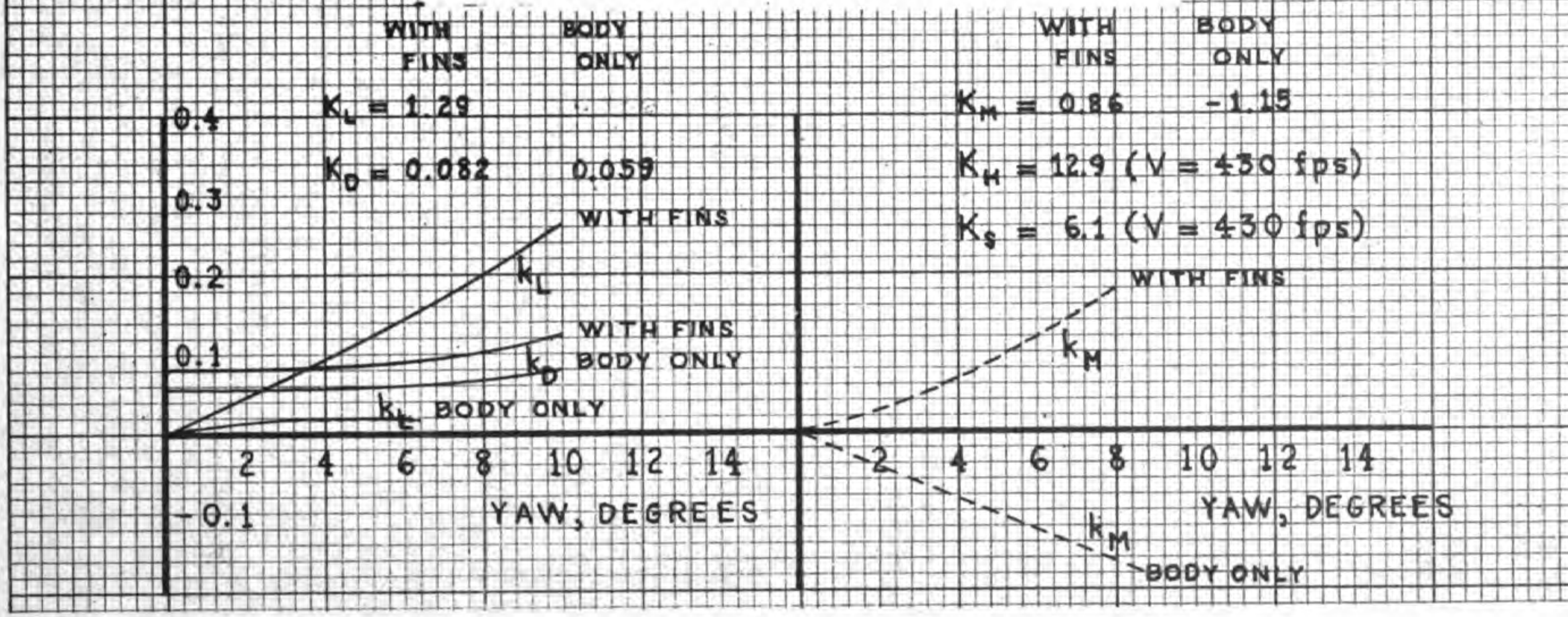
The latter two are related to the coefficient K_L, K_M, in a very simple way. Namely, $k_L = K_L \sin\delta$ and $k_M = K_M \sin\delta$. Actually the use of $\sin\delta$ instead of δ is hardly justified. The departure of the coefficients k_L and k_M from linearity is usually more serious than the difference between $\sin\delta$ and δ. The most satisfactory way of obtaining K_L and K_M is to select an interval about 0° yaw on which k_L and k_M appear to be linear, and to fit a straight line to the observed data over this range. The values of K_L and K_M shown on Figure 1 were obtained this way.

The tremendous advantage of the wind tunnel method is the directness and rapidity of the measurement. The directness of observation is also extremely useful in another way. Certain phenomena may be observed in a qualitative fashion in such a way that a good bit of information is obtained about the probable performance of a model. In particular one can observe separation of the air flow. Around the front end of a projectile which is not too blunt, the air flow will be steady, that is, the vector velocity of the air at points near the projectile will not change with time. On the other hand behind the projectile the velocity at any place will vary rapidly with time, like the flow in the wake of a boat. It is highly desirable to design a projectile where separation occurs as far back as possible. If the flow about the fins is unsteady their restoring action is greatly dimin-

Figure III.4.1

Aerodynamic Coefficients vs. Yaw
Full Scale, Bomb, G.P., 500 Lb., AN-M43,
Twenty Foot Wind Tunnel, Wright Field

1. Suspension: Shielded sting
2. Corrections: For tare and interference
3. Velocity: 586 fps
4. Reynolds No.: 44.2×10^5



ished. Also early separation inevitably gives a very high drag. There are two methods commonly used to observe whether or not the flow is steady. The simplest method which can be used in a subsonic tunnel is to attach small tufts of yarn to the surface of a model. The tufts indicate the direction of flow and immediately show whether or not the motion is steady. Photographic methods permit a much more careful study of the flow pattern. These methods can be used best at supersonic and at relatively high subsonic speeds, since they depend on the fact that the density of the air varies from point to point in the air stream. The value of the observations of this sort is shown by photographs 2a, 2b. In the photograph 2a the angle between the sides of the model and the tail cone is too great, and the flow separates. In the second photograph the model has been modified and no separation occurs. In itself this fact is of no importance but it shows a reason for the extremely important fact that the first model is unstable, while the second is stable. That is, the first model has an overturning moment and the second has a restoring moment.

Interpretation of wind tunnel results requires considerable care. Although higher accuracy can be obtained, most wind tunnel work on projectiles contains uncertainties of the order of five or ten per cent. Some care is also necessary in setting up design requirements. A finned projectile is, in a certain sense, stable if K_M is negative; that is, the model has a restoring moment. This criterion of stability is actually inadequate in itself. Some definite margin of stability is needed. In Section 5 we shall discuss a more suitable criterion of stability which involves, naturally enough, both aerodynamic and physical properties of the projectile.

There is a limitation on the size of the model that can be tested in any particular wind tunnel. It is clear that if the model is too large the air flow will not resemble the flow about the projectile in free

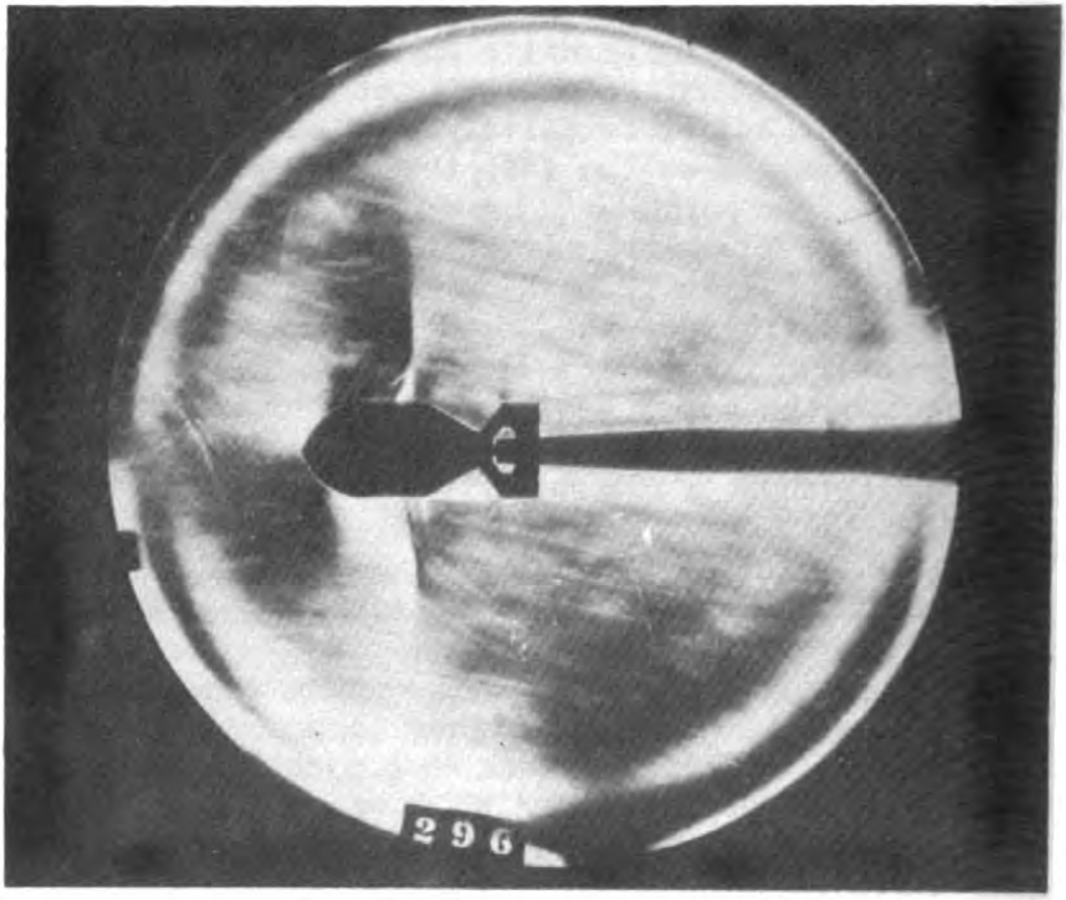


Figure III.4.2a

Schlieren Photograph of Bomb Model at Mach No. 0.85
Model MB 11, Air Flow Separates
Bomb Tunnel, Ballistic Research Laboratories

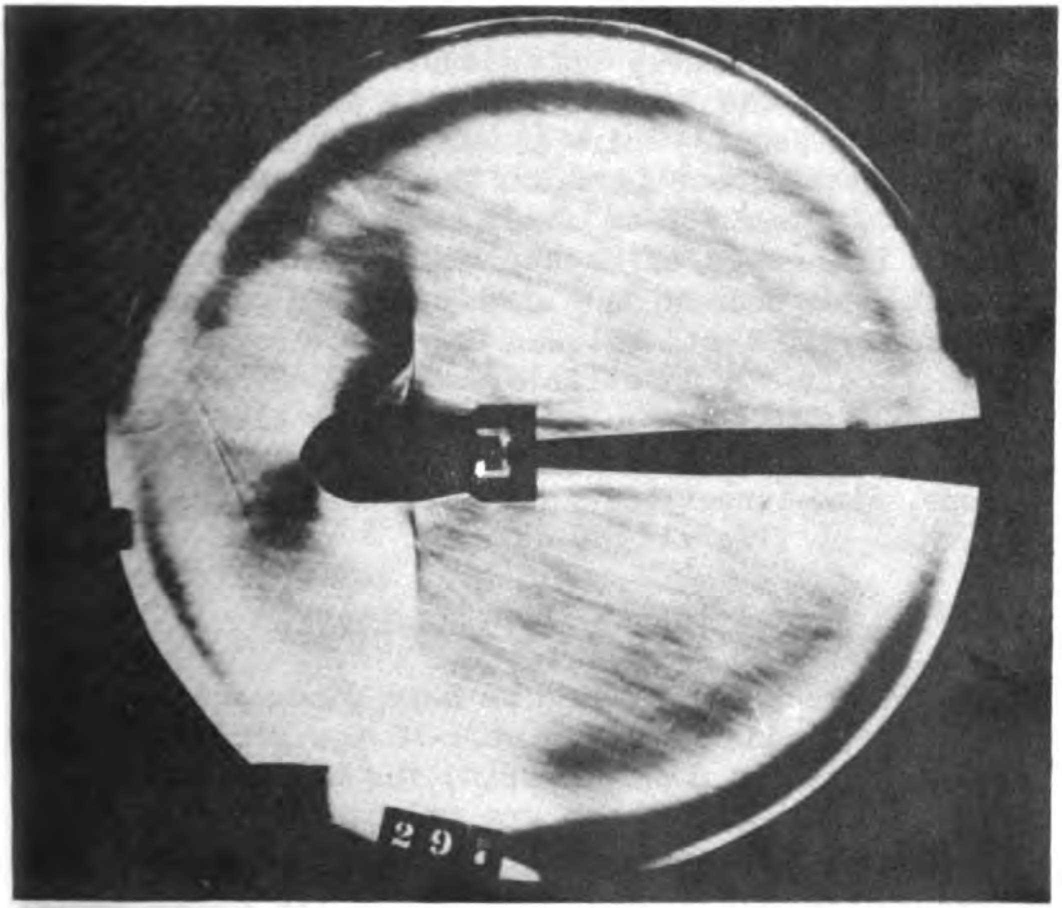


Figure III.4.2b

Schlieren Photograph of **Bomb Model at Mach** No. 0.85
Model MB 12, Air Flow Does Not Separate
Bomb Tunnel, Ballistic Research Laboratories

flight. In order to check on whether the model is too large it is only necessary to measure the pressure at intervals along the wall of the test section. If these measurements agree with the pressures obtained with the tunnel empty, it is reasonably certain that the model is sufficiently small.

It is particularly unfortunate that any model is too large if the tunnel speed is sufficiently near the speed of sound. There is always an interval in the range of Mach number in which the effect of the tunnel walls on the air stream about the model is not negligible. The range of Mach number in which tests are impossible depends on the size and shape of the model, but ordinarily tests between Mach numbers .85 and 1.1 are difficult and unreliable. It is unfortunate that the limiting speed of most of our bombs lies precisely in this blind range. For this reason, and for other reasons, wind tunnel tests on bombs must always be supplemented by free flight data.

5. Wind tunnel tests; dynamic tests.

In the preceding section we have discussed measurement of drag, lift and moment. If the projectile under consideration does not spin, the Magnus forces and couples vanish and there are only one force and one torque in addition. These are the damping moment, or the torque due to cross spin, and the force due to cross spin. All measurements of the damping torque, with the exception of a single series of tests in the Twenty Foot Wind Tunnel at Wright Field, have been made at the National Bureau of Standards. The only wind tunnel measurements of the force due to cross spin were made in the Wright Field series.

There are two methods which have been used to measure the damping moment. The first, the so-called log decrement method, was first used by H. L. Dryden and R. H. Heald. The second, the forced oscillation method, is due to G. B. Schubauer, also of the Bureau of Standards. We will discuss the log decrement method first.

The model is mounted upon a spindle which runs through the position corresponding to the position of the center of mass on the full scale bomb. The other end of the spindle is set in a bearing. The model is then turned from its equilibrium position and released, so that it oscillates and gradually comes to rest. A record is made of the angular position as a function of time. The differential equation governing the motion of the model is not difficult to derive.* Let I be the moment of inertia of the model and spindle about the spindle axis. Then $I\ddot{\delta}$ is the rate of change of angular momentum. The damping torque is $-\ d^4uK_H$, and the righting moment $d^3u^2K_M$. (Recall that K_M is negative for fin-stabilized projectiles.) The equation of motion is then

(1) $$I\ddot{\delta} = -\rho d^4uK_H\dot{\delta} + \rho d^3u^2K_M\delta.$$

For convenience let us define

(2) $$h = +\rho d^4uK_H,\quad m = -\rho d^3u^2K_M,$$

so that (1) becomes

(3) $$I\ddot{\delta} + h\dot{\delta} + m\delta = 0.$$

The quantities h and m are positive. The equation (3) is linear homogeneous with constant coefficients and its solution is of the form

$$\delta = a_1 \exp \lambda_1 t + a_2 \exp \lambda_2 t,$$

where a_1 and a_2 are constants and λ_1 and λ_2 are the

*In this equation account should be taken of friction. The friction can be assumed to be of the form (constant)(angular velocity) + (constant), and two constants can be determined experimentally. The analysis of the experiment in this case is more tedious but not essentially more difficult than the idealized analysis which we give.

solutions of

$$(4) \qquad I\lambda^2 + h\lambda + m = 0.$$

This can easily be checked by substitution. If both λ_1 and λ_2 are negative the bomb comes to rest without oscillation. The oscillation of a bomb in a water tunnel would generally be of this type. For most projectile models in a wind tunnel, $h^2 - 4mI$ is negative. The roots λ_1, λ_2 are then complex numbers, and the solution is oscillatory. We may write

$$\exp \lambda_1 t = \exp(-h + \sqrt{h^2 - 4mI})t/2I$$
$$= (\exp - ht/2I)(\cos \sqrt{4mI - h^2}t/2I$$
$$+ i \sin \sqrt{4mI - h^2}t/2I).$$

For most projectiles $4MI$ is very much larger than h^2, so that without loss in accuracy we may replace $\sqrt{4mI - h^2}/2I$ by $\sqrt{m/I}$. Since $\delta(t)$ is a real linear combination of $\exp \lambda_1 t$ and $\exp \lambda_2 t$ it must have the form

$$(5) \qquad \delta = c(\exp - ht/2I) \cos(\sqrt{m/I}\, t + \phi)$$

where c and ϕ are constant. Let us choose the origin of time at a point of maximum yaw. Then for $t = 0$, $\dot{\delta} = 0$, $\delta = \delta_0$ and (5) must have the form

$$(6) \quad \delta = \delta_0(\exp - ht/2I) \cos(\sqrt{m/I}\, t + \phi)/\cos \phi,$$

where $\tan \phi = -h/2\sqrt{mI}$.

Let P be the time from maximum to maximum. Clearly,

$$(7) \qquad \text{Period} = P = 2\pi\sqrt{I/m}.$$

If the period is measured from the record of yaw against time, then solving (7) for m and solving (2) for K_M gives

$$(8) \qquad K_M = -4\pi^2 I/\rho d^3 u^2 P^2.$$

In order to evaluate K_H from the yaw record let us compute δ at successive maxima and minima.

$$\delta_{1st\ max} = \delta_0,$$

$$\delta_{1st\ min} = \delta_0 \exp(-hP/4I),$$

$$\delta_{2nd\ max} = \delta_0 \exp(-hP/2I),$$

$$\delta_{2nd\ min} = \delta_0 \exp(-3hP/4I).$$

Let

$$A_1 = \delta_{1st\ max} - \delta_{1st\ min},$$

$$A_2 = \delta_{2nd\ max} - \delta_{2nd\ min}.$$

These are better quantities to measure than δ itself, since the zero of the yaw record is usually uncertain. Substituting from (9) gives directly

(10) $$A_1/A_2 = \exp hP/2I.$$

Solving this equation for h, and then for K_H gives the required formula,

(11) $$K_H = \frac{2I \log_e A_1/A_2}{P\rho d^4 u}.$$

The accuracy which can be obtained in determining K_H by this method is not too high. The difficulty lies mainly in the fact that the yaw damps out very rapidly, and the error in K_H is directly proportional to the percentage error in the measurement of angle. The combination of a limited number of observations, and limited accuracy on each, leads to fairly substantial uncertainties.

A somewhat more satisfactory method was devised by G.B. Schubauer. The principal purpose of his mechanism was to set up a forced oscillation which would permit measurement of a steady instead of a transient phenomenon. As before, the model is mounted on a spindle, free to rotate. The bottom of the spindle is then attached, through a torsional spring, to an oscillating disc, as shown in Figure 1. In addition to the torques shown in equation (1) there is then an additional torque, which will be proportional to the

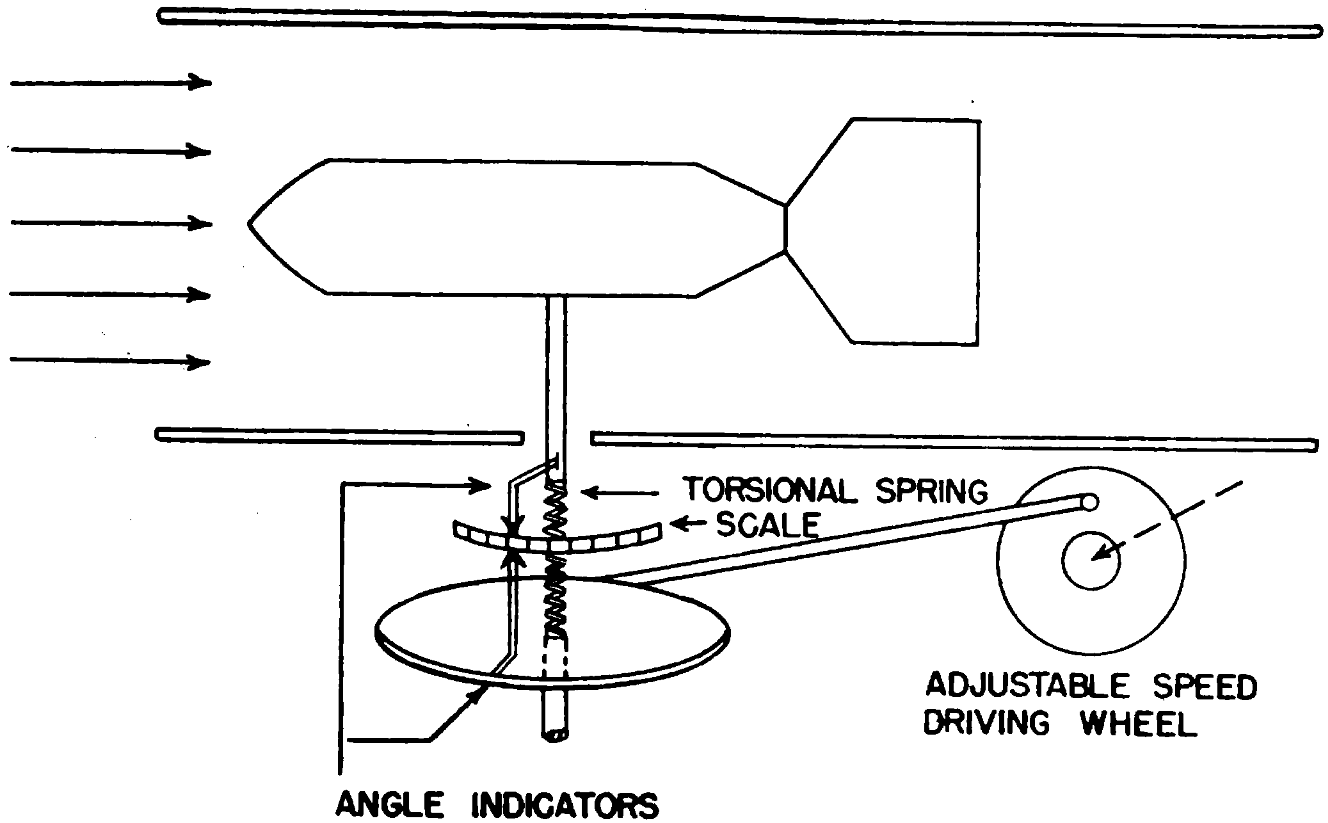


Figure III.5.1

Schematic Diagram of Bomb-oscillating Apparatus difference in angular position between the model and the driving disc. If the frequency of the driving oscillation is ν and its amplitude is α_0, the angular position is given by $\alpha_0 \cos \nu t$. The torque exerted on the model is then, according to Hooke's law, proportional to $\delta - \alpha_0 \cos \nu t$. The constant of proportionality will be denoted k, and is called the spring constant. The equation governing the motion is then obtained by adding this term to equation (1).

(12) $$I\ddot{\delta} = -h\dot{\delta} - m\delta + k(\delta - \alpha_0 \cos \nu t).$$

We assert that this equation has a solution of the form

(13) $$\delta = \delta_0 \cos (\nu t + \phi).$$

Here δ_0 is the amplitude of the model's oscillation, and ϕ is called the phase difference. Substitution of (13) in (12) leads to

(14) $$\begin{aligned} -I\delta_0 \nu^2 \cos (\nu t + \phi) \\ = h\delta_0 \nu \sin (\nu t + \phi) \\ + (k - m)\delta_0 \cos (\nu t + \phi) - k\alpha_0 \cos \nu t. \end{aligned}$$

In order to have this equation satisfied identically we replace $\cos \nu t$ by

$$\begin{aligned} \cos [(\nu t + \phi) - \phi] \\ = \cos (\nu t + \phi) \cos \phi + \sin (\nu t + \phi) \sin \phi \end{aligned}$$

and equate the coefficients of $\cos (\nu t + \phi)$ and $\sin (\nu t + \phi)$.

(15) $$\begin{aligned} -I\delta_0 \nu^2 = \delta_0(k - m) - k\alpha_0 \cos \phi, \\ 0 = h\nu\delta_0 - k\alpha_0 \sin \phi. \end{aligned}$$

These equations may readily be solved for δ_0 and ϕ which shows that there does exist a solution of the form (13). In other words, <u>for any frequency</u> ν <u>and any amplitude</u> α_0 <u>of the drive system there is a possible solution</u> $\delta = \delta_0 \cos (\nu t + \phi)$. For a fixed ν and α_0 there are, of course, as many solutions as there are sets of initial conditions. However, the difference between any two solutions of (12) is a solution of (1), and this difference is therefore damped out rapidly.

The experimental method based on this analysis is the following. The frequency ν of the driving system is varied, until a phase difference ϕ of 90° is observed.* When this phase difference is obtained the amplitude δ_0 of bombs oscillation and the frequency ν are measured. The second of equations (15), setting $\phi = 90^\circ$, gives

$$h = k\alpha_0/\nu\,\delta_0, \tag{16}$$

and hence

$$K_H = k\alpha_0/\nu\,\delta_0 \rho u d^4. \tag{17}$$

Since the spring constant k and the amplitude α_0 of the drive are known, the evaluation of K_H is accomplished.

This analysis is based on a linear theory. However, by varying the amplitude of the drive, the amplitude of the model's oscillation can be governed and an idea of the mean value of K_H as a function of yaw results. As might be expected, there is considerable variation with shape of model.

The determination of force due to angular velocity requires no new type of experiment. Suppose that K_H

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*In Schubauer's experiment indicators were linked to the model and the drive, with a 90° phase difference built into the linkage. The desired value of the frequency then caused the indicators to oscillate together. In the Wright Field experiment an apparatus designed by Dr. A.C. Charters flashed a light at the extremes of the oscillation of the drive. This "stopped" the indicator of the model, revealing directly the phase difference.

is determined for a model and that the spindle is then shifted r calibers forward and the experiment repeated. If the resulting value of damping coefficient is K_H^*, according to (II.5.7)

(18) $$K_H^* = K_H - r(K_S + K_M) + r^2 K_N.$$

Hence,

(19) $$K_S = (K_H - K_H^* - rK_M + r^2 K_N)/r.$$

This method of determination of K_S was used in the Wright Field experiments. Due to uncertainties in K_H, the determination of K_S was probably subject to uncertainties of the order of 40 per cent.

The value of measurements of damping has been demonstrated by empirical stability tests. It should be noted that the present methods, though giving results somewhat less accurate than is desirable, are capable of considerable improvement through further development of the instrumentation.

Chapter IV

THE NORMAL EQUATIONS

1. The normal equations of the trajectory.

In Section 7 of the second chapter we have derived equations whose solution would describe the motion of a projectile with accuracy entirely adequate for the purposes of ballistics. However, these equations are far too complex to be manageable in the form given. They involve the density of the air at various places and this is a function of position which changes from day to day. They involve the winds, which also vary with time. They involve a large number of aerodynamic functions, varying from one model of projectile to another; and they also involve the initial conditions of motion. For any given values of the functions entering the equations it would be possible, though difficult, to solve the equations with any desired degree of accuracy, by the methods to be explained in Chapter VI. But this is evidently impossible in the field. The trajectories would have to be computed and tabulated in advance. This, on the other hand, would require anticipating every possible arrangement of winds, densities, etc., a manifestly impossible undertaking.

One feasible way of extricating ourselves from this difficulty is to split the problem into two parts. In the first stage, the problem is intentionally oversimplified. All the small terms in the equations of motion are omitted. The density is assumed to be always the same function of y, the vertical coordinate, alone, and likewise the temperature. The wind is always the same constant value. For each one of a selected few drag coefficient functions K_D, it is now possible to compute a collection of trajectories corresponding to the remaining parameters in the equations of motion, namely (d^2/m) and the initial speed and angle. From these the pertinent data, that is the range, time of flight, angle of fall and striking velocity can be found and tabulated against the parameters. In particular, for a given shell or bomb the initial conditions needed to produce a hit on a target in a known position can be found; in other words, a firing or bombing table can be prepared for the projectile, for use in the field. This however still leaves the second part of the problem, which is to devise usable methods for calculating the effects of the small forces which were ignored and for differences between the actual conditions of temperature, density, etc., prevailing at the time of firing and the conditions which were assumed to hold when the trajectories were computed. This latter part of the problem will form the subject matter of Chapters VII to IX. This chapter and the next two will be devoted to a study of the first part.

Consider first the question of selecting a standard density function. There are two aims that we must keep in mind. First, the function must be such that on any given day the actual density is unlikely to differ much from our chosen standard. The reason for this is that the method which will be used in calculating corrections for departures from standard has its greatest percentage accuracy when the departures are small, and can show greater errors when the departures become large. The second aim is an obvious one; the function should be chosen so as to be convenient to use in computation. Exactly similar considerations apply to the selection of a standard temperature. It does not matter to the ballistician that his chosen standards may not be compatible with the physics of a stable atmosphere of the composition of the earth's atmosphere. If he wishes, he may think of his standard atmosphere as a temporary structure, lasting only long enough for the projectile to traverse it and then going over into turbulent motion. But this is really not essential. The purpose of the selection of the standard structure is to give a mathematical point of departure, from which the desired ballistic data can be found by applying corrections appropriate to the structure actually existing at some particular moment.

Since most artillery trajectories start from points not vastly higher than sea-level, it is reasonable to assume as a standard condition that the trajectory starts from the point (0, 0, 0) and that this point is at sea-level. The standard air density at sea-level will be denoted by ρ^*. In U.S. ballistics this is taken to be .075126 pounds per cubic foot. It is also assumed that the density is a function of y alone, and its ratio to ρ^* is denoted by H(y), so that

$$\rho(y) = \rho^* H(y). \tag{1}$$

(Cases in which the start of the trajectory is not near sea-level will be discussed in the next section.) Two different standards for H(y) are frequently used. In the case of trajectories whose highest and lowest points have altitudes differing only by a few hundred feet, the assumption is usually made that H(y) is constantly equal to 1, so that $\rho(y)$ is constantly equal to ρ^*. For trajectories extending through a considerable depth of atmosphere the standard assumption is

$$H(y) = e^{-hy}, \tag{2}$$

where the quantity h has dimensions $[L]^{-1}$. The standard value of h in the United States is .000045 $\log_{10} e$ if y is in meters; that is

$$h = .0001036 \text{ m}^{-1} = .00003158 \text{ ft}^{-1}. \tag{3}$$

Since the exponential law loses its accuracy in the stratosphere, other standard density ratios have been discussed which depart from the exponential at large values of y. However, up to the present little if any ballistic computation in the United States has been based on laws other than the constant or the exponential H(y).

Before we discuss the choice of a standard temperature law, it might be advisable to allay the possible puzzlement of a reader who has not noticed any temperature mentioned in the equations of motion. The drag coefficient K_D is a function of Mach number, which is the ratio of the speed of the projectile with respect to the air to the velocity of sound in the undisturbed air near the projectile. It is accurate enough for our purposes to suppose that the velocity of sound in air is proportional to the square root of the absolute temperature of the air. As a matter of fact, we could appropriately say that we are choosing not a standard law for the variation of temperature with height, but a law for the velocity of sound as a function of height. We shall continue to give lip service to the traditional statement that we are choosing a temperature standard; but in case the distinction should ever become necessary, we should keep in mind that it is really a standard selected velocity of sound that we actually make use of.

For trajectories whose highest and lowest points have little difference in altitude it is reasonable to suppose that the temperature is constant, and this is usually done. It was also done at the Ballistic Research Laboratory at Aberdeen Proving Ground when the trajectories were prepared which form the basis for the bomb ballistic reduction tables, used in preparing the bombing tables for the war which ended in 1945. There was a valid computational reason for this choice, to be explained in the next section. However, for trajectories extending through a great depth of atmosphere it is customary to assume a temperature law more nearly representing recorded temperatures. The law now taken as standard at Aberdeen is that the absolute temperature in degrees Kelvin (that is, absolute Centigrade) is a function of y alone, and is

(4) $$\Theta = 288e^{-2a_1 y},$$

where

$$(5) \qquad a_1 = 2h/21,$$

h having been defined in (3). This is a striking example of selection in accordance with the two aims discussed in the third paragraph of this section. The law selected is serviceable in that on any given day the temperature at height y is not likely to depart very greatly from this standard; this is especially true at great altitudes. And as for the second aim, the fraction 2/21 in (5) was chosen because this is an easy gear ratio to obtain with the gears available for use with the differential analyzer at the Ballistic Research Laboratory.

Equation (5) leads to the law for the speed of sound at height y

$$(6) \qquad u_s(y) = u_s(0)e^{-a_1 y},$$

where u_s is the speed of sound at height y and the speed of sound at height 0 is taken to be that corresponding to a temperature of 15° Centigrade (= 288° Kelvin = 59° Fahrenheit). This is approximately 1120 feet per second.

In the United States the ballistic standard gravitational acceleration is taken to be

$$(7) \qquad g = 9.8 \text{ m/sec}^2$$

exactly.

Our final simplifying assumption is that K_D, which is a function of Mach number, Reynolds number and shape parameters (see Sections I.15 and II.2), is independent of Reynolds number and for a given shape of projectile depends on Mach number alone. This is certainly untrue if widely different Reynolds numbers are compared. Experiments have shown a ten per cent discrepancy between the values of K_D at the same Mach number for two similar projectiles whose diameters were approximately six inches and one-half inch respectively. But little precise information is now available, and we are not now in a position to improve on the assumption that K_D depends on Mach number and shape alone. After all these simplifications, our equations of motion (II.8.2) take the form

$$\ddot{x} = -(d^2/m)\rho(y)\, K_D(v/u_s(y))\, v\dot{x},$$

(8) $$\ddot{y} = -(d^2/m)\rho(y)\, K_D(v/u_s(y))\, v\dot{y} - g,$$

$$\ddot{z} = -(d^2/m)\rho(y)\, K_D(v/u_s(y))\, v\dot{z},$$

where

(9) $$v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}.$$

It is interesting to note to what extent we have committed ourselves by choosing this form for the equations. We have discarded all the small terms, such as the forces perpendicular to the trajectory and the change in gravity and the Coriolis forces, and have retained only drag and gravity. We have disregarded any possible dependence of drag on Reynolds number; and we have assumed that the air is stationary with respect to the axes. We have not needed to assume that the axes were stationary with respect to the earth. They could be in uniform translation with respect to the earth, provided that the air moves with them. We have not yet, in (8), committed ourselves to any particular standard density or temperature laws, such as (2) or (4). So if the projectile is moving through an air mass which is itself moving uniformly with respect to the earth, and we know the density and temperature distribution in the air mass, we could solve (8) to obtain an approximate trajectory. It would still be in error to the extent of the effects of the small forces and the effect of the dependence of K_D on Reynolds number.

In the nineteenth century it was not understood that temperature had any effect, and it was consequently believed that K_D depended on v alone. Thus for motion through air at standard density the first of equations (8) would have been mistakenly written in the form

$$\ddot{x} = -(d^2/m)[\rho^* K_D(v)v]\dot{x},$$

and the others in a corresponding way. The factor d^2/m was called the ballistic coefficient on the European continent; in the English-speaking countries that name was given to m/d^2. The expression in square brackets was called $G(v)$, and several tables prepared. We shall now show how we can make practical use of these tables without involving ourselves in the fallacy that K_D is a function of v. This is a necessary task because it is traditional in ballistics to transform equations (8) into a form involving G. There are two methods by which we can correctly bring (8) into traditional form; these methods are formally different, but produce the same sets of numerical values when the units used in the computations are those customarily used in ballistics in the United States. Both methods are frequently referred to, so it is desirable to explain both of them.

In the first approach, we recall that $G(v)$ is what the expression $\rho K_D v$ would be if density and temperature were standard. Sometimes the product ρK_D is also used, under the name $B(v)$. Accordingly, we define the ballistic coefficient C by the equation

(10) $$C = m/d^2,$$

and we define B, G to be what ρK_D and $\rho K_D v$ would be at velocity v provided temperature and density were both standard:

(11) $$B(v) = \rho^* K_D(v/u_s(0)),$$

(12) $$G(v) = \rho^* K_D(v/u_s(0))v.$$

These are named the retardation coefficient and the drag function respectively. For compactness of notation we also define the relative sound velocity function to be

(13) $$a = \sqrt{\Theta/\Theta_s(0)},$$

where Θ is absolute temperature and $\Theta_s(0)$ is standard absolute temperature at sea-level (ordinarily taken to be 288°K). It is obvious that the velocity of sound is

$$u_s = au_s(0). \tag{14}$$

Hence when we have selected a standard temperature law, standard temperature at altitude y being $\Theta_s(y)$, (13) furnishes us with a standard relative sound velocity law, related to $u_s(y)$ by equations (14). By (11) and (14),

$$\begin{aligned} \rho^* K_D(v/u_s(y)) &= \rho^* K_D([v/u_s(0)]/a) \\ &= B(v/a), \end{aligned} \tag{15}$$

whence by (12),

$$\begin{aligned} \rho^* K_D(v/u_s(y))v &= [v/a]B(v/a)a \\ &= aG(v/a). \end{aligned} \tag{16}$$

With the help of these equations, (10) and (11), equations (8) take the form

$$\begin{aligned} \ddot{x} &= -(H/C)\ B(v/a)\ v\dot{x} \\ &= -(H/C)\ G(v/a)\ a\dot{x}, \\ \ddot{y} &= -(H/C)\ B(v/a)\ v\dot{y} - g \\ &= -(H/C)\ G(v/a)\ a\dot{y} - g, \\ \ddot{z} &= -(H/C)\ B(v/a)\ v\dot{z} \\ &= -(H/C)\ G(v/a)\ a\dot{z}. \end{aligned} \tag{17}$$

The second form, involving G, has been much more frequently used than the first, involving B. As a rule, it is somewhat more adapted to numerical work. However, many trajectories have been computed on the differential analyzer with the first form of the equations. It is evident that the choice between the two forms is purely a matter of convenience. When computing machines (or logarithmic tables) are used, it is more convenient to tabulate B or G against $v^2/100$

rather than against v. The reason is that in the process of numerical integration the velocity components $\dot{x}$ and $\dot{y}$ are found, and it is easier to compute v^2, which is the sum of their squares, than to compute v, which would require finding the square root of v^2.

With the definitions just given, it is clear that C has dimensions $[M]/[L]^2$, H and a are dimensionless, while B has dimensions $[M]/[L]^3$ and G has dimensions $[M]/[L]^2[T]$. Equations (17) are valid in any consistent set of units. Unfortunately, they are not so used, and some care is needed to avoid being lost in a maze of assorted units. Except in some of the most recent tables prepared at the Aberdeen Proving Ground, the units used in (10) are the pound and the inch; standard air density at sea-level is .075126 pounds per cubic foot; and v is expressed in meters per second, so that either x, y and z must be expressed in meters or else a conversion factor must be applied in computing v^2 in order to find the value of G (or of B, in case the first form of (17) is used).

The situation with regard to the units is more easily seen through if we adopt the second of the two sets of definitions mentioned above. In this approach, a <u>standard projectile</u> is defined. This standard projectile is geometrically similar to the one whose trajectory is being discussed, but it is supposed to have a standard mass and a standard diameter. The standard projectile for U.S. ballistics is defined to have a diameter of one inch and a mass of one pound. For each shape of projectile there will of course be a standard projectile, but all the different standard projectiles have the same diameter and mass. We denote the standard diameter by d^* and the standard mass by m^*, and we define

$$(18) \qquad C = \frac{m/d^2}{m^*/d^{*2}},$$

$$(19) \qquad B(v) = (d^{*2}/m^*)\rho^* K_D(v/u_s(0)),$$
$$(20) \qquad G(v) = vB(v).$$

With these definitions we again obtain equations (17) for the motion of the projectile. Now however, C is dimensionless, B has dimensions $[L]^{-1}$ and G has dimensions $[T]^{-1}$. If we express all lengths in feet, all masses in pounds and all times in seconds, the first factor in the right member of (19) is 1/144 and the second is .075126 when the standard U.S. values are chosen, so that (19) becomes

$$(21) \qquad B(v) = K_D(v/1120)/1916.7,$$

while (20) becomes

$$(22) \qquad G(v) = vK_D(v/1120)/1916.7.$$

The constants 1120 and 1916.7 in these equations are values in the units chosen of the quantities $u_s(0)$ and $(m^*/d^{*2})/\rho^*$, which have dimensions $[L]/[T]$ and $[L]$ respectively. Thus if the meter is to be used as the unit of length, equation (22) must be replaced by

$$(23) \qquad G(v) = vK_D(v/341.4)/583.95,$$

and (21) must be similarly amended. Equations (22) and (23), though they appear different, must yield the same value for G corresponding to any velocity of the projectile, provided that the velocity is expressed in feet per second in (22) and in meters per second in (23). This is evident from the fact that G has dimensions $[T]^{-1}$. Thus if a table of values of G has been prepared with argument in meters per second, all that is needed in order to obtain a table of G with argument in feet per second is to convert the argument; the values of G will not need change.

Equations (17) can be still more compactly written by introducing the symbol

$$(24) \qquad E = aH(y)G(v/a)/C = H(y)vB(v/a)/C.$$

Then they become

$$
(25) \qquad \begin{aligned} \ddot{x} &= - E\dot{x}, \\ \ddot{y} &= - E\dot{y} - g, \\ \ddot{z} &= - E\dot{z}. \end{aligned}
$$

The only excuse for introducing so trivial a variant of (17) is that the notation (24) is traditional and is convenient in the computation of trajectories.

Although the use of the ballistic coefficient C has been traditional for many years in the United States, it is not a particularly convenient parameter. In many instances it has proved convenient to use, instead of C, its reciprocal

$$
(26) \qquad \gamma = 1/C.
$$

For example, at a fixed altitude and initial velocity the range of a bomb is fairly nearly a linear function of γ, except for the very lightest models of bombs. This makes interpolation in tables easy. In terms of the reciprocal ballistic coefficient, equation (24) becomes

$$
(27) \qquad E = \gamma H(y)aG(v/a) = \gamma H(y)vB(v/a),
$$

equations (25) remaining valid.

If the functions H(y), a(y) and G(v) (or B(v)) have continuous derivatives, as we shall always assume, equations (25) have exactly one solution for each set of initial values $x(0)$, $y(0)$, $z(0)$, $\dot{x}(0)$, $\dot{y}(0)$, $\dot{z}(0)$. In particular, if $z(0) = \dot{z}(0) = 0$, and we solve the first two of equations (25) with $\dot{z} = 0$ to obtain x and y and then set z identically equal to zero, we obtain a solution. This is then the only solution with the given initial values. Henceforth, in discussing the normal equations, we shall always suppose that the coordinate system has been so chosen that the initial velocity is in the (x, y)-plane. Then z and $\dot{z}$ vanish at time t = 0, and therefore remain identically equal to zero. Thus if we wish we may retain only the first two of the normal equations, keeping in mind that the trajectory will be entirely in the plane $z = 0$. This justifies a statement made in Section 8 of Chapter II. However, when we come to compute the effects of the small forces which we have ignored, the effects of cross winds, etc., the third equation must rejoin the other two.

2. Applications of the normal equations to bomb and aircraft artillery trajectories.

Although it is not mathematically necessary, it is evidently convenient to consider the trajectory as always starting from the origin and to take the time as 0 at the time of starting, so that the initial conditions

$$(1) \qquad x(0) = y(0) = z(0) = 0$$

will be taken as standard. This is appropriate in coast and field artillery problems, in which the start of the trajectory is usually from some point not greatly above sea-level. However, in problems concerning firing and bombing from aircraft we meet a difficulty. The air density at the origin, in the normal equations, is taken to be standard air density at sea-level. But bomb trajectories may start from as much as six miles above sea-level, where the air density is roughly a third as great as at sea-level. This clearly is not the sort of difference that can be corrected for by means of a small correction term. It is at this point that the exponential density law proves extremely convenient.

Suppose that the origin of coordinates is at an altitude Y above sea-level. The point (x, y, z) will have altitude $Y + y$ above sea-level, and if the standard density law has been well chosen the air density at this point will not differ greatly from $\rho^* H(Y + y)$, where as before ρ^* denotes standard air density at sea-level. So a trajectory computed with air density

$\rho(y) = \rho^* H(Y + y)$ will be nearly enough accurate to allow making a small correction for the departure of conditions at time of firing from the assumed conditions. In the same way, the ratio of the velocity of sound at the point (x, y, z) to the standard velocity of sound at sea-level is approximately $a(Y + y)$. Thus the equations which we wish to solve are obtained from (1.25) by replacing $H(y)$ by $H(Y + y)$ and $a(y)$ by $a(Y + y)$, and so have the form

$$\begin{aligned} \ddot{x} &= -\gamma H(Y+y)\, G(v/a(Y+y))\, a(Y+y)\, \dot{x}, \\ \ddot{y} &= -\gamma H(Y+y)\, G(v/a(Y+y))\, a(Y+y)\, \dot{y} - g, \qquad (2) \\ \ddot{z} &= -\gamma H(Y+y)\, G(v/a(Y+y))\, a(Y+y)\, \dot{z}. \end{aligned}$$

If we select the standard density law (1.2) we find

$$H(Y - y) = e^{-hY} e^{-hy} = e^{-hY} H(y). \qquad (3)$$

Substituting this in (2) yields

$$\begin{aligned} \ddot{x} &= -[\gamma e^{-hY}]\, H(y)\, G(v/a(Y+y))\, a(Y+y)\, \dot{x}, \\ \ddot{y} &= -[\gamma e^{-hY}]\, H(y)\, G(v/a(Y+y))\, a(Y+y)\, \dot{y} - g, \qquad (4) \\ \ddot{z} &= -[\gamma e^{-hY}]\, H(y)\, G(v/a(Y+y))\, a(Y+Y)\, \dot{z}. \end{aligned}$$

These are not identical with (1.25). But if we are willing to use the standard temperature law $a = 1$, thus assuming the same temperature at all altitudes, equations (4) differ from (1.25) only in that the constant γ in (1.25) is replaced by the constant $[\gamma e^{-hY}]$ in (4). In bombing ballistics it is customary to define the summital ballistic coefficient C_s by the equation

$$C_s = Ce^{hY}, \qquad (5)$$

so that its reciprocal, the <u>reciprocal summital ballistic coefficient</u> or summital γ for short, is

$$\gamma_s = \gamma e^{-hY}. \qquad (6)$$

Thus equations (4) differ from (1.25) only in that is replaced by γ_s.

This remark has great importance in connection with bomb ballistics. Let us suppose that we are undertaking the task of preparing a table showing the range and time of flight of a bomb launched horizontally, as a function of its initial velocity, its reciprocal ballistic coefficient and the altitude of release. We would anticipate having to prepare a collection of trajectories for several values of each of these variables, spaced closely enough to permit trustworthy interpolation. But if we are willing to assume standard sea-level temperature throughout the trajectory, so that a is constantly equal to 1, we need only prepare a collection of trajectories for a set of values of γ and initial velocity, which constitutes a great reduction in the number of trajectories to be computed. Now, given any values of γ, altitude Y of release and velocity v_0 at release, we first compute γ_s by (6). Then with this γ_s and v_0 we find the time at which the bomb has fallen a distance Y and we find the x-coordinate (which is the range) of the bomb at this time. The work involved in computing the trajectories is still considerable; but without this simplification it would hardly be within the bounds of possibility for human computers equipped with ordinary computing machines.

In computing bomb trajectories one further small modification of equations (4) is customary. Since the trajectory is entirely below the point of release, the y-coordinate in the system we have been using would always be negative. To avoid an incessant repetition of minus signs, we change the coordinate system by choosing downwards as the positive y-direction, x and z being unaltered. If we also adopt the standard

$$a = 1,$$

equations (4) transform into

$$\ddot{x} = -\gamma_s e^{hy} G(v) \dot{x},$$

$$(7) \qquad \ddot{y} = -\gamma_s e^{hy} G(v) \dot{y} + g,$$

$$\ddot{z} = -\gamma_s e^{hy} G(v) \dot{z}.$$

As previously remarked, the third of these is superfluous if we assume $z(0) = \dot{z}(0) = 0$. The first two are the equations whose solution forms the basis for bomb ballistic tables.

3. Ballistic tables and firing tables.

Since considerable confusion has been observed in discussions of ballistic tables and firing tables, it is worth while to devote a few lines to distinguishing the two. A ballistic table consists essentially of a summary of the pertinent information gained from a collection of solutions of the normal equations. A collection of "normal trajectories," that is solutions of the normal equations (1.25) or (2.4), is computed. From these the range and time of flight corresponding to various initial conditions and various values of Υ are obtained by interpolation and are tabulated against the initial conditions and reciprocal ballistic coefficient as arguments. This table is the heart of the ballistic table. In addition it may contain tables of striking velocity and angle of fall against initial conditions and Υ, and if quite complete should also contain the means of computing the effects of departures from the standard conditions assumed in the normal equations. Thus a ballistic table is computed for a set of conditions, the "ballistic table conditions," which assume a flat non-rotating earth with one fixed value of g, no forces acting on the projectile other than drag and gravity, no wind, and a standard distribution of density and temperature.

A firing table expresses primarily the relation between angle of departure and the range of a particular projectile, launched in a specified way. For artillery the table will relate to a particular combination of projectile and gun, with a specified powder charge. For bombing the table will relate to a particular bomb, giving range and time of flight in terms of speed of launching and altitude of release. For aircraft rockets the table will refer to a particular rocket launched from a particular airplane with a particular propellant charge. In addition to the primary table of range and time against angle of departure, etc., it will also contain tables of corrections to be applied for that particular weapon when the wind is known and not zero, when the temperature is not standard, when the projectile is above or below standard weight, etc. The firing or bombing table is made for the use of the man using the weapon, and its basic purpose is to produce a hit on a target with the particular weapon. The ballistic table is made for the use of the man who prepares the firing table, and its basic purpose is to permit him to prepare an adequately accurate firing table fast enough to meet the needs of the using service.

4. Errors in firing tables produced by erroneous choice of drag function.

People who have some knowledge of the order of accuracy of the determination of drag functions, but who are inexperienced in the use of ballistic tables, have frequently expressed astonishment or amusement about the number of significant places carried in such tables. It seems obvious, at first sight, that if the drag function is not known to better than one per cent, there is no sense whatever in using it to compute a table in which the entries are given to five significant figures; the ranges could not be determined more accurately than the drag function justifies. This would in fact be correct if the ballistic table were used in a highly stupid manner. This naive method would be to measure the diameter d and the mass m of the projectile and then to compute $\gamma = d^2/m$. Now corresponding to any initial velocity

$$v_0 = \sqrt{\dot{x}(0)^2 + \dot{y}(0)^2} \tag{1}$$

and to any angle of departure

$$\theta_0 = \text{arc tan } \dot{y}(0)/\dot{x}(0) \tag{2}$$

we look in the ballistic tables to find the range and the time of flight. These can almost be guaranteed to be too much in error to be of any use.

In actual practice, the ballistic tables are not used in this way. Suppose that a projectile is being tested whose drag function $G(v)$ is not the same as the drag function $G^*(v)$ on which the ballistic tables were based, but happens to be a constant multiple of it, say

$$(3) \qquad G(v) = (1/i)G^*(v)$$

In this case the number i is called the <u>form factor</u> of the projectile with respect to the drag function G^*. In order to avoid complicating the ideas involved, let us imagine that the projectile is fired on a day on which all the conditions are the same as ballistic table conditions or differ so little from them as to produce no significant effect. The initial velocity v_0 and the angle of departure θ_0 are measured, and also the range X. The trajectory of the projectile will not differ significantly from the solution of the normal equations

$$(4) \qquad \ddot{x} = -[d^2/m]\, H(y)\, G(v/a)\, a\dot{x},$$

etc. But by (3) these are identical with

$$(5) \qquad \ddot{x} = -[id^2/m]\, H(y)\, G^*(v/a)\, a\dot{x},$$

etc. That is the trajectory coincides with the normal trajectory based on G^* with

$$(6) \qquad \gamma = id^2/m;$$

for the three equations of which (5) is the first are exactly the same as (1.25) with the definition (6) for γ. Now, if we are quite certain that (3) is correct, we are finished. For the range and the time of flight of the projectile corresponding to any initial conditions would be those in the ballistic table corresponding to the same initial conditions and to reciprocal ballistic coefficient defined by (6). The firing determined the range and time of flight corresponding to one set of initial conditions, and by using the tables we can find the γ which with the initial conditions produces the observed range. This determines i by (6). With the same γ we can enter the ballistic table to find the range and time of flight corresponding to any other initial conditions. This shows that it is possible for a table based on a drag function everywhere in error by a factor 1/i to yield exactly correct results, if properly applied.

If the shape of the projectile being fired differs at all from that of the projectile whose drag function is G^*, it would be most unlikely that equation (3) would hold exactly. It would be far more probable that the ratio $G(v)/G^*(v)$ would be a smooth and slowly varying function of the velocity v. Suppose again that the projectile is fired on a day on which all conditions are normal, and that the initial velocity v_0 and the angle of departure θ_0 are measured. Along the trajectory the velocity will vary between certain limits, and between those limits the ratio $G(v)/G^*(v)$ will vary between a maximum value which we denote by $1/i_1$ and a minimum value which we denote by $1/i_2$. Then along the whole trajectory we have

$$(7) \qquad (1/i_2)G^* \leq G \leq (1/i_1)G^*.$$

If we are willing to accept the plausible statement that when two projectiles have the same mass and diameter but one has at all velocities a greater drag function than the other, then when they are fired with the same initial conditions the one with the smaller drag function will have the greater range, it is easy to see that equation (7) implies that the range of the projectile will lie between the ranges corresponding to the reciprocal ballistic coefficients $i_1 d^2/m$ and $i_2 d^2/m$. It will therefore correspond to a reciprocal ballistic coefficient

$$(8) \qquad \gamma_x = i_x d^2/m,$$

where i_x is between i_1 and i_2. (The subscript x connotes that the form factor was deduced from the range.)

The <u>form factor</u> i_x is a sort of mean value of the ratio G^*/G. It is a rather complicated sort of weighted mean; the value of the ratio at the beginning of the trajectory is evidently more important than its value near the end, since the deceleration of the projectile near the beginning of the trajectory affects all its subsequent flight while the deceleration near the end has not time to produce much of an effect on the position of the projectile. At this moment we do not care very much just how this mean can be found. But it is reasonable to suppose that it is an integral mean, weighted by some sort of function depending on the velocity and the air density. The point of greatest importance at this instant is that such a mean may be expected to vary in a smooth slow way if any of the initial conditions, such as angle of departure, are varied. Thus if range firings are conducted at several angles of departure, with ranges consisting of the greatest range, a short range, and two or three points between them, each of the firings will yield a value of i_x, or of γ_x. These values will not all be equal, but we may expect them to vary slightly and smoothly between the points at which they are determined. Even if we are unable to determine any formal law governing the change of i_x with range, when we have found the values of i_x corresponding to four or five different values of the range and discovered that the values are nearly equal and lie on a smooth curve, we are entitled to feel some degree of confidence that for values of the range between those used in the experimental firings the value of i_x can be accurately estimated by interpolating between the values experimentally determined. Even though the ratio G^*/G varies from 1 by quite a considerable amount, this process can yield a firing table of adequate accuracy.

The time of flight can be treated in the same way as the range. At each range at which an experimental firing is conducted, there will be found a value of γ such that the time of flight found in the ballistic tables corresponding to the given initial conditions and this γ is the same as the experimentally determined time of flight. The reciprocal ballistic coefficient thus determined will be denoted by γ_t. As long as the ratio of G to G^* is variable, there is no reason to expect that γ_t will be the same as γ_x.

If we regard the technique described above from an abstract point of view, it will appear that the ballistic tables constitute a means of effecting a change of variables from initial conditions and range (v_0, θ_0, X) to initial conditions and reciprocal ballistic coefficient (v_0, θ_0, γ). Unless the drag function is very ill-chosen, this transformation will have the property that for a given projectile, if $X(v_0, \theta_0)$ is the range corresponding to the initial conditions, the points (v_0, θ_0, γ) into which the sets (v_0, θ_0, $X(v_0, \theta_0)$) transform have the values of γ which vary slowly and smoothly with v_0 and θ_0. From this point of view (which at least temporarily ignores the question of corrections for departures from standard) it would not matter how the table is derived, or whether any trajectories or any drag function had anything to do with it. The ballistic table is merely a mathematical aid to interpolation between experimentally determined ranges. But to be useful such a table should be given to more decimal places than the experiments yield, and should be smooth to facilitate interpolation. This is the explanation for the five significant figures in a table which is based on a drag function whose experimental determination may have been several per cent in error.

A discussion like that just completed applies to the use of approximate methods of solution of the normal equations. In the next chapter we shall study the Siacci method, an important feature of which is the simplification of the equations of motion by replacing v by some constant multiple of $\dot{x}$ to permit a rapid approximate solution. If this method were used with $\gamma = d^2/m$, or even with the γ defined by (6) and determined with the help of the ballistic tables from an experimental firing, the ranges deduced would be grossly in error. Let us momentarily give the name "Siacci range" to the range computed for given reciprocal ballistic coefficient and initial conditions by use of the Siacci approximations. The firing having been done, we find the γ such that the "Siacci range" corresponding to this γ and the given initial conditions is the same as the experimentally determined range. This is done at several ranges; the values of γ thus found may be expected to form a slowly and smoothly varying set, and the values of γ corresponding to intermediate values of the range can be estimated by interpolation. This being done, the angle of departure needed to attain the given range with the given initial velocity is found by the same approximation method, and may be expected to be reasonably accurate.

Now it becomes necessary to put in a word of caution. By now the reader may have the feeling that there is no advantage in using the correct drag function or in using accurate methods of integration of the equations of motion. This is not the case. For one thing, if the drag function used is seriously in error the values of γ_x found by the firings at different ranges will vary more than they would if the drag function were nearly correct. The more variable the values of γ_x found by the firings, the less confidence can we feel in the results of interpolation between them. In other words, a badly chosen drag function may yield a fairly good firing table, but a well-chosen drag function will yield a better one. For another thing, there probably never was and never will be a day on which all conditions were the same as ballistic table conditions. Thus it is not feasible merely to measure the range and the time of flight of the projectile. We must also be able to correct for the departures from ballistic table conditions before we can use the tables to find γ_x or γ_t. Also we must be able to furnish correction tables in the firing tables so that the user can correct for the conditions at the moment of use of the weapon. Now it will be seen in Chapter VIII that these corrections depend on the drag function used. If the drag function is not of the right shape, the corrections of the experimental results for departures from standard conditions will be incorrect, and so the experimental evidence will be misinterpreted. Also the using service will be furnished with erroneous tables of corrections for the conditions of the time of firing. In fact, it is not necessary to await the discussion of small corrections in general before seeing the truth of this statement. One interesting example will illustrate it without any computation. Suppose that the drag function chosen in making the ballistic table were based on a constant K_D; this is in fact done in the Euler method. If we inspect the normal equations in the form (1.8) we see that the temperature enters only through the argument $v/u_s(y)$ of K_D, and if K_D is constant the value of this argument is of no importance. So in the ballistic tables no effect of temperature will be found. If the true drag function of the projectile is such that K_D is not constant, the trajectory will depend on the temperature. The use of the ballistic tables will furnish a firing table which may be quite accurate whenever the temperature structure is the same as that on the day of the experimental firing. But the ballistic tables furnish us no information at all that will enable us to correct the firing table entries when the temperature structure differs from that during the experimental firing.

5. Choice of drag function and of method.

It is quite evident that the ideal situation for the ballistician would be first, an exact knowledge of all the aerodynamic coefficients of each projectile fired, together with an exact knowledge of all other forces acting, and second, access to a computing machine which would compute the solutions of the equations of motion with great speed and accuracy. Some devices were designed during the course of the past war which come closer to the latter specification than anything available up to September 1945. However, the nearest approach to such a computing machine available for use in preparing firing tables during the war was the differential analyzer.

Insofar as the normal equations are concerned, the crucial aerodynamic coefficient is K_D. The choice of K_D, or of the drag function $G(v)$ corresponding to it, will be dictated by two considerations. First, we are clearly limited by our ignorance of the drag function of the projectile being studied. The most obvious instance up to the time of writing this chapter is the case of aircraft bombs. As yet (1945) there has been no single bomb for which a drag function has been determined with any satisfactory degree of accuracy. All the bombing tables have been based on the Gâvre drag function, not because of confidence that this is the best possible choice but because of ignorance of anything better. In the previous section it was explained how it is possible that in spite of this ignorance, the ballistic tables can nevertheless be used to prepare bombing tables accurate enough for all the purposes of warfare.

However, if a large number of highly accurate drag functions for various types of bombs had been available, the ballisticians of the U. S. Army and Navy would have been hard put to make adequate use of them. The differential analyzers at Aberdeen and at the University of Pennsylvania were busy night and day with computations for artillery tables, and the two differential analyzers at the Massachusetts Institute of Technology were also engaged to the fullest extent. So the computations for bombing tables were necessarily handled by older methods, that is by hand computation using computing machines. The preparation of a complete ballistic table by such means is a long and laborious task, and it would not be feasible to do it all over again for each new drag function. Some scheme would be needed by which the ballistic data corresponding to the new drag function could be obtained by appropriate modifications of the results already tabulated in the ballistic tables based on the Gâvre drag function, and the devising of such a scheme would call for a high degree of ballistic competence.

Consider next the problem of preparing a firing table for some gun of large caliber, with the help of the differential analyzer. Finding a drag function for a large projectile requires either the firing of the full-scale projectile or the making of extremely accurate models in a smaller size. In either case considerable time and expense are involved. Nevertheless, over half a dozen different drag functions for artillery projectiles have been determined. The most accurate way of using these with the differential analyzer is to make a template, on which the graph of the drag function or the drag coefficient K_D is represented in relief, so that when the computation has indicated a velocity v for the projectile the analyzer can "read" the value of G or of K_D corresponding to v/a. Making such a template requires some care, although it is of course nothing to compare with the labor of making a ballistic table. At Aberdeen templates have been prepared for the more than half a dozen drag functions previously referred to. Now when a given model of shell is to be fired for the preparation of a firing table, one of these drag functions is selected and the corresponding template set in the analyzer. By methods similar to those discussed in the previous section (we do not now care to discuss what is done to account for the effects of non-standard conditions at the time of the experimental firing) the ballistic coefficient is found for each range at which firing takes place, and the value for other ranges estimated by interpolation. At this stage we can tell whether the drag function has been well chosen. If it has, the values of C deduced from the several firings will be nearly equal, and we can feel confident of the results of the interpolation.

As a final example, let us consider the case of small caliber projectiles fired from aircraft. Developments in experimental techniques have proceeded to such an extent that it is no longer a major problem to find a drag function for such projectiles; on one occasion the necessary firings were completed and the drag function computed within the space of twenty-four hours. However, if we wish to use the differential analyzer we are faced with the difficulty that the trajectories start at a wide variety of altitudes and temperatures, and many trajectories are needed. Besides that, probably someone else is using the analyzer. The saving feature here is that the trajectories desired are flat, so that the method of Siacci (to be presented in the next chapter) is applicable with little error. In fact, the error made by using the approximate method of solution is often very considerably less than the error that would result from using any one of the half-dozen drag functions. So in this case it is possible to attain both accuracy and speed by using the Siacci method. This alone would be a sufficient reason for the continued use of an approximation method which might have been expected to be forgotten when the more accurate solution methods, based on numerical integration and introduced into U. S. ballistics by F. R. Moulton during the war of 1914-1918, became available for trajectory computation.

It need hardly be remarked that the authors sincerely hope that ballistics will never again have any use more serious than the computation of rocket trajectories to help the Post Office in the rapid delivery of mail. But if it should ever again happen that ballistics is a matter of vital importance to the nation, it would be ill-advised to put all our trust in the performance of any machine, however accurate and convenient and fast. For at the time at which the machine is of greatest importance, it might be presented with a flood of small problems, each one fairly easy for it or even for a good computer, but arriving at such a rate that the human beings who operate it would be overwhelmed with the task of changing the machine from problem to problem. Numerical integration methods did not entirely supplant the Siacci method, and the differential analyzer did not entirely supplant either. If new and better mechanical aids to computation become available, they will assist the ballistician. But they will be of the greatest assistance if they are operated by men who are familiar with all the methods of attack on problems of ballistics, and they probably will still leave over a multitude of problems to be handled by the older methods.

Chapter V

A P P R O X I M A T E   M E T H O D S

1. Change of independent variable.

The normal equations of the trajectory (IV.1.25) involve a function G whose values are determined experimentally. It is possible to approximate this function by analytic expressions to within any desired degree of accuracy, but such analytic expressions will naturally be quite complicated. This one fact alone is enough to indicate that there is no hope of effecting a solution of the equations in terms of a finite number of elementary functions. So in order to obtain usable results, many early ballisticians tried various methods of replacing the normal equations by approximate forms which were amenable to treatment by elementary methods. It would be (and has been) easy to fill long chapters with the details of such attempts. However, most of them have lost their interest, and we shall therefore present only three methods of obtaining approximate solutions. These three were selected because each of them was found useful in problems which arose during the war of 1939-45. The first which we shall discuss, and to which we shall attach the name of Siacci, is useful in cases in which the trajectory is "flat," meaning that its tangent does not turn through a large angle. The second, historically the oldest, is applicable to projectiles whose velocity is well below that of sound and which do not extend through a great depth of atmosphere. This is the method of Euler. The third approximation device was designed especially for use with dive-bombing and yields a rather simple formula for range and time of flight of projectiles launched at low velocity at angles considerably below the horizontal.

In the treatment of these methods it is found desirable to use several different choices of independent variable. Also, in order to avoid duplication of effort we shall not discuss the normal equations alone, but shall also include terms representing various departures from standard conditions, for use when we discuss differential corrections in Chapters VII, VIII and IX. As before, the components of velocity with respect to the axes will be denoted by v_x, v_y, v_z. If the air has a velocity with respect to the axes, the components being w_x, w_y, w_z, the velocity of the projectile with respect to the air will have components u_x, u_y, u_z, where $u_x = v_x - w_x$, etc. The length of the vector (u_x, u_y, u_z) will be denoted by u; this is then the air speed, or speed with respect to air, of the projectile. With the symbols of Section 1 of Chapter IV, the drag will have components $-Eu_x$, $-Eu_y$, $-Eu_z$. The standard gravity vector will be $(0, -g, 0)$, where g is the standard value of gravitational acceleration. In addition to drag and standard gravity, there will be other accelerations, usually small, which we lump together and call the vector (a_x, a_y, a_z). This will include, for example, the difference between actual gravitational acceleration and standard, the Coriolis forces, and some lesser aerodynamic accelerations. The equations of motion now are

$$(1)\qquad \begin{aligned} &\dot{x} = v_x,\ \dot{y} = v_y,\ \dot{z} = v_z, \\ &\dot{v}_x = -Eu_x + a_x, \\ &\dot{v}_y = -Eu_y + a_y - g, \\ &\dot{v}_z = -Eu_z + a_z, \end{aligned}$$

where

$$E = \gamma HaG(u/a).$$

If we imagine these equations solved for some interval $t_0 \leq t \leq T$ of time, the six variables x, y, z, v_x, v_y, v_z all appear as functions of the time t, and so do other quantities such as the slope m and the angle of inclination θ of the line tangent to the trajectory. (Throughout this chapter m shall stand for the slope. No confusion should arise from the fact that it previously stood for the mass of the projectile, since the mass will not enter the discussions in this chapter.) Let p be a function of t which for $t_0 \leq t \leq T$ has a non-vanishing derivative. Then the equation $p = p(t)$ can be solved for t; to each p between $p(t_o)$ and $p(T)$ corresponds exactly one value of t, and the t thus determined is a differentiable function of p. Also, if F(t) is a differentiable function of t, F(t(p)) is a differentiable function of p, and

$$dF(t(p))/dp = [dF(t)/dt][dt/dp]. \tag{2}$$

We shall denote derivatives with respect to p by primes. By (2), equations (1) takes the form

$$
\begin{aligned}
x' &= v_x t', \\
y' &= v_y t', \\
z' &= v_z t', \\
v_x' &= - Eu_x t' + a_x t', \\
v_y' &= - Eu_y t' + a_y t' - gt', \\
v_z' &= - Eu_z t' + a_z t'.
\end{aligned}
\tag{3}
$$

In the normal equations the functions H and a are assumed to have their standard values, and the accelerations a_x, a_y, a_z and the wind vector (w_x, w_y, w_z) are supposed to be zero. Also the initial values of z and v_z are zero. This implies that the entire normal trajectory lies in the (x, y)-plane, so that its slope is

$$m = dy/dx = v_y/v_x. \tag{4}$$

Even in the disturbed case, in which z may not be identically zero, we retain this same definition of the symbol m, although it may differ slightly from the slope of the trajectory. From (4) it follows that

$v_y = mv_x$, so by differentiation we obtain

(5) $$m'v_x = v_y' - mv_x'.$$

Substituting from the fourth and fifth of equations (3) and also substituting $v_x - w_x$ for u_x and $v_y - w_y$ for u_y transforms equation (5) into

(6) $$m'v_x = t'[-g + a_y - ma_x + E(w_y - mw_x)].$$

As a first application, let us choose $p = x$. Then $x' = 1$, so $t' = 1/v_x$ by the first of equations (3). Now by (3), (4) and (6) we obtain

(7)
$$\begin{aligned} dt/dx &= 1/v_x, \\ dy/dx &= m, \\ dm/dx &= [-g + a_y - ma_x + E(w_y - mw_x)]/v_x^2, \\ dv_x/dx &= [-Eu_x + a_x]/v_x, \\ dz/dx &= v_z/v_x, \\ dv_z/dx &= [-Eu_z + a_z]/v_x. \end{aligned}$$

The corresponding normal equations are obtained by setting the departures from normality equal to zero, (that is, a and H are standard and a_x, a_y, a_z, w_x, w_y, w_z are zero) and z and v_z equal to zero at the start. Then z is identically zero, and we may omit mention of it from the normal equations. These are then

(8)
$$\begin{aligned} dt/dx &= 1/v_x, \\ dy/dx &= m, \\ dm/dx &= -g/v_x^2, \\ dv_x/dx &= -E. \end{aligned}$$

It is an interesting and important feature of these equations that the last three can be solved independently of the first. For t does not enter explicitly into E, so that the last three equations involve only y, m and v_x. The first equation can be solved later by a quadrature.*

Next let us take $p = m$. Then $m' = 1$, and from (6) we obtain the first of the system of equations

$$
(9)\qquad
\begin{aligned}
\frac{dt}{dm} &= \frac{v_x}{[-g + a_y - ma_x + E(w_y - mw_x)]},\\
\frac{dx}{dm} &= \frac{v_x^2}{[-g + a_y - ma_x + E(w_y - mw_x)]},\\
\frac{dy}{dm} &= \frac{mv_x^2}{[-g + a_y - ma_x + E(w_y - mw_x)]},\\
\frac{dv_x}{dm} &= \frac{[-Ev_x + Ew_x + a_x]v_x}{[-g + a_y - ma_x + E(w_y - mw_x)]},\\
\frac{dz}{dm} &= \frac{v_z v_x}{[-g + a_y - ma_x + E(w_y - mw_x)]},\\
\frac{dv_z}{dm} &= \frac{[-Ev_z + Ew_z + a_z]\,v_x}{[-g + a_y - ma_x + E(w_y - mw_x)]},
\end{aligned}
$$

From the first of equations (9), with the first of (3), we obtain the second of (9); and from this in turn, with (4), we obtain the third of (9). The fourth of (9) follows from the first of (9) and the fourth of (3). The remaining two follow from the first of (9) with the third and sixth of (3).

*A quadrature is the computation of the definite integral of a function of one variable. Several useful quadrature formulas are established in Section 3 of Chapter VI. The word "integration" has a broader meaning. It is used to refer to the finding of a solution of a differential equation. Methods of integration form the principal subject matter of Chapter VI.

The corresponding normal equations are

$$
(10)\qquad
\begin{aligned}
dt/dm &= -\,v_x/g,\\
dx/dm &= -\,v_x^{\,2}/g,\\
dy/dm &= -\,mv_x^{\,2}/g,\\
dv_x/dm &= Ev_x^{\,2}/g.
\end{aligned}
$$

These equations split up even more strikingly than did equations (8). The last two can be solved together as a system, since E is determined when m, v_x and y are known. Subsequently t and x can be computed by a quadrature (say by Simpson's rule) from the values of v_x, by virtue of the first two of equations (10).

The change to independent variable θ is most easily made from (9) and (10). We consider only the normal equations, since these are all we shall need. Since

$$m = \tan\theta$$

and

$$dm/d\theta = \sec^2\theta,$$

equations (10) become

$$
(11)\qquad
\begin{aligned}
dt/d\theta &= -\,v_x \sec^2\theta/g,\\
dx/d\theta &= -\,v_x^2 \sec^2\theta/g,\\
dy/d\theta &= -\,v\quad \sec\theta\,\tan\theta/g,\\
dv_x/d\theta &= E\,v_x^2 \sec^2\theta/g.
\end{aligned}
$$

On either the ascending or the descending branch of a trajectory it is permissible to select y as independent variable. This cannot be done on an arc containing the summit of the trajectory, since near the maximum value of y both x and t are double-valued functions of y. For simplicity we restrict our attention to the normal equations. If we choose p = y, the second of equations (3) yields $t' = 1/v_y$. Then (6) becomes

$$m' = -\,gt'/v_x = -\,gmt'/v_y = -\,gmt'^{\,2} = -\,gm/v_y^{\,2}.$$

So, with conditions normal, (3) takes the form

$$
(12)\qquad
\begin{aligned}
dm/dy &= -\,gm/v_y^{\,2},\\
dv_y/dy &= -\,E - g/v_y,\\
dx/dy &= 1/m,\\
dt/dy &= 1/v_y.
\end{aligned}
$$

Here the first two equations form a system which can be solved independently of the other two, the values of t and x being obtained afterward by quadratures. For level bombing this system would offer still another advantage, since the problem then is to determine the range and time of flight corresponding to a particular value of y, and the use of y as independent variable would avoid much annoyance in interpolation. However, along with these advantages equations (12) have a serious defect. At the summit, which is the starting point of the trajectory in level bombing, v_y has the value 0, and so has m. So the right members of (12) have infinite discontinuities, which have so far balked all attempts to utilize these equations in computing trajectories.

Finally, let c be a positive constant, and choose

$$(13)\qquad p = cv_x.$$

We shall again consider only the normal equations. By (13), $v_x' = 1/c$, so the fourth of equations (3) yields

$$1/c = -\,E(p/c)t',$$

or

$$(14)\qquad dt/dp = -\,1/pE.$$

By substituting (13) and (14) in the first of equations (3) and equation (6), and using (4), we obtain

$$(15) \quad \begin{aligned} dt/dp &= -1/pE, \\ dx/dp &= -1/cE, \\ dy/dp &= -m/cE, \\ dm/dp &= +cg/p^2E. \end{aligned}$$

2. The Hitchcock-Kent modification of the Siacci method.

During the past century a number of methods of approximation to the solution of the normal equations in the form (1.15) have been proposed, having as common elements the assumption that air density is constant and temperature also, and that in computing G(v) the velocity v is replaced by a multiple of v_x. Three of these methods were devised by F. Siacci, who also computed the tables of auxiliary functions needed to make the method readily applicable. Siacci's methods were designed to be applicable to flat trajectories with small angle of departure, say $\theta < 15$ degrees. More recently modifications of the Siacci method have been devised which permit its use in computing flat trajectories with high angle of departure, as in anti-aircraft fire. For instance, such a method was presented by Commandant Gazot. The version which we shall discuss is due to Hitchcock and Kent.* In its first phases, which will be developed in this section, it resembles the method of Gazot; but the later refinements permit closer approximation to the correct result.

Let us then make the following assumptions. First,

(1) The difference in height of the highest and lowest points of the trajectory is small enough so that we may assume that H(y) is constantly equal to its value

*H. P. Hitchcock and R. H. Kent, Applications of Siacci's Method to Flat Trajectories, Ballistic Research Laboratory Report No. 114 (Aberdeen Proving Ground: 1938).

at the origin.

We shall not assume that the origin is at sea-level, since we wish the results obtained to be applicable to fire from aircraft. Let us denote the height of the origin above sea-level (the muzzle altitude) by Y_m. The ratio of the density of the air at any altitude Y to the standard air density at sea-level is denoted by H(Y). In particular, at the muzzle this ratio has the value $H(Y_m)$. As in (IV.2.5) we define the "ballistic coefficient corrected for muzzle altitude" to be

(2) $$C_s = C/H(Y_m).$$

Second, we assume:

(3) The value of v is approximated by $v_x \sec \theta_o$ along the whole trajectory with sufficient accuracy to permit us to replace v by $v_x \sec \theta_o$ in computing E.

Since the equation

(4) $$v_x = v \cos \theta$$

is exact, this second assumption amounts to saying that $\cos \theta$ changes but little along the trajectory. It is therefore satisfied when the trajectory is sufficiently flat, in the sense that the error in the assumption tends to zero as the total curvature of the trajectory tends to zero.

We further assume that:

(5) The difference in height of the highest and lowest points of the trajectory is small enough so that the temperature may be taken as constant along the trajectory; that is, along the trajectory the ratio a of the velocity of sound to standard sea-level velocity of sound is a constant.

The "pseudo-velocity" p is defined to be

(6) $$p = v_x \sec \theta_o.$$

Under assumptions (1), (3) and (5), E is given by

$$E = G(p/a)a/C_s$$

(compare equation (IV.1.24)). But (6) is the same as (1.13), with $c = \sec \theta_0$. So equations (1.15) are satisfied with this same value of c:

(7)
$$\begin{aligned} dt/dp &= -C_s/\{apG(p/a)\}, \\ dx/dp &= -C_s \cos \theta_0/\{aG(p/a)\}, \\ dy/dp &= -C_s\, m \cos \theta_0/\{aG(p/a)\}, \\ dm/dp &= +gC_s \sec \theta_0/\{ap^2G(p/a)\}. \end{aligned}$$

In order to express the solutions of these equations in a convenient form, four functions are tabulated. A number U is selected and fixed, larger than the greatest velocity at which the projectile is likely to be used, and for values of $u < U$ the following four integrals are computed, by numerical quadrature:

(8)
$$\begin{aligned} S(u) &= \int_u^U \{1/G(u)\}\, du, \\ T(u) &= \int_u^U \{1/uG(u)\}\, du, \\ I(u) &= \int_u^U \{2g/u^2G(u)\}\, du, \\ A(u) &= \int_u^U \{I(u)/G(u)\}\, du. \end{aligned}$$

These are the "Siacci functions," or more specifically "the primary Siacci functions," for the drag function G(v

By (6), the initial value of p is $p_0 = v_0$, the initial velocity, since at the initial point

$$v_x = v_0 \cos \theta_0.$$

The other initial values are

$$x_o = y_o = 0, \; m_o = \tan \theta_o.$$

Integration of the members of the first of equations (7), with the substitution p = au on the right, yields

$$\begin{aligned} t(p) &= -(C_s/a) \int_{p_o}^{p} \{ 1/pG(p/a) \} \, dp \\ (9) \qquad &= -(C_s/a) \int_{p_o/a}^{p/a} \{ 1/uG(u) \} \, du \\ &= (C_s/a) \{ T(p/a) - T(p_o/a) \} . \end{aligned}$$

In a similar way we obtain

$$(10) \quad x(p) = C_s \cos \theta_o \{ S(p/a) - S(p_o/a) \},$$

$$(11) \quad m(p) = m_o - C_s \sec \theta_o \{ I(p/a) - I(p_o/a) \} / 2a^2.$$

Substitution of the value of m from (11) in the third of equations (7) and integration yields, with the help of (10),

$$(12) \quad \begin{aligned} y(p) = x \{ m_o + C_s \sec \theta_o I(p_o/a)/2a^2 \} \\ - C_s^2 \{ A(p/a) - A(p_o/a) \} / 2a^2. \end{aligned}$$

For the Mayevski drag function, the four primary Siacci functions and a number of secondary functions (not defined above) can be found tabulated in J. M. Ingalls' Ballistic Tables, Artillery Circular M (Fort Monroe: 1893; revised, 1917). Within the past five years the S- and T-functions have been computed and tabulated for a considerable number of drag functions, mostly corresponding to projectiles of small caliber.

Suppose, for instance, that a projectile is fired through a velocity-measuring apparatus and then travels on to a target at a known distance x, the time of flight from apparatus to target being also measured. Assuming that the trajectory is nearly level, as it is in ordinary range firings of this type, equations (9) and (10) take the form

$$(13) \qquad \begin{aligned} t(p) &= C_s\{ T(p/a) - T(p_o/a)\} /a, \\ x(p) &= C_s\{ S(p/a) - S(p_o/a)\} \end{aligned}$$

Here we know the left members, and also know $p_O = v_O$, and a (the temperature having been measured). From (13) follows

$$T(p/a) = \{ ta/x\} S(p/a) + T(p_o/a) - \{ ta/x\} S(p_o/a),$$

and the solution $p < p_o$ of this equation can be found by successive trials, ending with an interpolation between the two successive tabular values of p/a for which the difference between right and left members of the above equation has opposite signs. As soon as p/a is determined, so is C_s by either of equations (13), and from this and the density of the air C can be determined.

If a projectile passes before two motion picture cameras, its motion being nearly level and slow enough to permit photography, and the cameras are simultaneously photographing clocks, it is possible to determine the horizontal component of the velocity at two places a known distance apart. We then know both p_o/a and p/a, and C_s is determined by the second of equations (13).

Although the Siacci method is rather outmoded for field artillery, let us consider the problem of determining C from a range firing in which the angle of departure θ_O, the initial velocity $v_O = p_O$, the relative air density, the relative velocity of sound a and the range X have been measured. We suppose that there is no wind, or that its effects have in some way been allowed for. Impact is assumed to occur when y returns to 0. The first step is to determine the value of p at which y(p) vanishes, while x(p) = X. From (10),

$$C_s = X \sec \theta_o/\{S(p/a) - S(p_o/a) \}.$$

If we substitute this in (12) and set $y(p) = 0$, after a certain amount of manipulation we obtain

$$
\begin{aligned}
&A(p/a) - A(p_o/a) \\
(14) \quad &= \{ S(p/a) - S(p_o/a) \} \\
&\cdot \{ I(p_o/a) + a^2 \sin 2\theta_o [S(p/a) - S(p_o/a)] / X \}.
\end{aligned}
$$

By successive trials a solution p/a is found, after which (10) yields C_s; hence C is determined.

In Artillery Circular M this last problem is rendered easier of solution by use of a number of auxiliary tables, containing "secondary Siacci functions." This was a sound idea at the time when Siacci methods were dominant, and only a single drag function was in common use. Since almost all artillery problems would then involve the use of these tables, it was advisable to include all aids to computation that were feasible. However, the recent uses of the Siacci method have in large part been in computing firing tables for small caliber projectiles fired from aircraft. Experiments have indicated a considerable variability in the shape of the drag functions for these projectiles, and so tables have been needed for a variety of shapes of drag function. This makes it impracticable to consider computing a complete set of auxiliary tables of secondary functions for each drag function. In fact, the practice has been to compute the S- and T-functions only, and to replace equations (11) and (12) by others which leave a numerical quadrature to be performed to find y along each trajectory. This device will be discussed in Section 5 of this chapter.

3. Oblique coordinates.

In several of his papers on ballistics, K. Popoff has shown the usefulness of a certain system of oblique axes. Let the L-axis be tangent to the trajectory at its initial point, and let the D-axis be vertical, with the positive direction downwards. Then

(1) $$x = L\cos\theta_0, \quad y = L\sin\theta_0 - D$$

and

(2) $$L = x\sec\theta_0, \quad D = x\tan\theta_0 - y$$

are the equations of transformation between this system and the original (x, y)-system. Since we are dealing only with the normal equations, we need not discuss z. To have analogous notation for the two sets of axes, we define

(3) $$v_L = dL/dt, \quad v_D = dD/dt$$

as the oblique components of velocity of a moving point. (It should be observed that v_L is not the perpendicular component of the velocity vector along the L-axis; it is the component obtained by projecting the velocity vector onto the L-axis by vertical lines.)

Then

(4) $$v_L = v_x\sec\theta_0, \quad v_D = v_x\tan\theta_0 - v_y.$$

The normal equations obtained from (1.1) by setting a = 0 and omitting the equations for z and v_z are

(5) $$\begin{aligned} \dot{x} &= v_x, \quad \dot{y} = v_y, \\ \dot{v}_x &= -Ev_x, \\ \dot{v}_y &= -Ev_y - g. \end{aligned}$$

From these we readily deduce

(6) $$\begin{aligned} \dot{L} &= v_L \quad \dot{D} = v_D, \\ \dot{v}_L &= -Ev_L, \\ \dot{v}_D &= -Ev_D + g. \end{aligned}$$

The first two of these are the same as (3), the third is obtained by multiplying by sec θ_0 in the third of equations (5), and the fourth is obtained by multiplying the members of the third and fourth of equations (5) by tan θ_0 and - 1 respectively and adding.

Equations (6) differ from (5) only in the change of notation from x, y to L, D and in the replacement of g by - g. Thus all the manipulations performed in Section 1 of this chapter are equally applicable to the system (6). (However, it would hardly be advisable to use the word "slope" as a name for the quantity $m = dD/dL$ defined by (1.4) as rewritten in the notation of (6)). In particular, (1.8) holds in the form

$$
\begin{aligned}
dt/dL &= 1/v_L, \\
dD/dL &= m, \\
dm/dL &= g/v_L{}^2, \\
dv_L/dL &= - E,
\end{aligned}
\tag{7}
$$

while (1.13) and (1.15) become

$$p = cv_L \tag{8}$$

and

$$
\begin{aligned}
dt/dp &= - 1/pE, \\
dL/dp &= - 1/cE, \\
dD/dp &= - m/cE, \\
dm/dp &= - cg/p^2E
\end{aligned}
\tag{9}
$$

respectively.

If we again assume constancy of density and temperature along the trajectory, and also assume as before that $v_x \sec \theta_o$ is an adequate approximation for v in computing E, we see by (4) and (8) that in our present notation we are assuming that v_L is close enough to v to use in computing E, and that

$$E = aG(p/a)/C_s \tag{10}$$

is sufficiently accurate for our approximate computation of the trajectory, wherein $p = v_L$. The initial conditions are

(11) $p_o = v_o = v_{Lo}, \quad v_{Do} = 0, \quad L_o = D_o = 0.$

In (9) we are choosing $c = 1$, since we have chosen $p = v_L$. But now the angle of departure θ_o has disappeared entirely both from the equations (9) and (10) and from the initial conditions (11). Thus, to the degree to which our assumptions concerning the shortness and flatness of the trajectory are correct, the drop D, the time t and the rate of change of D with respect to L are all functions of L alone.

To find the specific form of these functions requires little additional work. For the special case $\theta_o = 0$, equations (2) reduce to $L = x$, $D = -y$, while $m_o = dD/dL$ has the value 0. So equations (9) to (12) of the preceding section yield

$$t(p) = C_s\{ T(p/a) - T(p_o/a) \}/a,$$

$$L(p) = C_s\{ S(p/a) - S(p_o/a) \},$$

$$(12) \quad dD/dL = C_s\{ I(p/a) - I(p_o/a) \}/2a^2,$$

$$D(p) = - C_s L I(p_o/a)/2a^2 + C_s^2 \{ A(p/a) - A(p_o/a) \}/2a^2.$$

Since these hold for $\theta_o = 0$ and the solution is independent of θ_o, they represent the solutions of the equations (under the approximation (10)) for all angles of departure.

There is a convenient geometric interpretation of this independence of θ_o. Imagine a sheet of graph paper in which the two systems of lines are each rigid, and the points of intersection are rigidly attached to the lines, but the angle of intersection can be varied. A piece of window-screening affords a good example. With the D-axis vertical and L-axis horizontal, graph a trajectory whose initial direction is tangent to the L-axis. If we now deform the paper, keeping the D-axis vertical but tilting the L-axis and all the lines parallel to it, the rectangles of the graph paper go into parallelograms, and by the preceding paragraph the graph of the trajectory continues to be the graph of a trajectory, with the same initial velocity and a non-zero angle of departure. That is, the solutions of the Siacci equations possess what may be termed "parallelogram rigidity." This property has sometimes proved extremely convenient when a number of trajectories with the same initial velocity but with different angles of departure had to be computed. When these can be handled accurately enough by the Siacci method, all that is needed is to prepare a single trajectory and obtain the others from it by "parallelogram rigidity."

It is an interesting mathematical fact that under the assumptions (2.1, 3, 5) there is in a sense only one trajectory for each drag function. That is, corresponding to each drag function we can compute a single trajectory, beginning at a velocity higher than any at which the projectiles of the type may be expected to be fired; and from this one trajectory, by forming linear combinations of terms with constant coefficients, we can find drop and time of flight for any L corresponding to any ballistic coefficient, any density, any temperature and any angle of departure. The angle of departure has already been taken care of in the preceding paragraph; if we have a trajectory with initial slope 0 we can find those with other angles of departure by use of parallelogram rigidity. Having chosen a value U of velocity greater than any which may reasonably be anticipated, we solve the equations

$$\text{(13)} \qquad \begin{aligned} d\tau/du &= -\,1/uG(u),\\ d\xi/du &= -\,1/G(u),\\ d\eta/du &= -\,\mu/G(u),\\ d\mu/du &= g/u^2G(u), \end{aligned}$$

with the initial values

(14) $$\tau(U) = \xi(U) = \eta(U) = \mu(U) = 0.$$

These equations could be solved by the method of Section 2, but this is not essential. Suppose now that we wish to obtain a solution of (2.7) with velocity of sound equal to a times standard sea-level velocity, ballistic coefficient C, air density $H(Y_m)$ times standard sea-level density and initial velocity v_0. As previously remarked, we may suppose that the angle of departure is 0. Then the initial conditions are

(15) $$p_o = v_o, \quad x_o = y_o = t_o = m_o = 0.$$

Let us define

(16) $$\tau_0 = \tau(p_o/a), \quad \xi_0 = \xi(p_o/a),$$ $$\eta_0 = \eta(p_o/a), \quad \mu_0 = \mu(p_o/a).$$

Then by substitution we find that the functions

(17) $$\begin{aligned} p &= ua \\ x &= C_s\{ \xi - \xi_0 \}, \\ t &= C_s\{ \tau - \tau_0 \}a, \\ m &= C_s^2\{ \mu - \mu_0 \}/a^2, \\ y &= C_s^2\{ (\eta - \eta_0) - \mu_0(\xi - \xi_0) \}/a^2 \end{aligned}$$

satisfy equations (2.7) with $\theta_0 = 0$ and also satisfy the initial conditions (15). Thus from the single trajectory (13) we can deduce any other trajectory by simple arithmetic operations, without any more processes of integration. Of course equations (17) are in effect a mere notational revision of the Siacci equations (2.9, 10, 11, 12). But the point of view is somewhat different. The Siacci functions S, T, I and A are shown to be the travel, time, slope and drop (except for a factor 2 in I) of a particular trajectory, namely the trajectory corresponding to standard temperature and density, ballistic coefficient 1, angle of departure $\theta_0 = 0$ and initial velocity U; and the Siacci equations (2.9, 10, 11, 12), by which the trajectory is computed with the help of these functions, show themselves to be the equations of a geometric tilting of axes and change of scale along the axes by which this fundamental trajectory is adapted to fit any initial conditions.

In order to have a comparison of the various methods to be presented in this chapter, they will all be applied to the same problem. This will be the computation of the trajectory of a projectile whose drag function is the Gâvre function, with respect to which it has ballistic coefficient 1. The initial velocity is 2700 feet per second and the angle of departure is 45°. The temperature is assumed to be sea-level standard at all altitudes, so that a(y) is identically 1. Here we compute the trajectory; the final results, with those obtained by other methods, are summarized in Section 10 and are there compared with the results of a much more accurate computation procedure.

With $p_0 = 2700$, we find from tables of the Siacci functions that

$$S(p_0) = 2681.74, \quad T(p_0) = 0.864, \qquad I(p_0) = 0.04931, \quad A(p_0) = 105.91. \tag{18}$$

The first two of these may be found in the tables at the end of this book. The other two are taken from tables available at the Aberdeen Proving Ground, but not reproduced here. The first column lists the value of L. Other columns are obtained by the following procedures. From (12), with $a = C_s = 1$,

$$S(p) = S(p_0) + L = 2681.74 + L.$$

The corresponding p is read from the S-table and entered in the third column, and the T(p) corresponding to this p is entered in the fourth column. Again by (12), with present values,

$$t(p) = T(p) - T(p_0) = T(p) - 0.864.$$

The values of I(p) are entered from the I-table (not reproduced in this book). The next column contains the values of A(p) from the A-table (not reproduced here), and the next is $C_s^2 A(p)/2$, which is A(p)/2. Next D(p) is computed by (12), which with present numerical values takes the form

$$D(p) = A(p)/2 - 0.02465\ L - 52.95.$$

The slope m(p) is computed by (2.11);

$$m(p) = 1 - (0.707107)\{I(p) - 0.04931\}.$$

In the next column we enter the value of the inclination $\theta(p) = \text{arc tan } m(p)$, found from a trigonometric table. The next column, sec $\theta(p)$, is also from a trigonometric table. If we observe that the horizontal component of velocity can be expressed either as p cos θ_0 or as v cos θ, we find the speed corresponding to slant range L (and therefore to the corresponding p) is

$$v(p) = p \cos \theta_0 \sec \theta\ (p).$$

This is computed and entered in the last column.

In this computation and all the subsequent ones of this chapter we have intentionally retained more significant figures than are justified by the accuracy of the methods. The reason is that we wish the comparison in Section 10 to exhibit the differences in the results of the methods themselves, unclouded by any rounding errors.

L (ft)	S(p) (ft)	p (ft/sec)	T(p) (sec)	t(p) (sec)
0	2681.74	2700.00	0.864	0.000
1000	3681.74	2407.18	1.261	0.397
2000	4681.74	2133.00	1.703	0.839
4000	6681.74	1643.40	2.773	1.909
6000	8681.74	1259.83	4.169	3.305
8000	10681.74	1031.92	5.940	5.076
10000	12681.74	910.72	8.013	7.149
12000	14681.74	824.82	10.324	9.460
14000	16681.74	754.39	12.861	11.997
16000	18681.74	692.93	15.629	14.765
18000	20681.74	637.59	18.639	17.775
20000	22681.74	586.90	21.910	21.046

L (ft)	I(p)	A(p) (ft)	$C_s^2A(p)/2$ (ft)	D(p) (ft)
0	0.04931	105.91	52.95	0.00
1000	0.05922	159.99	79.99	2.39
2000	0.07176	225.22	112.61	10.34
4000	0.10873	402.49	201.24	49.67
6000	0.17182	677.39	338.70	137.81
8000	0.27313	1115.65	557.82	307.63
10000	0.41133	1794.46	897.23	597.72
12000	0.58328	2783.38	1391.69	1042.88
14000	0.79051	4151.04	2075.52	1677.40
16000	1.03700	5971.58	2985.79	2538.36
18000	1.32854	8329.06	4164.53	3667.79
20000	1.67269	11320.67	5660.33	5114.28

L (ft)	m(p)	θ(p) (deg)	sec θ(p)	v(p) (ft/sec)
0	1.00000	45.00	1.41421	2700.00
1000	0.99299	44.80	1.40930	2398.83
2000	0.98413	44.54	1.40283	2115.84
4000	0.95798	43.77	1.38473	1609.15
6000	0.91337	42.41	1.35454	1206.68
8000	0.84174	40.09	1.30700	953.69
10000	0.74401	36.65	1.24642	802.67
12000	0.62242	31.90	1.17790	687.00
14000	0.47589	25.45	1.10747	590.76
16000	0.30159	16.78	1.04449	511.78
18000	0.09544	5.45	1.00454	452.89
20000	-0.14791	-8.41	1.01086	419.51

4. Formulas for approximating the drop.

From the first three of equations (1.8) it is easy to see that if $m_0 = 0$ the equations

(1)
$$m = -g\int_0^x v_x^{-2}\,dx = -g\int_0^t v_x^{-1}\,dt,$$
$$y = \int_0^x m\,dx = \int_0^t mv_x\,dt$$

are satisfied. Let us denote v_x by p, and assume that the Siacci method is applicable. Then by (2.7) we have

(2)
$$\dot{p} = -apG(p/a)/C_s.$$

For the special drag function with constant K_D, that is, with drag proportional to the square of the velocity, G(v) is a constant times v, and (2) takes the special form

$$\dot{p} = -cp^2, \tag{3}$$

c being a constant which we shall dispose of later. The solution of (3) is

$$(1/p) = ct + (1/p_o), \tag{4}$$

so by the first of equations (1)

$$m = -g\{(ct^2/2) + (t/p_o)\}. \tag{5}$$

When this is substituted in the second of equations (1) and the integration performed, it is found that

$$y = -g\{(t^2/4) + (t/2p_oc) - (1/2p_o^2c^2)\log(cp_ot + 1)\}. \tag{6}$$

By (4), $p_oc = \{(p_o/p) - 1\}/t$, so c can be eliminated from (6). The result is

$$y = -\tfrac{1}{2}gt^2\phi_2(p/p_o), \tag{7}$$

where

$$\phi_2(p/p_o) = \tfrac{1}{2} + \{(p_o/p) - 1\}^{-1} - \{(p_o/p) - 1\}^{-2}\log(p_o/p), \tag{8}$$

the logarithms being to the base e. A table of values of ϕ_2 will be found at the end of this book.

The choice of drag function was of course quite arbitrary, and we might reasonably fear that, if the drag function of some projectile is considerably different from a mere multiple of v, formula (8) might be seriously in error. So let us try another. For many modern projectiles, the drag at supersonic velocities is fairly well approximated by a multiple of the three-halves power of the velocity. If we should choose this resistance law instead of the quadratic, equation (3) would be replaced by

$$(9) \qquad \dot{p} = -2kp^{3/2},$$

where k is a positive constant. Integration yields

$$(10) \qquad p^{-1/2} = \chi(t) = kt + p_o^{-1/2},$$

where χ is merely an abbreviation for the linear function in the right member of (10). By (1),

$$(11) \qquad m = -g \int_0^t \chi^2 \, dt = -g(\chi^3 - \chi^3_o)/3k,$$

so by the second of equations (1)

$$(12) \qquad y = -(g/3k^2) \int_{\chi_o}^{\chi} \{ (\chi^3 - \chi_o^3)/\chi^2 \} \, d\chi = -(g/3k^2)\{ (\chi^2/2) + (\chi_o^3/\chi) - (3\chi_o^2/2) \}.$$

By (10), $k = (\chi - \chi_o)/t$. If we write $r = \chi_o/\chi$, equation (12) becomes

$$y = (gt^2/6)\{(1 - 3r^2 + 2r^3)/(1 - r)^2\} = (gt^2/6)(2r + 1).$$

That is, y satisfies the equation

$$(13) \qquad y = -\tfrac{1}{2}gt^2 \, \phi_{3/2}(p/p_o),$$

where

$$(14) \qquad \phi_{3/2}(p/p_o) = \{ 1 + 2\sqrt{p/p_o} \}/3.$$

The subscripts in (7) and (13) were of course chosen to serve as reminders of the drag functions on which they are based.

Formulas (8) and (14) seem to be totally unrelated It is interesting to observe that nevertheless for values of p/p_o not too widely different from 1 they give nearly the same values for the factor multiplying $gt^2/2$. For example, at $p/p_o = .75$ we find that

$\phi_2 = .91086$ and $\phi_{3/2} = .91068$. At $p/p_o = .5$ we find $\phi_2 = .80685$ and $\phi_{3/2} = .80474$. So even after the projectile has lost half of its x-component of velocity the two estimates of the drop differ only by about a fourth of one per cent.

This close agreement between the estimates derived from two very different drag functions would naturally lead us to hope that the drop as deduced by either formula might be reasonably close to the actual drop when the drag function is that of any projectile. As a matter of fact formula (7) has been tested under a variety of circumstances and usually gives estimates of the drop which are closer to the truth than one has any evident reason to expect. Some numerical examples are given in Section 10 of this chapter.

The preceding formulas have been written as though the initial tangent to the trajectory were horizontal. But in accordance with the results of the preceding section, we can immediately apply these formulas to any trajectory. For the drop is independent of the angle of departure, to the same order of accuracy as the Siacci method applies, and so it is given approximately by

$$D = \tfrac{1}{2}gt^2\phi(p/p_o), \tag{15}$$

where ϕ is defined either as in (8) or as in (14).

One obvious use of these formulas is in the case in which a trajectory of only moderate accuracy is called for, and only S- and T-tables are available. By the S- and T-tables we can find t and p as functions of travel L, and then we can find the drop approximately by (15). Another use to which the formulas and accompanying tables of ϕ_2 have been put is to furnish a running check on other computations. In these instances the drop was computed by the methods of the next section. At the same time it was also computed by (15). The difference between the two computations was small and slowly varying, so that any numerical error was quite conspicuous. It is interesting to note that these same computations furnished a check on the accuracy of (15), since the drop was also being computed by a more accurate method at the same time. For small caliber projectiles fired forward from aircraft, with ranges up to 2000 yards, the error remained under two yards.

In order to show the errors of the two approximation formulas, the drops were calculated by both formulas for the trajectory which is being used as a test sample for the various methods. The results will be found in Section 10.

5. The Hitchcock-Kent treatment of variable density and ballistic coefficient.

For trajectories of anti-aircraft shell fired at large angles of departure, assumption (2.1) of the Siacci theory may be seriously in error, since the shell may reach heights at which the air density is much less than it is at sea-level. If the trajectory is still flat, it can be handled by a method due to Hitchcock and Kent, which is so different from that of Section 2 as to merit the name of "Hitchcock-Kent method," rather than the name of a modification of the Siacci method.

Since the altitude of the projectile above sea-level is $Y_m + y$,

(1) $$E = H(Y_m + y)aG(v/a)/C,$$

and the second of the normal equations (1.15) can be written in the form

(2) $$c \left\{ H(Y_m + y)\ G(v/a)/C\ G(p/a) \right\} dx = - \left\{ 1/a\ G(p/a) \right\} dp.$$

As in Section 2 we choose

(3) $$c = \sec \theta_0.$$

Given any sort of first approximation for y as a function of x, say $y = y_o(x)$, the density ratio $H(Y_m + y)$ can be replaced approximately by a function of x. For flat trajectories a rather good first approximation to y is given by

$$(4) \qquad y_o(x) = x \tan \theta_o,$$

and this is used in examples by Hitchcock and Kent; but if in special instances a better approximation happens to be available, it could be used in place of (4) without changing any of the subsequent developments. As in the Siacci theory, it will now be assumed that the trajectory is flat enough so that $G(v/a)$ is nearly equal to $G(p/a)$. The normal equation (2) is thus replaced by the approximate form

$$(5) \quad \sec \theta_o \left\{ H(Y_m + y_o(x))/C \right\} dx = - dp/aG(p/a).$$

Let p be the pseudo-velocity corresponding to abscissa x. By integration of both members of (5) we find, with the help of (2.8),

$$(6) \quad \sec \theta_o \int_0^x \left\{ H(Y_m + y_o(x))/C \right\} dx = S(p/a) - S(p_o/a).$$

Here it is not necessary to regard C as a constant. In fact, since a projectile ordinarily has a certain amount of yaw near the muzzle, its drag near the muzzle is greater than the drag function alone would predict. In such cases it is sometimes desirable to regard the ballistic coefficient as varying with distance travelled, so that C appears as a function of x. This does not complicate (6).

Equation (6) determines p as a function of x. It remains to find the other elements of the trajectory. Here the Siacci methods do not seem to be helpful, and we abandon them. But since by equation (3) we have $v_x = p \cos \theta_o$, equations (1.8) can be used. In integrated form, these are

$$t = \sec \theta_0 \int_0^x (1/p)\, dx,$$

$$(7) \quad m = m_0 - g \sec^2\theta_0 \int_0^x (1/p^2)\, dx,$$

$$y = \int_0^x m\, dx$$

$$= m_0 x - g \sec^2\theta_0 \int_0^x \left\{ \int_0^x (1/p^2) \right\} dx.$$

Observe that these are not approximations. If the pseudo-velocity p is exactly determined as a function of x, these equations will yield exact solutions of the normal equations. Of course when the values of p are only approximately determined, as in the method now under discussion, the values of t, m and y determined by (7) will likewise only be approximations.

To use these formulas, it is convenient to subdivide the interval of values of x in which we are interested, cutting it into smaller equal subdivisions. At each of these subdivision points the value of p is found by (6), and $1/p$ and $1/p^2$ are computed and tabulated. Now the quadratures called for in equations (7) are performed, for example, by Simpson's rule. The result of the quadratures is to provide a value of t, m and y for each tabular value of x. In Section 3 of Chapter VI we shall discuss some quadrature formulas that are more appropriate to this problem than Simpson's, but nevertheless the latter is quite usable.

The method just described has a feature of great importance even when the trajectory is short enough so that the assumption of constant air density is accurate enough. The only one of the Siacci functions used is the S-function. Hence this method is applicable in the many cases in which the drag function is one of those for which only the S- and T-functions have been tabulated.

It is easy to transform to the oblique coordinates defined in the beginning of the third section of this chapter. The first approximation to the altitude $y_o(x)$ becomes $Y_o(L) = y_o(L \cos \theta_o)$, and by (3.2) the change of variable of integration from x to L in (6) yields

$$(8) \quad \int_0^L \{ H(Y_m + Y_o(L))/C \} \, dL = S(p/a) - S(p_o/a).$$

If we differentiate the third of equations (7) twice with respect to x and use (3.2), we obtain

$$(9) \quad \begin{aligned} d^2D/dL^2 &= - \cos^2\theta_o \, d^2y/dx^2 \\ &= + g/p^2. \end{aligned}$$

Since dD/dL vanishes at x = o, successive integrations of (9) yield the two equations

$$(10) \quad \begin{aligned} dD/DL &= g \int_0^L (1/p^2) \, dL, \\ D &= \int_0^L (dD/DL) \, dL \\ &= g \int_0^L \left\{ \int_0^L (1/p^2) \, dL \right\} dL, \end{aligned}$$

while the time is obtained by change of variables in the first of equations (7), which becomes

$$(11) \quad t = \int_0^L (1/p) \, d\omega.$$

Suppose, for example, that a trajectory has been computed for a projectile having a given initial velocity, D and t being tabulated against L. We assume that the trajectory is short and flat enough so that the assumptions (2.1, 3, 5) are adequate. Let a target have distance X from the muzzle and have angular altitude Φ above the horizontal line through the muzzle. We wish to find the angle of departure θ_0 needed to produce a hit. Let ε be the <u>superelevation</u>, that is

$$\varepsilon = \theta_0 - \Phi. \tag{12}$$

Then

$$L = X \cos \varepsilon + D \sin \theta_0. \tag{13}$$

Since the trajectory is flat, we may replace $\cos \varepsilon$ by 1, and in the small term $D \sin \theta_0$ we may replace the proper value D(L) by the approximately equal value D(X) and also replace θ_0 by Φ. Then (13) yields a rather accurate estimate of the value of L corresponding to X at angular altitude Φ. With this L we determine the corresponding D. Then by the law of sines

$$\sin \varepsilon = (D \cos \Phi)/L. \tag{14}$$

The D in this equation was determined to a second-order approximation, since the first approximation was D(X) and this was used with (13) and the tabulation of D against L to obtain the second approximation used in (14). However, this second approximation will not differ much, for reasonably flat trajectories, from the first approximation D(X), which was independent of Φ. So (14) will yield a solution $\sin \varepsilon$ which is very nearly proportional to the cosine of the angular altitude of the target, Φ.

This relation can be incorporated in the design of a gunsight for high-angle fire of small caliber weapons. Imagine, say, a machine gun with a bead-type front sight and a hinged leaf-type rear sight. We suppose that the hinge of the rear sight is at a distance b from the front sight, and that the line joining the

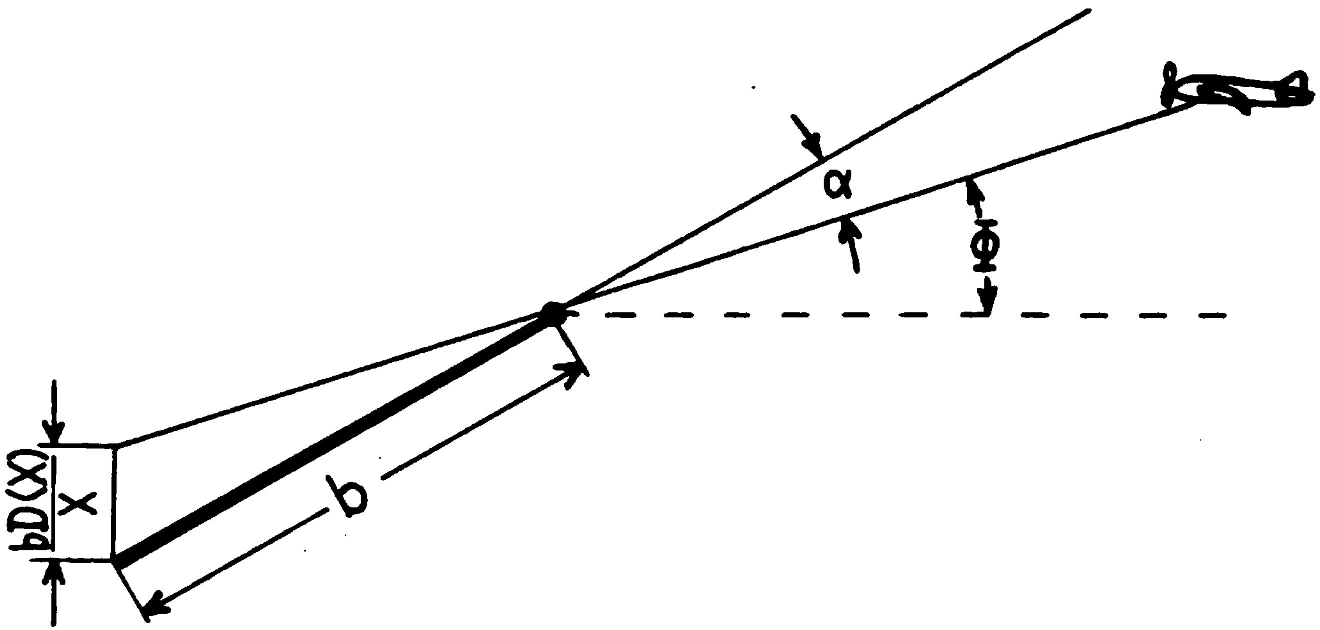

Figure V.5.1

hinge to the front sight is parallel to the bore of the gun. The point on the leaf at a distance bD(X)/X from the hinge is marked with the label "X"; it is this point which is to be collineated with the front sight and the target when the distance of the target is X. By means of a pantograph or a hanging weight or some other device, the leaf of the sight is forced to remain vertical, whatever the inclination of the gun barrel. From the figure we see that the super-elevation α of the gun barrel above the line of sight satisfies

$$\frac{\sin \alpha}{b\, D(X)/X} = \frac{\cos \Phi}{b},$$

whence we see that sin α satisfies the approximate form of (14) with X in place of L. Since this is a very good approximation at all ranges at which a machine gun is effective, the sight would be highly accurate. Such sights have been designed more or less independently by a number of ballisticians, and some have been constructed.

In order to test the accuracy of the method of this section it will be applied to the same trajectory as the other methods; that is, the drag function is the Gâvre, $C = C_s = 1$, $a = 1$, $v_o = 2700$ feet per second and $\theta_o = 45$ degrees. We again assume exponential density so that the left member of (8) is

$$\int_0^L \exp\,(-h \sin \theta_o L)\, dL$$

$$= \{1 - \exp\,(-h \sin \theta_o L)\}/h \sin \theta_o.$$

Since the numerator in the right member of this equation can be found from the table of H(y) the numerical quadrature can be avoided. However, in order to follow the pattern that would be used with other density laws, the integral was in fact computed by numerical quadrature. In the computation below, the integrand is tabulated in the column headed H/C, and its integral in the next column. As in the earlier computation, $S(p_o) = 2681.74$, from the S-table. The remaining values of S(p) are found by (8), which in this case takes the form

$$S(p) = S(p_o) + \int_0^L (H/C)\, dL.$$

The values of p are now found from the S-table and listed. Next its reciprocal 1/p is computed and listed, and by numerical quadrature we obtain (see (11)) the value of t. Next g/p^2 is computed; to avoid a multiplicity of zeros we list $10^6 g/p^2$ instead. A numerical quadrature furnishes us with dD/dL, as in (10); and a second numerical quadrature furnishes D. The second of equations (7) is easily transformed into

$$m = \tan \theta_o - (dD/dL) \sec \theta_o.$$

From this we compute

$$\sec \theta = \sqrt{1 + m^2},$$

and finally we compute v, as in Section 3, by the formula

$$v = p \cos \theta_o \sec \theta.$$

L (ft)	H/C	$\int_0^L (H/C)\,dL$ (ft)	S(p) (ft)	p (ft/sec)
0	1.00000	0.00	2681.74	2700.00
1000	0.97793	988.90	3670.64	2410.33
2000	0.95628	1956.00	4637.74	2144.68
4000	0.91456	3826.45	6508.19	1682.33
6000	0.87460	5615.41	8297.15	1322.89
8000	0.83642	7326.04	10007.78	1091.05
10000	0.79986	8962.17	11643.91	966.63
12000	0.76493	10526.57	13208.31	885.94
14000	0.73152	12022.91	14704.65	823.94
16000	0.69955	13453.61	16135.35	772.56
18000	0.66899	14822.04	17503.78	728.25
20000	0.63977	16130.46	18812.20	689.15

L (ft)	1/p (sec/ft)	t (sec)	$10^6 g/p^2$	dD/dL
0	0.0003704	0.0000	4.4104	0.000000
1000	0.0004149	0.3921	5.5342	0.004945
2000	0.0004663	0.8321	6.9901	0.011179
4000	0.0005944	1.8867	11.3603	0.029154
6000	0.0007559	3.2453	18.3723	0.058381
8000	0.0009165	4.9097	27.0098	0.103727
10000	0.0010345	6.8829	34.4104	0.165596
12000	0.0011287	9.0318	40.9639	0.240804
14000	0.0012137	11.3916	47.3608	0.329347
16000	0.0012944	13.8837	53.8698	0.430322
18000	0.0013731	16.5679	60.6247	0.544990
20000	0.0014511	19.3757	67.6990	0.673033

L (ft)	D (ft)	m	sec θ	v (ft/sec)
0	0.00	1.000000	1.41421	2700.00
1000	2.38	0.993001	1.40927	2401.91
2000	10.32	0.984190	1.40300	2127.68
4000	49.20	0.958770	1.38527	1647.91
6000	134.39	0.917438	1.35700	1269.38
8000	293.62	0.853308	1.31449	1014.12
10000	560.48	0.765812	1.25964	860.98
12000	964.69	0.659452	1.19782	750.38
14000	1532.71	0.534233	1.13373	660.53
16000	2200.21	0.391433	1.07310	586.66
18000	3263.27	0.229268	1.02594	528.31
20000	4478.93	0.048186	1.00116	487.87

Unless slopes are wanted, the computation can be somewhat simplified by using two formulas to be established in the next chapter. Nothing is altered in the columns preceding the one headed dD/dL. In this only the entries for L = 0 and L = 2000 appear; the latter is computed by Simpson's rule. The entry for D at L = 2000 is computed by (VI.3.12). From this stage on dD/dL is not needed. The second differences of D are computed by (VI.3.9), which in the present case states that the second difference of D on any line is the sum of 1/3 the second difference of $10^6 g/p^2$ on the same line and 4 times the value of $10^6 g/p^2$ on the preceding line. Thus for example on line L = 4000 we would have

$$\Delta^2 D = (1.7905/3) + 4(6.9901) = 28.5572,$$

while on line L = 6000,

$$\Delta^2 D = (2.6814/3) + 4(11.3603) = 46.3218.$$

The latter disagrees with the previous computation by about .01 ft., which is unimportant.

6. Successive approximations.

In the heyday of the Siacci method, at the end of the nineteenth century and the beginning of the twentieth, it was often applied at the margin of its domain of reliability. To help it out in such cases, an assortment of compensating factors was devised, some with a theoretical basis, others to reconcile the predictions with experimental firings. We shall not set forth any of these ad hoc corrections. When they are necessary, the Siacci method has begun to break down, and the best procedure is to abandon it and turn to one of the more accurate processes described in the following two chapters. However, it is true that a rather natural extension of the Siacci method can be made which yields a rapidly converging sequence of successive approximations to the trajectory. The effort involved in computing a trajectory to the maximum accuracy permitted by the fundamental tables would be rather more than twice as much as in computing the same trajectory by the method of Section 5.

From (2.4, 6) we have

(1) $$v \cos \theta = v_x = p \cos \theta_0.$$

Hence if we are using the rectangular (x, y)-coordinate system, from a knowledge of p and m we can find v,

(2) $$v = p \cos \theta_0 / \cos (\text{arc tan } m).$$

If we are using the oblique (L, D)-system, the slope m does not appear directly; instead, the computation process yields values of

(3) $$D' = dD/dL.$$

But from this we can readily find the value of m, since by (3.2)

(4) $$m = \tan \theta_0 - D' \sec \theta_0.$$

Let us suppose, to be specific, that we choose to use the oblique axis system. We make some sort of first approximation $Y_0(L)$ to the altitude which the projectile will have when its L-coordinate has value L, and as a first rough approximation to a(y) we take its value $a_0 = a(Y_m)$ at the origin. By (5.8, 9, 10, 11) we compute p, t, D' and D as functions of L. In order to proceed to the second approximation we transform equation (5.2) by substituting (5.3) and (3.2) and multiplying both members by $a\, G(p/a) / a_0 G(p/a_0)$. The result is

$$(5) \qquad \{ H(Y_m + y)\, G(v/a)\, a / C\, a_0\, G(p/a_0) \}\, dL = - \{ 1/a_0\, G(p/a_0) \}\, dp.$$

Integration yields

$$(6) \qquad \int_0^L \{ H(Y_m + y)\, a\, G/v/a(y)) / C\, a_0\, G(p/a_0) \}\, dL = S(p/a_0) - S(p_0/a_0).$$

We already have y approximated as a function of L, by the first stage of the solution, so we have approximate values of $a(Y_m + y)$ and $H(Y_m + y)$. Also, by (4) and (2) we have an estimate for v, so from the table of values of G or 1/G we can compute the value of the ratio $G(v/a(Y_m + y))/G(p/a_0)$. So the integrand in the left member of (6) is known to a higher precision than in the first approximation. We compute the integral by numerical quadrature, and by (6) determine a new approximation for p as a function of L. A repetition of the use of (5.9, 10, 11) yields improved estimates of t, D' and D. If desired the process can now be repeated to obtain a third approximation; the new values of D and D' give a closer estimate of the integral in the left member of (6), and this closer estimate furnishes a new estimate for p as a function of L.

In applying this method to a numerical example, it was found desirable to compute not the integral in (6), but the smaller amount by which this integral differs from the first estimate. For simplicity, we shall confine our attention to the case in which temperature effect is ignored, so that a is taken to be a constant along the trajectory. The first estimate for H is

$$H_1 = \exp\ [-h(L \sin \theta_o)];$$

the second approximation is

$$H_2 = \exp\ [-h(L \sin \theta_o - D)].$$

The first approximation to the left member of (6) was

$$\int_0^L C^{-1} \{ \exp[-h(L \sin \theta_o)]\} dL.$$

The sum of this and S(p(0)) gave a value of S which with the help of the S-table furnished the first approximation to p(L), which we shall call $p_1(L)$. That is, $p_1(L)$ was determined so that $S(p_1(L)) - S(p(0))$ was equal to the integral just above. The second approximation to the left member of (6) was

$$\int_0^L \{ \exp[-h(L \sin \theta_o - D)]\} \cdot \{ a\, G(v/a) \} \{ C\, a\, G(p_1(L)/a)\}^{-1} dL.$$

The sum of this and S(p(0)) gave a value of S which with the help of the S-table furnished the second approximation $p_2(L)$ to the correct value of p(L). That is, $p_2(L)$ was determined so that $S(p_2(L)) - S(p(0))$ was equal to the integral last written. So that $S(p_2(L)) - S(p_1(L))$ is equal to the difference between the two preceding integrals:

$$(\Delta S)_2 = S(p_2(L)) - S(p_1(L))$$

(7) $$= \int_0^L \{ \exp[-h(L \sin \theta_o)]\}\{ C\, G(p_1/a) \}^{-1} \cdot \{ (\exp hD)(G(v/a) - G(p_1/a) \} dL.$$

The second fraction in the integrand is a small quantity, and not many significant figures will be needed. After the integral is computed, it is then added to $S(p_1(L))$ to find $S(p_2(L))$, which in turn determines $p_2(L)$.

A similar modification was found desirable in passing from the second approximation to the third. The difference between second and third approximations can be investigated by a process like that which led us to (7), and which we shall not present in detail. The result is

$$(\Delta S)_3 = S(p_3(L)) - S(p_2(L))$$

$$(8) \quad = \int_0^L \{[\exp(hD_2)G(v_2) - G(p_2)]/G(p_2)$$

$$- [\exp(hD_1)G(v_1) - G(p_1)]/G(p_1)\}(H_1/C)\} dL.$$

In the preceding paragraphs we have taken the altitude of the muzzle to be 0. But as we have seen, when the exponential law of density is used this involves no loss of generality. The same method applies to any height of muzzle; as shown in Section 2 of the preceding chapter, all that is needed is to replace C by $C \exp(hY_m)$.

For flat trajectories the estimate (5.4) is a good one to start with. If, however, we should wish to use the method of successive approximations to compute a trajectory which departs considerably from flatness, as for example the complete trajectory from muzzle to ground of a projectile with angle of departure 45°, we could do better. For example, if we can form some rough estimate X^* of the range, for our first rough approximation we could assume the trajectory to be a parabola with slope 1 at the muzzle and intersecting the line $y = 0$ at $x = X^*$. This would yield

$$D = [\sin \theta_0 \cos \theta_0 / X^*] L^2,$$

or

$$(9) \qquad Y_0(L) = L \sin \theta_0 - L^2 [\sin \theta_0 \cos \theta_0 / X^*].$$

However, a still better technique is available. Suppose that the interval between successive values of L in the computation is λ, so that the tabular values of L are 0, λ, 2λ, ... The second approximation to t, etc., on the line $L = n\lambda$ is independent of the values of the first approximation on later lines of the computation. It is therefore possible to carry through the computation to the second approximation, and if desired to higher orders of approximation, on any given line before writing anything at all on the succeeding lines. Suppose that this has been done, and that the process has been carried out on line $L = (n - 1)\lambda$ until satisfactorily accurate values of t, D' and D have been reached. In the process the value of the integrand in the left member of (6) will also have been found with adequate accuracy. By inspecting the values of this integrand on line $L = (n - 1)\lambda$ and the preceding lines, we should be able to form a good guess as to the value which this integrand will have on line $L = n\lambda$. (Extrapolation will be discussed in more detail in Section 2 of Chapter VI). This guess is entered as a rough approximation, the value of the integrand computed by numerical quadrature, and the first approximation to p found by (6). In theory this process is not essentially different from that of using some a priori estimate like (9) or (5.4) for $Y_0(L)$. But by delaying the choice of the first rough estimate until the last possible moment, we are guided by the past behavior of the trajectory, and it is possible to make a more accurate selection of the first estimate.

By now our process has become one of numerical integration, and more properly belongs in the class of integration methods which will be discussed in Chapter VII. Of the original Siacci process nothing survives but the use of one of the four primary Siacci functions. We shall not pursue the subject further in this chapter, but shall leave for later investigation the question of the appropriate choice of methods of numerical integration, in particular whether the method here described is a good one for practical computation. It is however of some slight interest that a sequence of small changes and improvements has led us from the Siacci method of Section 2 to a method of numerical integration, having no limit to its mathematical accuracy except the number of significant figures carried and the errors in the numerical processes used, which are at the disposal of the computer.

In the application below to the same trajectory that has already been computed by other methods, the subscripts 1, 2, 3 refer to first, second and third approximations respectively. The first approximation is taken directly from the computation at the end of Section 5. The first five columns are self-explanatory. The column $d(\Delta S)_2/dL$ contains the values of the integrand of equation (7), computed from the preceding columns. The next column is obtained from this by quadrature, and the next by adding $(\Delta S)_2$ to the first approximation $S(p_1)$. The S-table furnishes the value $p_2(L)$ corresponding to this new estimate of S. From here on the computation of the second approximation is like that of the first approximation, in Section 5.

The third approximation follows closely the pattern of the second; the principal change is that immediately after $G(p_2)$ we tabulate the quantity

$$\{\exp(hD_2)\, G(v_2) - G(p_2)\}/G(p_2), \tag{10}$$

the column being headed merely (10) for brevity. From this we subtract the analogous quantity with subscripts 1, already computed in the second approximation; the difference is the integrand in (8), and the column is accordingly headed $d(\Delta S)_3/dL$. This function is now integrated by a numerical quadrature to obtain $(\Delta S)_3$, which is added to $S(p_2)$ to obtain $S(p_3)$.

SECOND APPROXIMATION

L (ft)	$\exp(hD_1)$	$G(v_1)$ (1/sec)	$G(v_1)\exp(hD_1)$ (1/sec)	$G(p_1)$ (1/sec)
0	1.00000	0.30271	0.30271	0.30271
1000	1.00008	0.28298	0.28300	0.28352
2000	1.00033	0.26472	0.26481	0.26592
4000	1.00155	0.22264	0.22298	0.22653
6000	1.00425	0.15867	0.15934	0.17146
8000	1.00932	0.07228	0.07295	0.09907
10000	1.01786	0.04212	0.04287	0.05974
12000	1.03093	0.03239	0.03339	0.04530
14000	1.04960	0.02746	0.02882	0.03820
16000	1.07196	0.02433	0.02608	0.03391
18000	1.10856	0.02216	0.02457	0.03101
20000	1.15195	0.02071	0.02386	0.02885

L (ft)	$d(\Delta S)_2/dL$	$(\Delta S)_2$ (ft)	$S(p_2)$ (ft)	p_2 (ft/sec)
0	0.000	0	2681.74	2700.00
1000	-0.002	-1	3669.64	2410.06
2000	-0.004	-4	4633.74	2145.74
4000	-0.014	-21	6487.19	1687.10
6000	-0.062	-87	8210.15	1337.94
8000	-0.220	-350	9657.78	1128.16
10000	-0.226	-822	10821.91	1021.25
12000	-0.201	-1254	11954.31	948.70
14000	-0.180	-1634	13070.65	892.24
16000	-0.162	-1976	14159.35	845.35
18000	-0.139	-2278	15225.78	804.51
20000	-0.111	-2528	16284.20	767.54

SECOND APPROXIMATION

L (ft)	$1/p_2$ (sec/ft)	t_2 (sec)	$10^6 g/p_2^2$	dD_2/dL
0	0.0003704	0.0000	4.4104	0.000000
1000	0.0004149	0.3921	5.5347	0.004946
2000	0.0004660	0.8335	6.9832	0.011314
4000	0.0005927	1.8847	11.2960	0.029093
6000	0.0007474	3.2229	17.9612	0.058066
8000	0.0008864	4.8639	25.2619	0.101361
10000	0.0009792	6.7377	30.8279	0.157957
12000	0.0010541	8.7687	35.7232	0.224226
14000	0.0011208	10.9487	40.3873	0.300696
16000	0.0011829	13.2439	44.9920	0.385735
18000	0.0012430	15.6789	49.6758	0.480717
20000	0.0013029	18.2207	54.5765	0.584583

L (ft)	D_2 (ft)	m_2	$\sec\theta_2$	v_2 (ft/sec)
0	0.00	1.000000	1.41421	2700.00
1000	2.38	0.993005	1.40928	2401.66
2000	10.46	0.984000	1.40295	2128.66
4000	49.43	0.958856	1.38543	1652.77
6000	134.37	0.917882	1.35739	1284.19
8000	291.36	0.856654	1.31676	1050.42
10000	548.82	0.776616	1.26615	914.33
12000	929.37	0.682897	1.21093	812.33
14000	1452.74	0.574753	1.15340	727.69
16000	2137.64	0.454490	1.09844	656.60
18000	3002.53	0.320165	1.05000	597.32
20000	4066.20	0.173277	1.01490	550.82

THIRD APPROXIMATION

L (ft)	$\exp(hD_2)$	$G(v_2)$ (1/sec)	$G(v_2)\exp(hD_2)$ (1/sec)	$G(p_2)$ (1/sec)
0	1.00000	0.30271	0.30271	0.30271
1000	1.00008	0.28296	0.28298	0.28350
2000	1.00033	0.26479	0.26488	0.26599
4000	1.00156	0.22320	0.22355	0.22705
6000	1.00425	0.16241	0.16310	0.17471
8000	1.00924	0.08408	0.08486	0.11328
10000	1.01748	0.04957	0.05044	0.07444
12000	1.02979	0.03713	0.03824	0.05587
14000	1.04695	0.03097	0.03242	0.04618
16000	1.06984	0.02727	0.02917	0.04036
18000	1.09947	0.02475	0.02721	0.03644
20000	1.13703	0.02298	0.02613	0.03356

L (ft)	(10)	$d(\Delta S)_3/dL$	$(\Delta S)_3$ (ft)	$S(p_3)$ (ft)	p_3 (ft/sec)
0	0.0000	0.0000	0.0	2681.7	2700.00
1000	-0.0018	0.0002	0.1	3669.7	2410.06
2000	-0.0040	0.0000	0.2	4633.9	2145.69
4000	-0.0141	-0.0001	0.1	6487.3	1687.07
6000	-0.0581	0.0039	3.9	8214.1	1337.25
8000	-0.2098	0.0102	18.0	9675.8	1126.14
10000	-0.2579	-0.0319	-3.7	10818.2	1012.52
12000	-0.2414	-0.0404	-76.0	11878.3	952.98
14000	-0.2180	-0.0380	-154.4	12916.3	899.45
16000	-0.1939	-0.0319	-224.3	13935.1	854.51
18000	-0.1694	-0.0304	-286.6	14939.2	815.08
20000	-0.1416	-0.0306	-347.6	15936.6	779.35

THIRD APPROXIMATION

L (ft)	$1/p_3$ (sec/ft)	t_3 (sec)	$10^6 g/p_3^2$	dD_3/dL
0	0.0003704	0.0000	4.4104	0.000000
1000	0.0004149	0.3921	5.5355	0.004946
2000	0.0004660	0.8328	6.9835	0.011291
4000	0.0005927	1.8847	11.2965	0.029094
6000	0.0007478	3.2225	17.9798	0.058057
8000	0.0008880	4.8660	25.3528	0.101473
10000	0.0009789	6.7417	30.8118	0.158192
12000	0.0010493	8.7679	35.4032	0.224142
14000	0.0011118	10.9336	39.7426	0.299637
16000	0.0011703	13.2125	44.0328	0.383079
18000	0.0012269	15.6135	48.3960	0.475817
20000	0.0012831	18.1198	52.9353	0.576781

L (ft)	D_3 (ft)	m_3	$\sec \theta_3$	v_3 (ft/sec)
0	0.00	1.000000	1.41421	2700.00
1000	2.38	0.993005	1.40928	2401.66
2000	10.33	0.984032	1.40297	2128.64
4000	49.28	0.958855	1.38543	1652.74
6000	134.20	0.917895	1.35740	1283.53
8000	291.27	0.856496	1.31666	1048.46
10000	549.12	0.776283	1.26594	914.42
12000	929.92	0.683016	1.21100	816.05
14000	1452.26	0.576250	1.15418	734.07
16000	2133.54	0.458246	1.10000	664.66
18000	2990.99	0.327095	1.05214	606.46
20000	4042.07	0.184311	1.01684	560.37

7. Interpolation between anti-aircraft trajectories.

When it becomes necessary to compute a family of trajectories as an auxiliary in the preparation of a firing table, it is usually desirable to organize the work in such a way that the desired data can be obtained from the smallest number of trajectories. One helpful precaution is to choose the parameters involved in the family of trajectories in such a way that the desired elements of the trajectory are nearly linear in the parameters. For example, in the ballistic tables used as a basis for bombing tables at Aberdeen Proving Ground, the range, time of flight, etc., are tabulated against reciprocal ballistic coefficient γ instead of against C. The reason is that these quantities are much more nearly linear in γ than in C, so that linear interpolation is possible with fewer entries in the table.

If we are about to compute a family of anti-aircraft trajectories on which to base a firing table, we know already from Section 3 that the oblique coordinates L and D might well be good ones to use. For in the simple Siacci theory D appears as a function of L, independent of the angle of departure. So we can feel sure that in a more precise method, the dependence of D on angle of departure is not a sensitive one, and this should help in interpolation between computed trajectories. We shall now show, with the help of the results of the preceding section, that the angle of departure itself is not a particularly good choice as parameter; it is better to regard D as a function of L and sin θ_o rather than as a function of L and θ_o. Of course either point of view is equally sound logically. But we shall show that for fixed L, the drop D is nearly linear as a function of sin θ_o, and therefore not very nearly linear as a function of θ_o.

Suppose that for the given shell, with its given initial velocity v_o, the trajectory with angle of departure $\theta_o = 0$ has been computed to a satisfactory degree of accuracy. This could be done by the method of the preceding section or by any other method; this point is of no importance. The functions $D'(L)$ and $D(L)$ thus determined make a good first approximation, which we shall use as a starting point to calculate a second approximation, for a trajectory with the same initial velocity but with angle of departure θ_o different from 0.

Since in first approximation we have

$$y = L \sin \theta_o - D(L),$$

the $D(L)$ being that determined from the trajectory already computed, our estimate of H will be

$$H(Y_m + y) = \exp \{ - h(Y_m - D(L)) \} \exp(- hL \sin \theta_o),$$

and unless L is very large this can be expanded in a rapidly convergent power series, the term involving $\sin \theta_o$ being quite small. By (6.2, 4) we find

(1) $$v = p \sqrt{1 - 2D' \sin \theta_o + D'^2},$$

and, if the trajectory is fairly flat, D'^2 will be much less than 1. So by the binomial theorem we find that v/p is nearly equal to one plus a small multiple of $\sin \theta_o$. By the same argument as was applied to H, we see that $a(Y_m + y)$ differs from $a(Y_m)$ by a small amount which is nearly proportional to $\sin \theta_o$. Hence for each L, $v/a(Y_m + y)$ differs from p/a_o by an amount nearly proportional to $\sin \theta_o$. Following through the process of computing the second approximation, the integral in the left member of (6.6) changes from the first estimate by an amount nearly proportional to $\sin \theta_o$, for each L; so does the corresponding value of p, and by the binomial theorem so do $1/p$ and $1/p^2$; and finally so do the second approximations to t, D' and D.

This approximate proportionality to $\sin \theta_o$ is a property of the trajectories, and has nothing to do with the particular method used to compute them. The successive approximations method is one in which this proportionality is easy to predict from the formulas used; but in any other computation method yielding accurate results the approximate proportionality will be evident in the final results. Thus, however we choose to compute the trajectories, if we use the oblique (L, D)-axis system and prepare trajectories at equal intervals of sin θ_0, a relatively small collection of these trajectories will enable us to interpolate to find the values of t, D' and D corresponding to a given L and to an angle of departure θ_0 between those for which the trajectories were computed.

8. The method of Euler.

In every age since the invention of artillery there has been some need for weapons delivering plunging fire on an enemy who cannot be reached by flat fire because of obstacles between gun and enemy. The weapons designed for such purposes are usually characterized by low velocity and high angle of departure. Trench mortars furnish an important example. Since the velocity of the projectile is low, its trajectory will not be very long, and therefore cannot extend through a great depth of atmosphere. Therefore it is possible to assume constant atmospheric density in computing the trajectories. However, the trajectories are highly curved, and the Siacci method is therefore inappropriate.

As long as the velocity of the projectile is well below that of sound, the drag coefficient K_D is a slowly varying function of the Mach number, and it is a good approximation to take the drag to be proportional to the square of the velocity of the projectile. If as usual we define γ to be the reciprocal of the ballistic coefficient C, and assume the relative air density H to be constantly equal to its value $H(Y_m)$ at the height of the muzzle, then by either (IV.1.10, 12) or (IV.1.16, 17) we find

(1) $$\gamma a G(v/a) = (d^2/m)\rho^* K_D v,$$

and by this and (IV.1.24)

(2) $$\begin{aligned} E &= [H(Y_m) d^2 \rho^* K_D/m] v \\ &= k_s v, \end{aligned}$$

where k_s is merely an abbreviation for the quantity in the square bracket in the preceding line. If we substitute this in equations (1.11), recalling that $v = v_x \sec \theta$, we find

(3) $$\begin{aligned} dt/d\theta &= - v_x \sec^2\theta/g, \\ dx/d\theta &= - v_x{}^2 \sec^2\theta/g, \\ dy/d\theta &= - v_x{}^2 \sec^2\theta \tan \theta/g, \\ dv_x/d\theta &= k_s v_x{}^3 \sec^3\theta/g. \end{aligned}$$

The last of these equations is separable, and its solution is easily expressed in terms of the function

(4) $$\begin{aligned} \xi(\theta) &= \int_0^\theta \sec^3\theta \, d\theta \\ &= \tfrac{1}{2} \{ \tan \theta \sec \theta \\ &\qquad + \log \tan (\tfrac{1}{2}\theta + \tfrac{1}{4}\pi) \} \end{aligned}$$

wherein angles are supposed to be expressed in radians and the logarithm is a natural logarithm. From the last of equations (3),

(5) $$v_x{}^{-3} \, dv_x = (k_s/g) \sec^3\theta \, d\theta,$$

whence, by integration from the initial point, where v_x and θ have the values of v_{xo} and θ_o respectively, to a later point, where they have the values v_x and θ, we find

(6) $$v_x{}^{-2} - v_{xo}{}^{-2} = - (2k_s/g) \{ \xi(\theta) - \xi(\theta_o) \} .$$

This can be written

(7) $$v_x^{-2} = (2k_s/g) \{ K - \xi(\theta) \},$$

where

(8) $$K = \xi(\theta_o) + (g/2k_s)v_{xo}^{-2}.$$

We assume as usual that t, x and y are all zero at the beginning of the trajectory, where θ has the value θ_o. If we substitute (7) in the first three of equations (3) and integrate, we find

$$\sqrt{2k_s g}\, t = -\int_{\theta_o}^{\theta} \sec^2\theta \{ K - \xi(\theta) \}^{-1/2} d\theta,$$

(9) $$2k_s x = -\int_{\theta_o}^{\theta} \sec^2\theta \{ K - \xi(\theta) \}^{-1} d\theta,$$

$$2k_s y = -\int_{\theta_o}^{\theta} \sec^2\theta \tan\theta \{ K - \xi(\theta) \}^{-1} d\theta.$$

It is interesting to note that the curves defined by equations (9) tend to asymptotes in both directions. Consider first the direction of increasing t, or decreasing θ. If we temporarily denote the integrand in the second of equations (9) by $\{ r(\theta) \}^2$, we see by (4) that $r(\theta)$ tends to 2 as θ approaches $-\pi/2$. Therefore x approaches a finite limit as θ tends to $-\pi/2$. But the integrand in the first of equations (9) is $r(\theta) \sec\theta$, and that in the last is $\{ r(\theta) \}^2 \tan\theta$. For θ in the neighborhood of $-\pi/2$ these are respectively greater than $(\sec\theta)/2$ and less than $-|\tan\theta|/2$, and therefore the integrals tend to $+\infty$ and to $-\infty$ respectively. So x cannot increase beyond bounds, and as t increases, x approaches a finite limit, while y tends to $-\infty$ and θ tends to $-\pi/2$. The curve thus has a vertical asymptote in the direction of increasing t.

As θ approaches $\pi/2$, the value of $\xi(\theta)$ tends to $+\infty$, and as θ approaches $-\pi/2$, the value of $\xi(\theta)$ tends to $-\infty$. So there exists an angle θ_a such that

$$K = \xi(\theta_a). \tag{10}$$

As θ approaches θ_a, the denominators of all three integrands in (9) tend to 0, while the numerators of the first two tend to non-zero limits, as does the numerator of the third integrand unless θ_a happens to be 0. Hence as θ approaches θ_a, both x and t approach $-\infty$, while y approaches $-\infty$ if $\theta_a > 0$ and approaches $+\infty$ if $\theta_a < 0$. In any case, the point (x, y) recedes unboundedly from the origin as θ tends to θ_a and t to $-\infty$. Consider now the point Q at which the y-axis is intersected by the line which passes through (x, y) and has inclination θ_a. The ordinate of this point is $y - x \tan \theta_a$, which by (9) and (10) is equal to

$$(1/2k_s)\int_{\theta_0}^{\theta} \sec^2\theta\{\tan \theta_a - \tan \theta\}\{\xi(\theta_a) - \xi(\theta)\}^{-1}d\theta. \tag{11}$$

As θ approaches θ_a, both numerator and denominator of the integrand tend to 0, and by de l'Hospital's rule the integrand approaches $\sec \theta_a$. Therefore the integral approaches a finite limit, and the point Q approaches a limiting position Q_a. The line with inclination θ_a is thus an asymptote of the trajectory as t tends to $-\infty$.

A table in form convenient for use could be prepared with the help of equations (9) as follows. Let us first define

$$T(\theta, K) = \int_0^{\theta} \sec^2\theta \{K - \xi(\theta)\}^{-\frac{1}{2}} d\theta,$$

$$X(\theta, K) = \int_0^{\theta} \sec^2\theta \{K - \xi(\theta)\}^{-1} d\theta, \tag{12}$$

$$Y(\theta, K) = \int_0^{\theta} \sec^2\theta \tan \theta \{K - \xi(\theta)\}^{-1} d\theta.$$

By numerical quadrature, we can prepare a double-entry table of values of these functions, corresponding to some chosen collection of values of K and to equally spaced values of θ between θ_a and $-\pi/2$. We next select values of θ_0, and for each of these values we proceed to tabulate the relevant ballistic data. First we must find the values of θ corresponding to standard ground impact, that is the value of θ for which y returns to zero. Each value of K furnishes a line of the table. By (9) and (12), the value of θ for standard ground impact is that for which

$$(13) \qquad Y(\theta, K) - Y(\theta_0, K) = 0.$$

The negative of this θ is tabulated as the angle of impact, with the designation ω. As soon as ω is determined, we can find and tabulate the quantities

$$(14) \qquad \begin{aligned} 2k_s X &= X(\theta_0, K) - X(-\omega, K), \\ \sqrt{2k_s g}\, T &= T(\theta_0, K) - T(-\omega, K), \end{aligned}$$

where X is the range and T the time of flight. Or instead of the latter we could tabulate

$$(15) \qquad T\sqrt{g/X} = \{T(\theta_0, K) - T(-\omega, K)\}\{X(\theta_0, K) - X(-\omega, K)\}^{-\frac{1}{2}}.$$

From (7) we find that

$$(16) \qquad 2k_s v^2/g = \sec^2\theta / \{ K - \xi(\theta) \},$$

and by setting first $\theta = \theta_0$ and second $\theta = -\omega$ we find $2k_s v_0^2/g$ and $2k_s v_\omega^2/g$, and also the ratio v_ω/v_0, where v_ω is the striking velocity. The height y_s of the summit is found from the last of equations (9) with $\theta = 0$. From this and (12),

$$(17) \qquad y_s/X = Y(\theta_0, K)/\{ X(\theta_0, K) - X(-\omega, K) \}.$$

It is worth noticing that the quantities specified above are all dimensionless, so that such tables are equally useful with any system of units. A table of the above quantities was prepared by J. C. F. Otto, and revised by Lardillon. Cranz presents a table in his Lehrbuch der Ballistik, Vol. I, Aussere Ballistik, 2nd ed. (Berlin: Julius Springer, 1925; photo-lithoprint reproduction, Ann Arbor, Mich.: Edwards Brothers, Inc., 1943); but the 5° interval in θ_0 is not very convenient for use, and there are errors in the table.

9. Approximate formulas for dive-bombing.

In the Siacci method of approximation, as in Section 3 of this chapter, an approximation to a solution was obtained under the assumptions that the trajectory was flat (i.e., had small curvature) and the atmospheric density nearly constant along it. In the Euler method, the simplifying approximations were that the trajectory, though curved, lay in regions of nearly constant atmospheric density, and that K_D is nearly constant. The trajectories encountered in problems of dive-bombing satisfy part of one of these sets of conditions and part of the other, so that neither method can be satisfactorily applied. For the trajectory, being steep, has little curvature. In fact, the aircraft used in the Second World War had a very restricted forward vision; in most of them it was impossible for the pilot to see more than eight degrees below the line of flight. Thus the whole trajectory would have to lie in an angular sector between the line of flight and a line eight degrees below it. Moreover, the speed of the bomb is well below that of sound, so that K_D can reasonably be taken to be approximately constant. But the assumption of nearly constant atmospheric density is a very inappropriate one to make, since dive-bombing may take place from as much as ten thousand feet. Thus if we wish to profit by the approximate constancy of K_D and the flatness of the trajectory, it behooves us to devise a new method of approximation.

It will be convenient to use oblique coordinates in the following computations. As in Section 3 of this chapter, the L-axis is tangent to the trajectory at its initial point, and the D-axis is vertical, with the positive direction downwards. The initial tangent to the trajectory, the line of flight of the aircraft, makes an angle θ_o with the horizontal. (θ_o will be negative for the problem of dive-bombing.) Using primes to denote differentiation with respect to L, the equations (3.7) which use L as independent variable can be written:

$$(1) \qquad \begin{aligned} t' &= 1/v_L, \\ D'' &= g/v_L^2, \\ v_L' &= -E = -\rho d^2 v K_D/m^*, \end{aligned}$$

where m^* is the mass of the projectile. Referring to equations (3.1) or (7.1) we can express v in terms of v_L and D' as follows:

$$(2) \qquad \begin{aligned} v^2 &= (dx/dt)^2 + (dy/dt)^2 \\ &= \{(x')^2 + (y')^2\} v_L^2 \\ &= (1 - 2D' \sin \theta_o + D'^2)v_L^2. \end{aligned}$$

The <u>resistance</u> <u>coefficient</u> is defined to be

$$(3) \qquad k_o = \rho_o d^2 K_D/m^*,$$

where ρ_o is the density at sea-level. The resistance coefficient at release is accordingly defined

$$k_r = \rho_r d^2 K_D/m^*,$$

where ρr is the density at the altitude of release. the equations (1) can be written in terms of k_r, assuming the usual exponential law for change of density with altitude, in the following form:

$$(4) \qquad \begin{aligned} t' &= 1/v_L, \\ D'' &= g/v_L^2, \\ v_L' &= -k_r v\, e^{h(D-L\sin\theta_o)} \\ &= -k_r v_L(1 - 2D' \sin \theta_o + D'^2)^{\frac{1}{2}}\, e^{h(D-L\sin\theta_o)}. \end{aligned}$$

The initial conditions for the differential equations are, when L = 0,

$$t = D = D' = 0, \; v_L = v_o.$$

We are going to expand t and D in a Taylor series in L. For convenience, v_L will first be eliminated from the equations (4). The system (4) then becomes

$$(5)\quad \begin{aligned} D'' &= gt'^2, \\ t'' &= k_r t'(1 - 2D' \sin \theta_o + D'^2)^{1/2} e^{h(D-L\sin\theta_o)}, \end{aligned}$$

with initial conditions

$$L = 0,\ D = D' = 0,\quad t = 0,\ t' = 1/v_o.$$

Differentiating equations (5), the first equation twice and the second once, gives:

$$(6)\quad \begin{aligned} D''' &= 2gt't'', \\ D^{iv} &= 2gt''^2 + 2gt't''', \\ t''' &= k_r t''(v/v_L)e^{h(D-L\sin\theta_o)} \\ &\quad - k_r t'D''\sin \theta_o (v/v_L)^{-1} e^{h(D-L\sin\theta_o)} \\ &\quad + k_r t'D'D''(v/v_L)^{-1} e^{h(D-L\sin\theta_o)} \\ &\quad + k_r t'h(D' - \sin \theta_o)(v/v_L)e^{h(D-L\sin\theta_o)}, \end{aligned}$$

where the ratio v/v_L in the last equation is determined in terms of v_L, D' and θ_o by (2):

$$v/v_L = (1 - 2D' \sin \theta_o + D'^2)^{1/2}.$$

The equations (6) will be employed only in obtaining initial values of the third- and fourth-order derivatives of D and T which are needed to form the corresponding Taylor-MacLaurin expansions in powers of L. By using the initial conditions and then equations (5) and (6) successively, it is possible to evaluate the successive derivatives of D and t at L = 0. Thus writing in successive lines the computed values:

at $L = 0$,

$$t = 0, \qquad D = 0,$$

$$t' = 1/v_o, \qquad D' = 0,$$

(7) $$t'' = k_r/v_o, \quad D'' = g/v_o^2,$$

$$t''' = \{ k - (g \sin \theta_o/v_o^2) - h \sin \theta_o\} k_r/v_o,$$

$$D''' = 2gk_r/v_o^2,$$

$$D'^{v} = \{2k_r - (g \sin \theta_o/v_o^2) - h \sin \theta_o\} 2gk_r/v_o^2.$$

The Taylor-MacLaurin expansions can now be written explicitly. We recall that for a function f of L the expansion is

$$f(L) = f(0) + Lf'(0) + L^2 f''(0)/2! + L^3 f'''(0)/3! + \ldots.$$

Applying this formula to the functions D and t then gives the following:

(8) $$\begin{aligned} D &= gL^2/2v_o^2 + gk_r L^3/3v_o^2 \\ &\quad + (gk_r L^4/12v_o^2)(2k_r - g \sin \theta_o/v_o^2 - h \sin \theta_o) + \ldots, \\ t &= L/v_o + k_r L^2/2v_o \\ &\quad + (k_r L^3/6v_o)(k_r - g \sin \theta_o/v_o^2 - h \sin \theta_o) + \ldots. \end{aligned}$$

In each case* the expansion is carried out to the first terms that include θ_o explicitly. As might be expected, since the approximation for D is of fourth order, the

---

*If k is considered to be a function of v, the formula for D is:

$$\begin{aligned} D &= gL^2/2v_o^2 + gk_r L^3/3v_o^2 \\ &\quad + (gk_r L^4/12v_o^2) \{ 2k_r - \sin \theta_o (h + g/v_o^2) \} \\ &\quad - (g^2 L^4/12v_o^2) \sin \theta_o kr(dk_r/dv) + \ldots. \end{aligned}$$

In this form the formula is a generalization of the Piton-Bressant formula used by the Gâvre commission in preparation of a resistance coefficient.

formula for D is useful over a much wider range than that for t. The formula for t is essentially that used in computation of drag coefficient from firings in the spark range. The approximation for D is extremely useful for dive-bombing. For bombs of weight between 100 and 1000 pounds an "average" value for the complicated third term is $(gk_rL^4/12v_o^2)(1.50 \cdot 10^{-4})$ if the units used are feet and seconds. This gives the further approximation,

(9) $D = (gL^2/2v_o^2)(1 + 2k_rL/3 + k_rL^2/4 \cdot 10^{-4})$.

In many cases it is actually adequate to simplify this further to

(10) $D = (gL/2v_o^2)(1 + 2k_r/3)$.

The following table gives an indication of the range of validity of the formulas (9) and (10) for D.

Alt.	Speed	Dive Angle	Bomb, 100 Lb. Error Using (9)	Bomb, 100 Lb. Error Using (10)	Bomb, 500 Lb. Error Using (9)	Bomb, 500 Lb. Error Using (10)
(ft)	(mph)	(deg)	(ft)	(ft)	(ft)	(ft)
1000	200	40	0	0		
2000	200	40	1	4		
2700	200	40	3	9		
3000	200	40			3	2
4000	200	70	1	6	1	1
7000	200	70	8	18	1	1
10000	200	70	24	55	3	11
4000	450	40	4	4	1	2*
7000	450	40	29	33	4	8*
10000	450	40	112	122	11	14*
4000	450	70	1	1	0	1*
7000	450	70	5	7	1	2*
10000	450	70	17	27	2	1*

*All errors are short in range except * entries which are over.

The approximation (10), and its companion formulas for t, $t = (1 + k_r L/2)L/v_0$, have been used also in a mathematical way. A large number of trajectories for dive-bombing have been computed by numerical integration for a range of dive angle, speed and k_r. However, the tabulation of these data in a useful form was extremely difficult, because of the large number of arguments and the rapid variation of D and t with θ_0, v_0 and k_r. On the other hand, the differences between the actual values D and t and the approximations given by these formulas varied much more slowly with variation in the arguments. Thus, these differences, or errors in the formulas, proved to be convenient functions for systematic tabulation.

10. Comparison of accuracy of methods.

The methods of approximate solution of the normal equations described in Sections 3, 5 and 6 and the two approximations to the drop given in Section 4 have been applied to a trajectory based on the Gâvre drag function, for a projectile with ballistic coefficient 1 fired at initial velocity 2700 feet per second, and angle of departure 45°. It is assumed that the relative sound velocity is 1 at all altitudes. In order to have a standard of comparison with a satisfactory degree of accuracy, the same trajectory was calculated by a modification of the method of Moulton, and also by a numerical integration method to be explained in the next chapter. Each of these methods is inherently very accurate. The results indicated that a high degree of accuracy was in fact attained by both methods, since for values of L up to 20,000 feet the altitudes as computed by the two methods differed by at most half a foot and the times of flight differed by at most .001 sec.

The quantities compared are the pseudo-velocity p, the velocity v, the slope m, the time t and the drop D. The last two are of course ballistically the most important, but the others are not without interest. In the case of the drop D the approximations by the methods of Section 4 are included; the corresponding columns are of course missing from the other four tables. The abbreviations at the heads of the columns have the following meanings:

Mod. M	Modification of F. R. Moulton's method.
Ch. VI	Numerical integration method of Chapter VI.
Siacci	Siacci method as in Section 3.
H.-K.	Hitchcock-Kent method as in Section 5.
2nd Ap.	Second approximation by method of Section 6.
3rd Ap.	Third approximation by method of Section 6.
By ϕ_2	Drop as approximated by use of ϕ_2, Section 4.
By $\phi_{3/2}$	Drop as approximated by use of $\phi_{3/2}$, Section 4.

p
(ft/sec)

L (ft)	Mod. M.	Ch. VI	Siacci	H.-K.	2nd Ap.	3rd Ap.
0	2700.0	2700.0	2700.0	2700.0	2700.0	2700.0
1000	2410.6	2410.6	2407.2	2410.3	2410.1	2410.1
2000	2145.7	2145.7	2133.0	2144.7	2145.7	2145.7
4000	1686.8	1686.8	1643.4	1682.3	1687.1	1687.1
6000	1336.7	1336.8	1259.8	1322.9	1337.9	1337.4
8000	1127.1	1127.0	1031.9	1091.1	1128.2	1127.4
10000	1021.9	1021.8	910.7	966.6	1021.3	1021.0
12000	953.4	953.4	824.8	885.9	948.7	952.9
14000	900.3	900.4	754.4	823.9	892.2	899.4
16000	855.8	855.8	692.9	772.6	845.4	854.5
18000	816.7	816.7	637.6	728.3	804.5	815.0
20000	781.2	781.3	586.9	689.2	767.5	779.3

v
(ft/sec)

L (ft)	Mod. M.	Ch. VI	Siacci	H.-K.	2nd Ap.	3rd Ap.
0	2700.0	2700.0	2700.0	2700.0	2700.0	2700.0
1000	2402.1	2402.2	2398.8	2401.9	2401.7	2401.7
2000	2128.8	2128.8	2115.8	2127.7	2128.7	2128.7
4000	1652.4	1652.4	1609.2	1647.9	1652.8	1652.8
6000	1283.1	1283.2	1206.7	1269.4	1284.2	1283.7
8000	1049.2	1049.2	953.7	1014.1	1050.4	1049.6
10000	915.0	914.9	802.7	861.0	914.3	914.1
12000	816.4	816.4	687.0	750.4	812.3	816.0
14000	734.9	735.0	590.8	660.5	727.7	734.1
16000	665.7	665.8	511.8	586.7	656.6	664.6
18000	607.8	607.9	452.9	528.3	597.3	606.4
20000	561.9	562.0	419.5	487.9	550.8	560.3

m

L (ft)	Mod. M.	Ch. VI	Siacci	H.-K.	2nd Ap.	3rd Ap.
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1000	0.9930	0.9930	0.9930	0.9930	0.9930	0.9930
2000	0.9842	0.9842	0.9841	0.9842	0.9840	0.9840
4000	0.9588	0.9588	0.9580	0.9588	0.9589	0.9589
6000	0.9180	0.9180	0.9134	0.9174	0.9179	0.9179
8000	0.8564	0.8564	0.8417	0.8533	0.8566	0.8566
10000	0.7767	0.7767	0.7440	0.7658	0.7766	0.7764
12000	0.6830	0.6831	0.6224	0.6595	0.6829	0.6830
14000	0.5769	0.5769	0.4759	0.5342	0.5748	0.5764
16000	0.4587	0.4589	0.3016	0.3914	0.4545	0.4582
18000	0.3284	0.3286	0.0954	0.2293	0.3202	0.3272
20000	0.1858	0.1859	-0.1479	0.0482	0.1733	0.1843

t
(sec)

L (ft)	Mod. M.	Ch. VI	Siacci	H.-K.	2nd Ap.	3rd Ap.
0	0.000	0.000	0.000	0.000	0.000	0.000
1000	0.392	0.392	0.397	0.392	0.392	0.392
2000	0.832	0.832	0.839	0.832	0.833	0.833
4000	1.885	1.885	1.909	1.887	1.885	1.885
6000	3.223	3.223	3.305	3.245	3.223	3.223
8000	4.867	4.867	5.076	4.910	4.864	4.865
10000	6.738	6.739	7.149	6.883	6.738	6.740
12000	8.768	8.768	9.460	9.032	8.769	8.768
14000	10.928	10.929	11.997	11.392	10.949	10.932
16000	13.208	13.208	14.765	13.884	13.249	13.212
18000	15.602	15.601	17.775	16.568	15.679	15.612
20000	18.106	18.106	21.046	19.376	18.221	18.120

L (ft)	D (ft) Mod. M.	Ch. VI	Siacci	H.-K.
0	0.0	0.0	0.0	0.0
1000	2.4	2.4	2.4	2.4
2000	10.3	10.3	10.3	10.3
4000	49.2	49.2	49.7	49.2
6000	134.1	134.0	137.8	134.4
8000	291.4	291.0	307.6	293.6
10000	548.8	548.7	597.7	560.5
12000	929.2	929.1	1042.9	964.7
14000	1451.4	1450.9	1677.4	1532.7
16000	2131.6	2131.2	2538.4	2200.2
18000	2987.9	2987.4	3667.8	3263.3
20000	4036.9	4036.5	5114.3	4478.9

L (ft)	D (ft) 2nd Ap.	3rd Ap.	By ϕ_2	By $\phi_{3/2}$
0	0.0	0.0	0.0	0.0
1000	2.4	2.4	2.4	2.4
2000	10.5	10.4	10.4	10.4
4000	49.4	49.4	49.2	49.1
6000	134.4	134.3	134.4	134.0
8000	291.4	291.4	292.2	290.7
10000	548.8	549.1	546.6	542.8
12000	929.4	929.7	909.0	901.4
14000	1452.7	1452.0	1393.4	1379.8
16000	2137.6	2133.2	2010.4	1987.9
18000	3002.5	2990.6	2776.1	2741.4
20000	4066.2	4041.6	3701.1	3649.9

Chapter VI

NUMERICAL INTEGRATION OF DIFFERENTIAL EQUATIONS

1. Notation.

Although the normal equations of the trajectory involve the second derivatives of the coordinates, as in (IV.1.17), it is possible to rewrite them so that only first derivatives appear. In fact, we have done this in (V.1.3); and in general, equations involving derivatives of order higher than the first can always be replaced by systems of equations involving only first-order derivatives, by the simple expedient of introducing new symbols to stand for the first, second, ..., derivatives, up to but not including the derivative of highest order. Thus any system of ordinary differential equations can be transformed into a system of equations, each of which expresses the first derivative of one of the dependent variables as a function of the independent variable and all the dependent variables. If we choose to represent the independent variable by x and the dependent variables by $y_1, \ldots, y_n$, the equations have the form

$$\begin{aligned} y_1' &= f_1(x, y_1, \ldots, y_n), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ y_n' &= f_n(x, y_1, \ldots, y_n). \end{aligned} \tag{1}$$

A set of functions $y_1(x), \ldots, y_n(x)$, defined on an interval $x_1 \leq x \leq x_2$, is a solution of these equations if the functions $y_i(x)$ have derivatives for each x in the interval, and equations (1) are identically satisfied if $y_1, \ldots, y_n$ are replaced by the functions $y_1(x), \ldots, y_n(x)$ respectively. We shall use the standing assumption that the functions $f_i(x, y_1, \ldots, y_n)$ are continuous, although as a matter of fact it would cause no trouble to admit the possibility of the existence of a finite number of values of x at which the functions f_i have simple jump discontinuities.

Both for theoretical investigation and for numerical computation, it is preferable to replace equations (1) by their integrated form. Let ξ be a number lying in the interval $x_1 < x < x_2$ on which the solution is sought, and let $\eta_1, \ldots, \eta_n$ be n numbers. If the functions $y_i(x)$ are a solution of (1) and at $x = \xi$ the functions $y_i(x)$ have the values η_i, then by integration of both members of (1) we find

$$
(2) \qquad
\begin{aligned}
y_1(x) &= \eta_1 + \int_{\xi}^{x} f_1(x, y_1(x), \ldots, y_n(x))\,dx, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
y_n(x) &= \eta_n + \int_{\xi}^{x} f_n(x, y_1(x), \ldots, y_n(x))\,dx.
\end{aligned}
$$

A glance at this equation will show that if we wish to solve it by numerical methods, we must have available formulas for computing the definite integral of a function. The main purpose of this section and the next two is to establish such formulas, together with allied formulas for interpolation between tabulated values of a function. For this it is desirable to introduce a certain amount of symbolism.

All the tables most commonly used in ballistics are of the type in which there is a constant difference between successive values of the independent variable (also called the argument). If a is one of these values, and ω is the constant difference between successive values of the argument, the values against which the function is tabulated will be

$$x_{-1} = a - \omega;\ x_0 = a;\ x_1 = a + \omega,\ \ldots;$$

(3) $$x_n = a + n\omega;\ \ldots.$$

The functional value corresponding to x_n will be denoted by $f(x_n)$, or $f(a + n\omega)$, or f_n, interchangeably.

Given a tabular interval ω, to each function f corresponds a new function, its first difference Δf, defined by the equation

(4) $$\Delta f(x) = f(x) - f(x - \omega).$$

This notation is not quite the same as that in most books on finite differences. Usually the quantity in the right member of (4) is denoted by an inverted delta, and the symbol we have used has a different meaning attached to it (namely, $f(x + \omega) - f(x)$). But the difference defined in (4) is sufficient for our needs in ballistics; we do not need to distinguish between two concepts, one of which we shall not use, and so we prefer the simpler symbol for the one which will be used.

The first difference of f is itself a function of x, and so it has a first difference, which is called the second difference of f; and this in turn has a first difference, which is the third difference of f; and so on. In symbols,

(5) $$\begin{aligned} \Delta^2 f(x) &= \Delta f(x) - \Delta f(x - \omega) \\ &= f(x) - 2f(x - \omega) + f(x - 2\omega), \\ \Delta^3 f(x) &= \Delta^2 f(x) - \Delta^2 f(x - \omega), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \Delta^n f(x) &= \Delta^{n-1} f(x) - \Delta^{n-1} f(x - \omega), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

This list of equations suggests introducing the alternative symbols:

(6) $$\Delta^1 f(x) = \Delta f(x); \quad \Delta^0 f(x) = f(x),$$

which we shall use whenever convenient. With this notation the last of equations (5) holds for all positive integers n.

The customary method of writing out these successive differences is exemplified in the following table:

x	f(x)	Δ^1	Δ^2	Δ^3
1	-6			
2	-1	5		
3	-2	-1	-6	
4	-3	-1	0	6
5	2	5	6	6
6	19	17	12	6

A simple but nevertheless important property of these difference operators is expressed in the equations:

(7) $$\Delta^n[af(x)] = a\Delta^n f(x),$$
$$\Delta^n[f(x) + g(x)] = \Delta^n f(x) + \Delta^n g(x),$$

where a is an arbitrary constant.

The proofs in the next section can be rendered more compact by the use of a sequence of polynomials $Q_n(x)$, which we now define:

(8) $$Q_k(x) = x(x+\omega)\ldots(x+[k-1]\omega)/\omega^k k!, \; k = 1, 2, \ldots;$$
$$Q_0(x) = 1;$$
$$Q_h(x) = 0, \; h = -1, -2, \ldots.$$

These polynomials satisfy the identity

(9) $$\Delta Q_n(x) = Q_{n-1}(x)$$

for all x and all integers n. For $n > 1$ this can be seen as follows:

$$\begin{aligned}\Delta Q_n(x) &= Q_n(x) - Q_n(x-\omega)\\ &= \{x(x+\omega)\ldots(x+[n-1]\omega)\\ &\qquad - (x-\omega)x\ldots(x+[n-2]\omega)\}/\omega^n n!\\ &= \{x+[n-1]\omega - (x-\omega)\}\\ &\qquad \cdot\{x(x+\omega)\ldots(x+[n-2]\omega)\}/\omega^n n!\\ &= Q_{n-1}(x).\end{aligned} \tag{10}$$

For $n = 1$ it is obvious, since $Q_1(x) = x$; and for $n \leq 0$ both members of (9) are identically zero.

From (9) we obtain by an obvious induction

(11) $$\Delta^k Q_n(x) = Q_{n-k}(x), \quad k = 0, 1, 2, \ldots,$$

(The statement (11) for $k = 0$ is merely a tautology, since both members are $Q_n(x)$.)

Any polynomial

(12) $$p(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n$$

can be expressed as a linear combination of the polynomials in (8),

(13) $$p(x) = c_0 Q_n(x) + \ldots + c_n Q_0(x).$$

If $n = 0$ this merely states that every constant is a multiple of 1. By induction, suppose it true for all polynomials of degree less than n. Then

$$p(x) - \omega^n n!\, a_0\, Q_n(x)$$

is of degree less than n, and by hypothesis can be expanded in a linear combination of the polynomials $Q_{n-1}, \ldots, Q_0$, which completes the proof.

From this it readily follows that for the polynomial p(x) defined by (12) the equations

$$\Delta^n p(x) = \omega^n n!\, a_0, \tag{14}$$
$$\Delta^k p(x) = 0, \ k > n$$

are satisfied. For by (11)

(15) $$\Delta^k p(x) = c_0 Q_{n-k}(x) + \ldots + c_n Q_{-k}(x).$$

If $k > n$, all terms on the right vanish by (8). If $k = n$, the right member reduces to c_0, by (8); and we have already noticed that $c_0 = \omega^n n! a_0$. If we wish, we can combine the pair of equations (14) into

(16) $$\Delta^k p(x) = \omega^n p^{(k)}(x), \ k \geq n.$$

For the n-th derivative of p(x) is $n! a_0$, and all higher derivatives are zero.

2. Interpolation.

The only kind of interpolation to be considered here is polynomial interpolation. Suppose that a function is tabulated for certain values of an argument, and we wish to estimate its value corresponding to some value x^* of the argument not included among those tabulated. We can form such an estimate by selecting certain tabular arguments $x_1, \ldots, x_m$ lying on both sides of x^*, finding the polynomial which coincides with the tabulated function at the points $x_1, \ldots, x_m$, and then finding the value of this polynomial corresponding to x^*. (If all the points $x_1, \ldots, x_m$ lie on the same side of x^*, the estimate thus formed is an extrapolation, not an interpolation.) The simplest case is that familiar to every one who has used logarithmic or trigonometric tables, the polynomial then being of the first degree.

There are thus two aspects of interpolation in which the computer should be interested. One is the developing of convenient formulas for evaluating the polynomials used in the interpolation process. The other is finding means of estimating the amount by which the interpolated value differs from the true value of the function. A general expression for the polynomial of degree m - 1 which coincides with f(x) at $x_1, \ldots, x_m$ was given by Lagrange; it is

$$(1)\quad L(x) = \sum_{i=1}^{m} \frac{f(x_i)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_m)}{(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_m)}$$

At each x_k all but one of the m terms on the right vanish, because of containing a factor $x - x_k$ which vanishes at x_k. The remaining term, corresponding to i = k, takes the value $f(x_k)$, proving that L(x) coincides with f(x) at $x_1, \ldots, x_k$. The factors multiplying the $f(x_i)$ in (1) are the "Lagrangean interpolation coefficients." For equally spaced arguments $x_1, \ldots, x_m$ these coefficients have been tabulated in several different publications, the values of m ranging as high as 11 in at least one of them. When such tables are available, the Lagrange formula (1) offers a practical means of interpolation. Nevertheless, for the purposes of trajectory computation it is less convenient than some others which we shall derive shortly.

Concerning the remainder, or error, in interpolation by polynomials we now prove a theorem.

(2) Theorem. Let f(x) be defined and possess derivatives of all orders up to and including the m-th on an interval of values of x. Let $x_1, x_2, \ldots, x_m, x^*$ be distinct points lying in this interval, and let L(x) be the polynomial of degree m - 1 which coincides with f(x) at $x_1, x_2, \ldots, x_m$. Then there exists a number ξ between the least and the greatest of the numbers $x^*, x_1, \ldots, x_m$ such that

$$(3)\quad f(x^*) - L(x^*) = f^{(m)}(\xi)(x^* - x_1)\ldots(x^* - x_m)/m!.$$

Let us define

$$(4) \qquad R(x) = f(x) - L(x).$$

By the definition of $L(x)$, this "remainder" $R(x)$ vanishes for $x = x_1, \ldots, x_m$. Next we define another function

$$(5) \quad g(x) = R(x) - \frac{R(x^*)(x - x_1)(x - x_2)\ldots(x - x_m)}{(x^* - x_1)(x^* - x_2)\ldots(x^* - x_m)}.$$

Both terms on the right vanish at $x_1, \ldots, x_m$, and they cancel each other when $x = x^*$. So $g(x)$ vanishes at $m + 1$ distinct points. By Rolle's theorem, between each pair of consecutive points the first derivative $g'(x)$ must vanish. So $g'(x)$ has m distinct zeros. Again by Rolle's theorem, between each pair of consecutive zeros of $g'(x)$ the derivative $g''(x)$ must vanish. Continuing the process, we find eventually that the m-th derivative $g^{(m)}(x)$ must vanish at some point ξ between the least and the greatest of the numbers $x^*, x_1, \ldots, x_m$, so that

$$(6) \qquad g^{(m)}(\xi) = 0.$$

Since $L(x)$ is a polynomial of degree $m - 1$, its m-th derivative vanishes identically, and by (4) the m-th derivative of R is identical with that of $f(x)$. In the second term in (5), the denominator and the first factor of the numerator are constants, and the remaining factors in the numerator represent a polynomial with leading term x^m. Therefore,

$$(7) \quad g^{(m)}(x) = f^{(m)}(x) - m!R(x^*)/(x^* - x_1)\ldots(x^* - x_m).$$

If we substitute this in (6) and rearrange the terms, we obtain (3) and the theorem is established.

If we know that the value of $f^{(m)}(x)$ does not change greatly on the interval in which we are interested, it is clearly to our advantage to make the coefficient of the m-th derivative in the right member of (3) as small as possible, to keep the error R small. If we do not know how the m-th derivative behaves, it is still safest to keep this coefficient as small as possible. This we do, after having chosen the number m, by selecting the m points x_i nearest to the x^*. For instance, in a table of a function tabulated against integer values of x, to estimate f(2.5) with the help of a third-degree polynomial (m = 4) we should select 1, 2, 3 and 4 for the x_i; the coefficient of the m-th derivative is then 9/16. If we had selected 2, 3, 4 and 5 instead, the coefficient would have been 15/16; and if we had chosen to extrapolate, using the table for x = 3, 4, 5 and 6, the coefficient would have been 105/16. It is clear from this, or better from (3) itself, that extrapolation even for short distances can lead to far greater errors than interpolation in the same table.

We now turn to the interesting special case of tables with equally spaced arguments, and begin by showing that

$$(8) \qquad \Delta^m f(x) = f^{(m)}(\xi)\,\omega^m,$$

where ξ is in the interval between $x - m\omega$ and x. Let L(x) be the polynomial of degree m coinciding with f(x) at $x, x - \omega, \ldots, x - m\omega$. Then $f(x) - L(x)$ vanishes at m + 1 points, and by the reasoning used in establishing Theorem (2) its m-th derivative vanishes at some point between the least and the greatest of these zeros, that is between $x - m\omega$ and x. From this and (1.16),

$$(9) \qquad f^{(m)}(\xi) = L^{(m)}(\xi) = \omega^{-m}\Delta^m L(x).$$

But L(x) coincides with f(x) at all of the points $x, x - \omega, \ldots, x - m\omega$ used in computing the difference $\Delta^m L(x)$, so $\Delta^m L(x) = \Delta^m f(x)$. This and (9) establish (8).

The point x^* at which we wish to estimate the function lies between two consecutive tabular values of the argument. In order to simplify the notation somewhat, we suppose that the smaller of these tabular values of the argument is subtracted from all the values of the argument, so that the point at which we wish to interpolate lies between 0 and ω. It is also convenient to introduce the concept of the "phase" n, which is simply x/ω. Then integer values of n correspond to tabular values of the argument, and we are interested in estimating the function for some non-integral n lying between 0 and 1.

We can now establish a theorem which contains as special cases all the interpolation formulas that we shall need in trajectory computation.

(10) Theorem. Let $c_{-2}, c_{-1}, c_0, c_1, \ldots$ be a sequence of integers such that each of the differences $c_1 - c_0$, $c_2 - c_1, \ldots$ is either 0 or 1. Let $y_0, y_1, y_2, \ldots$ be arbitrary real numbers such that no two consecutive y_i are different from zero. Let $f(x)$ be a function defined at the points

(11) $$c_m\omega, (c_m - 1)\omega, \ldots, (c_m - m)\omega.$$

Then if $y_{m+1} = 0$ the function

(12) $$P(x) = \sum_{j=0}^{m} [Q_j(x-c_{j-1}\omega) + y_j Q_{j-1}(x-c_{j-2}\omega)] \cdot [\Delta^j f(c_j\omega) - y_{j+1}\Delta^{j+1} f(c_{j+1}\omega)]$$

is the polynomial of degree m which coincides with $f(x)$ at the points (11).

The first step in the proof is to show that in (12) each y_i either is zero or is multiplied by zero. Suppose that for some integer k the number y_k is not zero; then by hypothesis y_{k+1} is zero, and so is y_{k-1} unless $k = 0$. In this latter case y_k occurs only in the first term of the sum, where it is multiplied by Q_{-1}, which is identically zero. Otherwise y_k occurs in two terms; first, in the term with $j = k - 1$, where it has coefficient

$$- Q_{k-1}(x - c_{k-2}\omega)\,\Delta^k f(c_k\omega),$$

and second in the term with $j = k$, where it has coefficient

$$+ Q_{k-1}(x - c_{k-2}\omega)\,\Delta^k f(c_k\omega).$$

So in either case it is multiplied by zero. It therefore makes no difference in (12) if we replace all non-zero y_j by 0, since this merely changes the number by which 0 is multiplied. We therefore need only prove the special case of (12) in which all the y_j are 0.

Let $G(x)$ stand for the polynomial which coincides with $f(x)$ at the points (11); our task is to show that

$$(13)\qquad G(x) = \sum_{j=0}^{m} Q_j(x - c_{j-1}\omega)\,\Delta^j f(c_j\omega).$$

The reasoning used to establish (1.13) can be repeated to show that there are numbers $b_0, \ldots, b_m$ such that

$$(14)\qquad G(x) = \sum_{j=0}^{m} b_j Q_j\ (x - c_{j-1}\omega).$$

By taking the k-th difference and recalling (1.8,11) we find

$$(15)\qquad \Delta^k G(x) = \sum_{j=0}^{m} b_j Q_{j-k}(x - c_{j-1}\omega).$$

Again by (1.8), the terms with $j < k$ vanish and the term with $j = k$ is equal to b_k. In (15) we now set $x = c_k\omega$.

For each $j > k$, the argument of Q_{j-k} is

$$(16)\quad \begin{aligned} c_k\omega - c_{j-1}\omega \\ = -\omega[(c_{j-1} - c_{j-2}) + (c_{j-2} - c_{j-3}) + \\ + (c_{k+1} - c_k)], \end{aligned}$$

and by the hypothesis concerning the numbers c_i the quantity in square brackets is one of the numbers 0, 1, ..., $j - k - 1$. But by (1.8), Q_{j-k} vanishes when x has any of the values (16). So when we put $x = c_k\omega$ all the terms in (15) vanish except the one with $j = k$, and that one is equal to b_k. We have therefore shown that

$$(17)\qquad b_k = \Delta^k G(c_k\omega).$$

The tabular arguments involved in computing the difference $\Delta^k G(c_k\omega)$ are

$$c_k\omega,\ (c_k - 1)\omega,\ \ldots,\ (c_k - k)\omega,$$

and these are included in the set (11) at which $f(x)$ and $G(x)$ coincide. Therefore,

$$(18)\qquad \Delta^k G(c_k\omega) = \Delta^k f(c_k\omega).$$

Equations (14), (17) and (18) combine to yield (13), and the theorem is proved.

As a first special case of this theorem, we take all the y_j equal to zero and choose $c_j = j$. If we substitute these in (12), and also make the change of notation

$$(19)\qquad x = n\omega$$

in the right member, we obtain

$$(20)\quad \begin{aligned} P(x) = f(0) + n\,\Delta^1 f(\omega) \\ + [n(n-1)/2!]\Delta^2 f(2\omega) + \ldots \\ + [n(n-1)\ldots(n-k+1)/k!]\Delta^k f(k\omega) + \ldots. \end{aligned}$$

This is the Newton-Gregory "forward" formula.

Next we select all $y_j = 0$, $c_{-2} = c_{-1} = c_0 = 0$, all other $c_j = 1$. With the same substitutions as before, we obtain

$$P(x) = f(0) + n\,\Delta^1 f(\omega) + [n(n-1)/2!]\,\Delta^2 f(\omega) + \ldots \tag{21}$$
$$+ [(n-1)n\ldots(n+k-2)/k!]\,\Delta^k f(\omega) + \ldots.$$

This is the Newton-Gregory "backward" formula. Had we chosen all $c_j = 0$, we would have obtained another formula, often called the Newton-Gregory "backward" formula; but it would have been an extrapolation formula for positive values of n.

If we select $c_{2k} = c_{2k+1} = k$ for all integers k, all y_j again being set equal to 0, equation (12) takes the form

$$\begin{aligned} P(x) &= f(0) + n\,\Delta^1 f(0) + [n(n+1)/2!]\,\Delta^2 f(\omega) + \ldots \\ &+ [(n-k)\ldots(n+k)/(2k+1)!]\,\Delta^{2k+1} f(k\omega) \\ &+ [(n-k)\ldots(n-k+1)/(2k+2)!]\,\Delta^{2k+2} f([k+1]\omega) \\ &+ \ldots. \end{aligned} \tag{22}$$

This is the Newton-Gauss "backward" formula.

Five more well-known interpolation formulas can be derived from the choice

$$c_{2k-1} = c_{2k} = k \tag{23}$$

for all integers k. If we substitute these in (12) and use (19), (1.8) and (1.5), we find

(24) The term of (12) corresponding to $j = 2k$ is

$$[1/(2k)!](n-k+1)\ldots(n+k-1)(n-k+2ky_{2k})$$
$$\cdot[(1+y_{2k+1})\,\Delta^{2k} f(k\omega) - y_{2k+1}\,\Delta^{2k} f([k+1]\omega)],$$

(25) The term of (12) corresponding to $j = 2k + 1$ is

$$[1/(2k+1)!](n-k)\ldots(n+k-1)[n+k+(2k+1)y_{2k+1}]$$
$$\cdot[(1-y_{2k+2})\,\Delta^{2k+1} f([k+1]\omega) + y_{2k+2}\,\Delta^{2k+1} f(k\omega)].$$

If we set all the y_j equal to 0, (12) becomes

$$P(x) = f(0) + n\Delta^1 f(\omega) + [n(n-1)/2!]\Delta^2 f(\omega) + \dots$$
$$+ [(n-k)\dots(n+k-1)/(2k)!]\Delta^{2k} f(k\omega)$$
$$+ [(n-k)\dots(n+k)/(2k+1)!]\Delta^{2k+1} f([k+1]\omega) \quad (26)$$
$$+ \dots.$$

This is the Newton-Gauss "forward" formula.

With the same choice (23) for the c_j, let us choose $y_{2k} = 1/2$, $y_{2k+1} = 0$ for all integers k. Then from (12) we obtain

$$P(x) = f(0) + n[\Delta^1 f(0) + \Delta^1 f(\omega)]/2 + \dots$$
$$+ [1/(2k)!][n^2-(k-1)^2]\dots[n^2]\Delta^{2k} f(k\omega) \quad (27)$$
$$+ [(n-k)\dots(n+k)/2(2k+1)!]$$
$$\cdot[\Delta^{2k+1} f(k\omega) + \Delta^{2k+1} f([k+1]\omega)] + \dots.$$

This is Stirling's formula.

Again with the same choice (23) for the c_j, let us choose

$$y_{2k} = 0, \; y_{2k+1} = -1/2$$

for all integers k. Equation (12) becomes*

$$P(x) = [f(0) + f(\omega)]/2 + [n-\tfrac{1}{2}]\Delta^1 f(\omega) + \dots$$
$$+ [(n-k)\dots(n+k-1)/(2k)!]$$
$$\cdot[\Delta^{2k} f(k\omega) + \Delta^{2k} f([k+1]\omega)]/2 \quad (28)$$
$$+ [(n-k)\dots(n+k-1)(n-\tfrac{1}{2})/(2k+1)!]$$
$$\cdot\Delta^{2k+1} f([k+1]\omega)$$
$$+ \dots.$$

This is Bessel's formula.

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*The first term in (28) is rather easier to derive from (12) itself than from (24).

Our next choice is

$$y_{2k} = 0, \; y_{2k+1} = -(n+k)/(2k+1)$$

for all integers k. By (25), the terms in (12) corresponding to odd j all vanish. By (24), the term corresponding to j = 2k is

$$(29) \quad \begin{aligned} &- [1/(2k+1)!](n-k-1)(n-k)\dots(n+k-1)\,\Delta^{2k}f(k\,\omega) \\ &+ [1/(2k+1)!](n-k)(n-k+1)\dots(n+k)\,\Delta^{2k}f([k+1]\,\omega). \end{aligned}$$

This can be given a more symmetric appearance by defining

$$(30) \quad E_{2k}(n) = (n-k)(n-k+1)\dots(n+k)/(2k+1)!.$$

The coefficient of the second term in (29) is then $E_{2k}(n)$, while that of the first term is $-E_{2k}(n-1)$. But E_{2k} is an odd function of n, as its definition shows, so

$$-E_{2k}(n-1) = E_{2k}(1-n).$$

It is customary at this point to introduce a symbol, say $\bar{n}$, for 1-n,

$$(31) \quad \bar{n} = 1-n.$$

Then the expression (29) becomes

$$E_{2k}(\bar{n})\,\Delta^{2k}f(k\,\omega) + E_{2k}(n)\,\Delta^{2k}f([k+1]\,\omega),$$

and the entire expansion (12) takes the form

$$(32) \quad \begin{aligned} P(x) = \bar{n}f(0) &+ nf(\omega) + \dots \\ &+ E_{2k}(\bar{n})\,\Delta^{2k}f(k\,\omega) \\ &+ E_{2k}(n)\,\Delta^{2k}f([k+1]\,\omega) + \dots . \end{aligned}$$

This is Everett's formula.

Our final choice is

$$y_0 = 0, \; y_{2k+1} = 0$$

for all integers k,

$$y_{2k} = (k-n)/2k$$

for all integers $k \neq 0$. By (24), all terms corresponding to even positive values of j vanish, and (12) becomes

$$\begin{aligned} P(x) = f(0) &+ [n(1-n)/2!]\,\Delta^1 f(\omega) \\ &+ [n(1+n)/2!]\,\Delta^1 f(0) + \dots \\ &+ [1/2(k+1)(2k+1)!](n-k)\dots(n+k-1) \\ &\cdot[(k+1+n)\,\Delta^{2k+1} f([k+1]\,\omega) \\ &+ (k+1-n)\,\Delta^{2k+1} f(k\omega)] + \dots . \end{aligned} \tag{33}$$

This is Steffensen's formula.

Each of the eight formulas derived above from equation (12) can be written to as many terms as desired. In actual use they must be broken off somewhere, and it is important to be able to tell when it is safe to stop. Let us suppose that we have selected a number ε, for example half a unit of the last significant figure tabulated in the table of functional values, and that we wish to stop as soon as we can be reasonably sure that the remainder does not exceed ε. If we stop the interpolation formula at a value of m such that

$$y_{m+1} = 0,$$

as required in Theorem (10), the first omitted term is

$$Q_{m+1}(x-c_m\omega)\,[\Delta^{m+1} f(c_{m+1}\,\omega) - y_{m+2}\Delta^{m+2} f(c_{m+2}\,\omega)]. \tag{34}$$

This alone could not be depended upon to furnish a sufficiently trustworthy estimate of the error.

However, in all eight formulas the quantity in square brackets in (34) is either the difference of order $m + 1$ formed from the functional values corresponding to the values (11) extended one step forward or backward, or else is a weighted mean of two such differences. By (8), each such difference is the product of ω^{m+1} and the value of the derivative $f^{(m+1)}$ at some point ξ between $(c_m + 1)\omega$ and $(c_m - m - 1)\omega$, provided that the derivative in question exists; and the weighted mean of two such products also must have the same form, since the weighted mean of the two values of the derivative lies between the two, and is itself a value of the derivative at some intermediate point. Thus for each of the eight formulas we know that there exists between $(c_m + 1)\omega$ and $(c_m - m - 1)\omega$ a number ξ_1 such that

$$\Delta^{m+1}f(c_{m+1}\omega) - y_{m+2}\Delta^{m+2}f(c_{m+2}\omega) = \omega^{m+1}f^{(m+1)}(\xi_1). \tag{35}$$

The first omitted term (34) is then

$$Q_{m+1}(x - c_m\omega)\,\omega^{m+1}f^{(m+1)}(\xi_1). \tag{36}$$

By Theorem (2), there is a number ξ between $(c_m - m)\omega$ and $c_m\omega$ such that the remainder $R(x)$, which is the error in assuming that $P(x)$ is equal to $f(x)$, is

$$R(x) = Q_{m+1}(x - c_m\omega)\,\omega^{m+1}f^{(m+1)}(\xi). \tag{37}$$

If ξ were equal to ξ_1, as in general it is not, we would have the remainder equal to the first omitted term. If we know nothing at all about the $(m + 1)$-th derivative of $f(x)$, we likewise know nothing at all about the size of the remainder (37). But in most ballistic problems we are not in complete ignorance about $f(x)$ and its derivatives. For example, if x stands for the time and $f(x)$ for the range at time x, we may feel confident from the form of the equations of motion that any given derivative of $f(x)$ will change from increasing to decreasing at only a relatively few points, so that the trajectory can be cut into a few subarcs on each of which the given derivative is either steadily rising or steadily falling. Moreover, these arcs will ordinarily be considerably longer than the interval of time used in computing the trajectory, that is the interval between successive times at which the range is listed. Consequently, if we know the derivative of order m + 1 of f(x) at a number of points to each side of a given point x^*, we may feel confident that at x^*, the derivative in question lies somewhere between its greatest and its least values at the points where it is known.

Suppose then that we compute the term (36), and then re-compute it with each of the differences in (34) replaced by that on the line above; we would then obtain an expression like (36) but with a different number, say ξ_2, in place of ξ_1. We repeat this with the differences two lines above those in (34), and so on until we reach m + 1 lines above those we started with. We repeat the whole procedure, going this time in the other direction. We thus obtain a collection of values of expressions like (36), but with ξ_1 replaced by an aggregate of numbers extending to both sides of the number ξ in (37). The numerically largest of the numbers thus found should exceed the quantity (37) in absolute value.

Consequently, in interpolation we may be guided with reasonable safety by the following rule. It is safe to omit a given term (and of course all succeeding terms) in an interpolation formula if it is not only **true that the value of this term is below the allowable amount ε, but also that if the differences used in computing** this last term are replaced by those on nearby higher or lower lines (as in computing $f(x - \omega)$, $f(x + \omega)$ instead of computing $f(x)$), the value is still less than ε. By "nearby" we mean not more than m + 1 lines up or down, where m + 1 is the order of the difference involved in the term being tested.

For example, if we wish to interpolate for $f(1/2)$ in the table

x	$f(x)$	Δ^1	Δ^2	Δ^3
-1	-1			
0	0	1		
1	1	1	0	
2	8	7	6	6
3	27	19	12	6

by the Newton-Gauss "forward" formula, the coefficients of the first, second and third differences are 1/2, - 1/8 and - 1/16 respectively. The second of these is multiplied by 0, but this does not permit us to omit it and all succeeding terms if we wish errors to remain below 0.1; because only one line lower the second difference is 6, whose product with the coefficient - 1/8 exceeds the allowance. Nor can we omit the third-difference term. However, the fourth difference is zero on all lines exhibited (and would be on any others with the same law of formation $f(x) = x^3$), so that the interpolation formula may stop with third differences.

The preceding rule applies to Bessel's formula if we are testing to see if we can stop with some difference of odd order, but it does not apply at once if we wish to see if we can stop with a difference of even order, because of the hypothesis $y_{m+1} = 0$ needed in the foregoing discussion. However, this is a triviality. If, for example, we find that we can stop with the difference of order 5 by applying the rule, and moreover the term involving the fifth difference contributes a negligible amount, we may omit it too.

The interpolation formulas based on differences have the slight disadvantage as compared with the

Lagrange formula that the differences must be computed before the interpolation can be done. However, they have several important advantages. The terms in the formulas rapidly decrease, so that except for the first two or three the numbers involved are small, which helps to avoid errors. The rules just discussed allow a ready determination of the order of differences needed for a given degree of accuracy, while this is much less easy to see when using the Lagrange formula. And if at some stage it is decided that higher differences are needed, these can be introduced without changing any of the work already completed, whereas increasing the degree of the polynomial in Lagrange interpolation requires making a fresh start.

The discussion of the proper stopping place in using an interpolation formula showed that accuracy of the formula is essentially the same thing as rapidity of convergence of the coefficients to their limit 0. The Newton-Gregory formulas fail in this respect; the coefficients decrease only slowly on the interval $0 < n < 1$. The Newton-Gauss "forward" formula ending with a difference of odd order m uses the functional values at the nearest $m + 1$ tabular points if $0 < n < 1$, which by the remark after Theorem (2) ensures that the remainder is as small as possible for the given order of difference and a given bound on the $(m + 1)$-th derivative of $f(x)$. Otherwise stated, the Newton-Gauss "forward" coefficients decrease rapidly. The Newton-Gauss "backward" coefficients are not as good in this respect (they would be at their best if we were interpolating for $- 1 < n < 0$ instead of for $0 < n < 1$, so that the "backward" formula would be a good one for "backward" interpolation). The Bessel formula always uses an even number of tabular points, and these are always the nearest to x if $0 < n < 1$. It has the advantage over the Newton-Gauss "forward" formula that the coefficients are slightly smaller, but the disadvantage that an extra arithmetic operation of addition is needed in each term involving differences of even order. The Everett formula also uses an even number of points, the nearest ones to x, but it is not generally regarded as being as convenient as the Bessel or Newton-Gauss "forward" formulas. The Stirling and Steffensen formulas always use an odd number of points, the nearest to x if $-1/2 < n < 1/2$, but are inferior to the Bessel and Everett formulas respectively if we are interpolating for $0 < n < 1$. So the two formulas which pass the test most successfully are the Newton-Gauss "forward" and the Bessel, and these are in fact the two most commonly used in practice.

The Newton-Gregory "forward" and "backward" formulas possess the apparent advantage that they can be used at the ends of a table, where the differences needed to use the Newton-Gauss, Bessel or Stirling formulas are not all available. But this advantage is illusory. Suppose that we wish to interpolate between the second-last and last lines of a table, using the Newton-Gregory formula to m-th differences. This would provide us with the value at the place x of a polynomial of degree m coinciding with the tabulated function f(x) at the last $m + 1$ tabular values. But if the last entry in the column of m-th differences is copied on the next m lines, and with the definition (1.5) the other differences and the functional values are built up from the m-th differences on all these lines, the functional values thus constructed will all be those of one and the same polynomial of degree m, since the difference of order $m + 1$ is identically zero. Thus the polynomial of degree m passing through the last $m + 1$ tabular entries is extended on for m more lines. By any of our interpolation formulas carried as far as m-th differences this polynomial will be evaluated at the place x, so that the number found will necessarily be identical with what the Newton-Gregory formula would have given us. So the Newton-Gauss or Bessel formulas work just as well as the Newton-Gregory at the ends of the table, and do better in the interior of the table; and therefore there is no real need of ever using the Newton-Gregory formulas.

3. Quadrature formulas.

If a function f(x) is approximated by any one of the polynomial formulas of the preceding section, the error remaining less than some number ε throughout an interval $x_1 \leq x \leq x_2$, the integral of f(x) between the limits x_1 and x_2 can be estimated by integrating the polynomial, and the error will be less than $\varepsilon(x_2 - x_1)$. As in the preceding section, we shall use the notation $x = n\omega$, and we shall be interested in formulas for the integral of f(x) between 0 and ω, or between $-\omega$ and ω.

Suppose first that we have approximated f(x) by means of Stirling's formula (2.27). On integrating from $-\omega$ to ω, and replacing dx by ω dn, we find

$$\int_{-\omega}^{\omega} f(x)\,dx = \omega[2f(0) + (1/3)\,\Delta^2 f(\omega) - (1/90)\,\Delta^4 f(2\omega) + \ldots]. \tag{1}$$

Replacing the second difference by (1.5) changes this into

$$\int_{-\omega}^{\omega} f(x)\,dx = (\omega/3)[f(-\omega) + 4f(0) + f(\omega) - (1/30)\,\Delta^4 f(2\omega) + \ldots]. \tag{2}$$

By discarding the terms involving differences of order four and higher, we obtain Simpson's formula. The term involving the fourth difference serves to furnish some sort of estimate of the size of the error made by using Simpson's rule.

Next we approximate f(x) by Everett's formula, (2.32), wherein it is now convenient to replace $\bar{n}$ by $1 - n$, which is the same thing by definition.

On integrating from 0 to ω (the integration being facilitated by the substitution $n^2 = v$ in the terms containing $E_{2k}(n)$ and the substitution $(1 - n)^2 = v$ in the others) we obtain

$$\int_0^{\omega} f(x)\, dx$$

$$(3) \quad = \omega\{(1/2)[f(0) + f(\omega)] - (1/24)[\Delta^2 f(\omega) + \Delta^2 f(2\omega)] + (11/1440)[\Delta^4 f(2\omega) + \Delta^4 f(3\omega)] + \dots\}.$$

If we omit all the terms involving differences, this simplifies to the "trapezoidal rule." If we retain the second difference terms, we have a quadrature formula comparable with Simpson's rule. On the whole, it is less convenient than Simpson's rule; for one thing, it involves values of the function lying outside of the interval of integration. If the fourth difference is nearly constant, the errors in (2) and (3) respectively introduced by omitting fourth and higher differences are of opposite sign, and the error in (3) is roughly 11/8 as great in absolute value as the error in (1) or (2). Nevertheless it has one advantage. If a function is to be tabulated by the process of quadrature of another function, the use of Simpson's rule alone has the difficulty of segregating the even-numbered lines from the odd-numbered lines. The accumulation of small errors, such as rounding errors, will cause the even-numbered and the odd-numbered lines to wander independently from the correct values, so that the first differences of the integrated quantity are alternately too high and too low. This does not happen with (3). Consider therefore the situation, such as occurs in the computation of a trajectory, in which a quadrature is an essential part of a long computation scheme, the quadrature requiring only a small part of the entire computing time. Both for accuracy and for detection of errors in calculation it would be wise to compute each entry both by Simpson's rule and by (3). If the fourth differences are under 40 the two results should not differ by more than 1, in units of the last figure carried. Rounding errors may easily increase this discrepancy to 3, and occasionally even to 4. A larger discrepancy should call for a verification of the computation. If the discrepancy is not too large, the number entered should be the average of the two results, throwing the .5 toward the result of Simpson's rule if the discrepancy is odd.

Suppose next that F(x) is obtained from f(x) by quadrature, and G(x) in turn obtained from F(x) by quadrature, so that

$$F(x) = F(0) + \int_0^x f(x)\,dx, \tag{4}$$

$$G(x) = G(0) + \int_0^x F(x)\,dx. \tag{5}$$

Then

$$\begin{aligned} \Delta^1 G(0) &= \int_{-\omega}^{0} F(x)\,dx, \quad \Delta^1 G(\omega) = \int_0^{\omega} F(x)\,dx, \\ \Delta^2 G(\omega) &= \int_0^{\omega} F(x)\,dx + \int_0^{-\omega} F(x)\,dx. \end{aligned} \tag{6}$$

By integration by parts, using (4), this becomes

$$\begin{aligned} \Delta^2 G(\omega) &= \int_0^{\omega} (\omega - x)\,f(x)\,dx \\ &\quad + \int_0^{-\omega} (-\omega - x)\,f(x)\,dx, \end{aligned} \tag{7}$$

and by substitution of variable of integration in the last term (changing x to − x) we obtain

$$\Delta^2 G(\omega) = \int_0^{\omega} (\omega - x)\,[f(x) + f(-x)]\,dx. \tag{8}$$

Let us replace x by $n\omega$ and approximate f(x) and f(- x) by means of Stirling's formula, (2.27). All differences of odd order vanish from the integrand, since they are multiplied by coefficients which are odd functions of n. The remaining terms are even functions of n, so that in these terms the contributions of f(x) and of f(- x) are equal. Hence

$$\Delta^2 G(\omega)$$

$$(9) \quad = 2\omega^2 \int_0^1 (1-n) \{ f(0) + (n^2/2)\, \Delta^2 f(\omega) + [n^2(n^2-1)/4!]\Delta^4 f(2\omega) + \dots \} \, dn$$

$$= \omega^2 \{ f(0) + (1/12)\, \Delta^2 f(\omega) - (1/240)\, \Delta^4 f(2\omega) + \dots \}.$$

By integration by parts, the second of equations (6) becomes

$$(10) \quad \Delta^1 G(\omega) = (x - \tfrac{1}{2}\omega)F(x)\Big|_0^\omega - \int_0^\omega (x - \tfrac{1}{2}\omega) f(x)\, dx.$$

In the second term on the right we replace x by $n\omega$ and approximate f(x) by means of Bessel's formula, (2.28). If we make the substitution $v = n - \frac{1}{2}$ in the coefficients of the differences of even order, these are seen to be the integrals of odd functions of v between limits $-\frac{1}{2}$ and $\frac{1}{2}$, and therefore the integrals are zero. The remaining terms are

$$\Delta^1 G(\omega)$$

$$= \omega [F(\omega) + F(0)]/2$$

$$(11) \quad - \omega^2 \int_0^1 \{ (n-\tfrac{1}{2})^2 \Delta^1 f(\omega) + [n(n-1)(n-\tfrac{1}{2})^2/3!]\Delta^3 f(2\omega) + \dots \} \, dn$$

$$= \omega [F(0) + F(\omega)]/2 - (\omega^2/12) \{ \Delta^1 f(\omega) - (1/60)\, \Delta^3 f(2\omega) + \dots \}.$$

If the terms involving differences of third and higher orders are omitted, this simplifies to

(12) $\Delta^1 G(\omega) \doteq (\omega/2)[F(0) + F(\omega) - (\omega/6)\,\Delta^1 f(\omega)]$,

a formula devised by C. B. Morrey in 1943. (The authors find no earlier reference to this formula.)

Formulas (9) and (12) are useful when an iterated quadrature must be effected. This happens, for instance, in trajectory computation when y' and y are to be deduced from y". If y' is first found from y" by a quadrature, using either or both of (2) and (3), it would be possible to use the same formulas to compute y from y'. But it is quite advantageous to use (12) instead. If we apply (3) to several successive intervals between multiples of ω and take differences, we see that the fourth difference of y' is approximately ω times a mean third difference of y". The error term in (12) is approximately $(\omega^2/720)$ times a third difference of y". If the tabular interval were halved, the error in Simpson's rule would be $(\omega/90)$ times the new fourth difference of y', which is about $(\omega/1440)$ times a fourth difference of y' with the original interval, or about $(\omega^2/1440)$ times a third difference of y". Thus Morrey's formula (12) has only about twice the error of Simpson's rule with intervals half as long. Moreover, it does not develop the oscillations which occur with Simpson's rule, discussed after (3).

It is not so easy to compare (12) with (9). The error in any one line is smaller when (9) is used. But since (9) gives the second difference, an error on any one line enters the corresponding first difference and remains there permanently, being added in in every later term. Thus the errors in using (9) accumulate; an error on any line causes an error m times as great in the value of y, m lines below. After a number of trials of the two formulas, it was decided that the most satisfactory procedure was to rely on (12), retaining (9) as a check formula.

One more important quadrature formula will be recorded here without proof, although it will not be used in the methods of this chapter. This is Weddle's rule,

$$(13)\quad \int_0^6 f(x)\,dx \doteq \frac{3\omega}{10}[f(0) + 5f(\omega) + f(2\omega) + 6f(3\omega) + f(4\omega) + 5f(5\omega) + f(6\omega)].$$

This is correct as far as fifth differences. For details, the reader is referred to L. M. Milne-Thomson's Calculus of Finite Differences (London: MacMillan and Company, Ltd., 1933).

Equation (9) can be used to obtain a variant form of Everett's (or, alternatively, of Bessel's) interpolation formula which has a certain advantage in connection with trajectory computations. When a second-order differential equation has been solved by numerical integration, a set of values of the solution-function and of its second derivative are obtained in the process of solution, and the first, second and perhaps the third differences of the second derivatives are also tabulated. Differences of the solution, of order higher than the second, will not ordinarily be at hand. Hence there is an advantage in having an interpolation formula which uses only the differences automatically available. In the notation of this section, G and f are tabulated for values $-\omega$, 0, ω, 2ω, ... of the independent variable If we first replace f by G in Everett's formula (2.32) and then substitute (9) in the result, we find that the polynomial approximation to G is

$$
\begin{aligned}
&P(x)\\
&= \bar{n}G(0) + nG(\omega)\\
&\quad + \omega^2 \{ E_2(\bar{n})f(0) + E_2(n)f(\omega)\\
(14)\qquad &\qquad + [E_4(\bar{n})+E_2(\bar{n})/12]\Delta^2 f(\omega)\\
&\qquad + [E_4(n)+E_2(n)/12]\Delta^2 f(2\omega)\\
&\qquad + [E_6(\bar{n})+E_4(\bar{n})/12-E_2(\bar{n})/240]\Delta^4 f(2\omega)\\
&\qquad + [E_6(n)+E_4(n)/12-E_2(n)/240]\Delta^4 f(3\omega) + \dots \}.
\end{aligned}
$$

The intervals must be small enough to be usable with the process of numerical integration, and this will insure that the contribution of the fourth differences of f is negligible. For n between 0 and 1, we find that $E_4(\bar{n}) + E_2(\bar{n})/12$ and $E_4(n) + E_2(n)/12$ differ from each other by less than .0004. Therefore if we denote their mean by M_4'', when we replace both of them by M_4'' we introduce an error less than $.0002\omega^2$ times the third difference of f, which is about a five-thousandth of the fifth difference of G. Hence we have the approximate formula

$$G(x) = \bar{n}G(0) + nG(\omega) + \omega^2\{E_2(\bar{n})\,f(0) + E_2(n)\,f(\omega) + M_4''(n)[\Delta^2 f(\omega) + \Delta^2 f(2\omega)]\}. \tag{15}$$

In the following table the values of E_2 are exact; those of M_4'' are rounded to the fourth decimal place.

n	$\bar{n}$	$E_2(n)$	$E_2(\bar{n})$	$M_4''(n) = M_4''(\bar{n})$
0.0	1.0	0.0000	0.0000	0.0000
.1	.9	- .0165	- .0285	.0020
.2	.8	- .0320	- .0480	.0039
.3	.7	- .0455	- .0595	.0053
.4	.6	- .0560	- .0640	.0062
0.5	0.5	-0.0625	-0.0625	0.0065
$\bar{n}$	n	$E_2(\bar{n})$	$E_2(n)$	$M_4''(\bar{n}) = M_4''(n)$

This shows that we have gained an incidental advantage; M_4'' is only about half as large as E_4. Consequently the term with M_4'' as factor in (15) may often be omitted, simplifying the computations.

As an application, we perform an interpolation in a column of the trajectory sheet which furnished the numbers designated by "Ch. VI" in the tables in Section 10 of Chapter V. For G we take x, and for f we take $\ddot{x}$.

Part of the computed results are as follows:

t	x	$\ddot{x}$	$\Delta^2\ddot{x}$
16	3949.38	-3.18	
18	4292.72	-2.88	
20	4624.52	-2.65	-7
22	4945.69	-2.50	-8

To find x when t = 18.2, we use n = 0.1, and by (15) and the table we obtain

$$x = (.9)(4292.72) + (.1)(4624.52)$$
$$+ (4)[(- .0285)(- 2.88) + (- .0165)(- 2.65)$$
$$+ (.0020)(- .15)]$$
$$= 4326.42.$$

Since the second differences of x must be (by (9)) about four times - 3 with tabular interval 2 seconds, when we cut down to tabular interval .2, the second difference would be a hundredth as great, or about .12. This is well within the limit (eight) permitting linear interpolation, so for t between 18 and 18.2 we can interpolate linearly for x. In particular, since the angle of departure of the trajectory was 45°, corresponding to slant range L = 20,000 feet we have x = 14142.05 feet or 4310.51 meters. By linear interpolation this corresponds to time t = 18.106 seconds, as was already listed in the comparison table in Section 10 of Chapter V.

All our quadrature formulas so far have applied to integrals $\int f\,dx$. However, the Stieltjes integrals $\int f(t)\,dg(t)$ studied in Chapter I also have to be estimated numerically, especially in the computations of differential effects which are to be discussed in Chapter VIII. Suppose then that both f(t) and g(t) are known for three values of t, say t_1, t_2, and t_3, where $t_1 < t_2 < t_3$. To condense the notation, we write f_1, f_2, f_3 for $f(t_1)$, $f(t_2)$, $f(t_3)$ respectively, and likewise for g. We shall suppose that $g_1 < g_2 < g_3$, but the resulting formula will also be applicable if $g_3 < g_2 < g_1$. The equations $x = g(t)$, $y = f(t)$ are the parametric equations of a curve in the (x, y)-plane which passes through (g_1, f_1), (g_2, f_2) and (g_3, f_3). Assuming that g is steadily increasing from t_1 to t_3, this curve can also be written in the form $y = Y(x)$, where to find $Y(x)$ we first solve $x = g(t)$ for t and then substitute this value of t in $f(t)$. The integral

$$I = \int_{t_1}^{t_2} f(t)\, dg(t)$$

which we seek is the same as the integral of Y from g_1 to g_3. As an approximation, we shall compute the integral (from g_1 to g_3) of the quadratic function of x which agrees with $Y(x)$ at g_1, g_2, and g_3. If $g(t)$ is quadratic, having second derivative q'' and assuming values $q(a)$, $q(b)$ at a and b respectively, its equation is

$$q(t) = (t - a)q(b)/(b - a) + (t - b)q(a)/(a - b)$$
$$+ (q''/2)(t - a)(t - b),$$

and so its integral is

$$\int_a^b q(t)\, dt = [q(b) + q(a)](b - a)/2 - (q''/12)(b - a)^3.$$

Now the quadratic function $q(x)$ which agrees with $Y(x)$ at g_1, g_2 and g_3 has slope $(f_2 - f_1)/(g_2 - g_1)$ halfway between g_1 and g_2, and has slope $(f_3 - f_2)/(g_3 - g_2)$ halfway between g_2 and g_3, so the rate of change of slope, q'', is

$$2\,\frac{\dfrac{f_3 - f_2}{g_3 - g_2} - \dfrac{f_2 - f_1}{g_2 - g_1}}{g_3 - g_1}.$$

We substitute this in the preceding formula. If we first integrate $q(x)$ from g_1 to g_2, and then integrate it from g_2 to g_3, and add the results, the sum is the following approximation to I:

$$I \doteq (g_2 - g_1)(f_1 + f_2)/2 + (g_3 - g_2)(f_2 + f_3)/2$$
$$- \frac{(g_2 - g_1)^3 + (g_3 - g_2)^3}{6(g_3 - g_1)} \left\{ \frac{f_3 - f_2}{g_3 - g_2} - \frac{f_2 - f_1}{g_2 - g_1} \right\} .$$

In the last term, we factor the sum of cubes and replace the denominator by $(g_3 - g_2) + (g_2 - g_1)$. The result can be written in the form

$$I \doteq (g_2 - g_1) \left\{ \frac{f_1 + f_2}{2} - (f_3 - f_2) \frac{r - 1 + 1/r}{6} \right\}$$

(16)

$$+ (g_3 - g_2) \left\{ \frac{f_3 + f_2}{2} + (f_2 - f_1) \frac{r - 1 + 1/r}{6} \right\} ,$$

where r is the ratio of the smaller of $g_2 - g_1$ and $g_3 - g_2$ to the larger of them (or vice versa!).

It is clear that if we discarded the terms with the factor $r - 1 + 1/r$ we would have the trapezoidal formula. If $g_3 - g_2$ and $g_2 - g_1$ are equal, then $r - 1 + 1/r = 1$ and (16) becomes Simpson's rule. Formula (16) is at its best when the intervals $g_2 - g_1$ and $g_3 - g_2$ are fairly nearly equal. In any case it may be expected to be considerably more accurate than the trapezoidal rule.

The auxiliary computation of $r - 1 + 1/r$ in (16) can be advantageously done with a slide rule. If $g_2 - g_1$ on the C-scale is set against $g_3 - g_2$ on the D-scale, the numbers on each of these scales opposite the "1" of the other scale will be r and $1/r$.

4. The existence theorem.

In order to avoid irrelevant analytical complexities, let us assume that the functions $f_i(x, y_1, \ldots, y_n)$, $(i = 1, \ldots, n)$ are continuous in all variables for x in an interval $a \leq x \leq b$ and for all y, and moreover that these functions possess partial derivatives with respect to the variables y_i for the same range of arguments, and that these partial derivatives remain less in absolute value than a constant M on the entire range of the variables x, y_i just mentioned. Let ξ be a number in the interval $a \leq x \leq b$, and let $\eta_1, \ldots, \eta_n$ be any n numbers. We shall prove* that there exists a solution of the differential equations (1.1) which at $x = \xi$ has the values $(\eta_1, \ldots, \eta_n)$.

The central feature of the proof is the construction of a sequence of successive approximations

$$(1) \qquad y_1(x;j), \ldots, y_n(x;j), \qquad j = 0, 1, 2, \ldots.$$

(The enumerating index j has been written inside the parenthesis merely in order to avoid complicated subscripts.) The first approximation, consisting of the n functions $y_1(x;0), \ldots, y_n(x;0)$, consists of any n functions defined and continuous on the interval $a \leq x \leq b$ and taking on the sequence of respective values $\eta_1, \ldots, \eta_n$ at $x = \xi$. The later approximations are defined inductively, each in terms of the preceding one, by the equation

$$(2) \qquad y_i(x;j+1) = \eta_i + \int_{\xi}^{x} f_i(x, y_1(x;j), \ldots, y_n(x;j))\, dx,$$

$$i = 1, \ldots, n.$$

---

*The proof below is essentially the same as that in G. A. Bliss' Lectures on the Calculus of Variations (Chicago: The University of Chicago Press, 1936), p. 274.

In order to discuss these functions, we first notice that for any two sets of n numbers, $y_1, \ldots, y_n$ and $Y_1, \ldots, Y_n$, the theorem of mean value tells us that

$$(3)\quad \begin{aligned} &f_i(x, Y_1, \ldots, Y_n) - f_i(x, y_1, \ldots, y_n) \\ &\quad = (\partial f_i/\partial y_1)(Y_1 - y_1) + \ldots + (\partial f_i/\partial y_n)(Y_n - y_n), \end{aligned}$$

the partial derivatives being evaluated at some point on the line segment joining $(y_1, \ldots, y_n)$ to $(Y_1, \ldots, Y_n)$. But by hypothesis all these partial derivatives have absolute values less than a constant M, so by (3)

$$\begin{aligned} &|f_i(x, Y_1, \ldots, Y_n) - f_i(x, y_1, \ldots, y_n)| \\ &\qquad \leq M \sum_1^n |Y_k - y_k|. \end{aligned}$$

If we add these n inequalities for i running from 1 to n, we obtain

$$(5)\quad \begin{aligned} &\sum_1^n | f_i(x, Y_1, \ldots, Y_n) - f_i(x, y_1, \ldots, y_n)| \\ &\qquad \leq Mn \sum_1^n |Y_k - y_k|. \end{aligned}$$

From (2) and (5),

$$(6)\quad \begin{aligned} &\sum_1^n | y_i(x;j+1) - y_i(x;j) | \\ &= \sum_1^n \left| \int_\xi^x [f_i(x, y_1(x;j), \ldots, y_n(x;j)) - f_i(x, y_1(x;j-1), \ldots, y_n(x;j-1))] dx \right| \\ &\leq \left| \sum_1^n \int_\xi^x |f_i(x, y_1(x;j), \ldots, y_n(x;j)) - f_i(x, y_1(x;j-1), \ldots, y_n(x;j-1))| dx \right| \\ &\leq \left| Mn \int_\xi^x \sum_1^n | y_i(x;j) - y_i(x;j-1) | \ dx \right|. \end{aligned}$$

Since both $y_i(x;0)$ and $y_i(x;1)$ are continuous functions, the inequality

$$(7) \qquad \sum_1^n |y_i(x;1) - y_i(x;0)| < K$$

is satisfied for some constant K, as x ranges from a to b. We can now show that

$$(8) \qquad \sum_1^n |y_i(x;j+1) - y_i(x;j)| \leq K(Mn)^j |x - \xi|^j/j!$$

for all x in the interval from a to b and all non-negative integers j. For j = 0 this follows at once from (7). For other j we prove it by induction. If (8) is true for a value k - 1 of j, then by (6) and (8) for j = k - 1 we have

$$\sum_1^n |y_i(x;k+1) - y_i(x;k)|$$
$$\leq \left| Mn \int_\xi^x K(Mn)^{k-1} |x - \xi|^{k-1}/(k-1)! \, dx \right|$$
$$= K(Mn)^k |x - \xi|^k/(k!),$$

so that (8) holds for j = k also. This completes the induction.

As a consequence of (8),

$$(9) \qquad \sum_1^n |y_i(x;j+1) - y_i(x;j)| \leq K(Mn)^j (b - a)^j/j!$$

for all x in the interval from a to b and for all non-negative integers j. But the right member is the general term of a convergent series of positive terms, so the left member of (9) is also the general term of a convergent series. In particular, for each one i of the integers 1, ..., n the sum

$$\sum_{j=0}^{\infty} |y_i(x;j+1) - y_i(x;j)|$$

converges. It still converges if we remove the absolute value sign. Therefore

$$\sum_{j=0}^{m} [y_i(x;j+1) - y_i(x;j)] = y_i(x;m+1) - y_i(x;0)$$

has a limit as m increases without bound, and consequently $y_i(x;m+1)$ tends to a limit as m tends to ∞. Let $y_i(x)$ denote this limit. If ε is a positive number, we can select an m so large that the sum of the right members of (9) from $j = m$ to ∞ is less than ε/Mn. Then for every pair of numbers p and $q > p$ both greater than m we have

$$|y_i(x;p) - y_i(x;q)| \leq \sum_{j=p}^{q-1} |y_i(x;j+1) - y_i(x;j)|$$

(10)
$$\leq \sum_{j=m}^{\infty} K(Mn)^j(b - a)^j/j!$$

$$< \varepsilon/Mn.$$

We now let q tend to ∞, holding p fixed, and find

(11) $$|y_i(x;p) - y_i(x)| \leq \varepsilon/Mn.$$

From this and (4),

(12) $$\begin{aligned}|f_i(x,y_1(x;p), \ldots, y_n(x;p))\\ - f_i(x, y_1(x), \ldots, y_n(x))| \leq \varepsilon.\end{aligned}$$

If we integrate this from ξ to x, we find that the values of the integrals from ξ to x of the functions inside the absolute value sign in (12) differ by at most $\varepsilon|x - \xi|$, which cannot exceed $\varepsilon(b - a)$.

Therefore

$$(13)\quad \lim_{p \to \infty} \int_{\xi}^{x} f_i(x, y_1(x;p), \ldots, y_n(x;p))\, dx = \int_{\xi}^{x} f_i(x, y_1(x), \ldots, y_n(x))\, dx.$$

Now in (2), as j tends to ∞ the left member tends to $y_i(x)$, while by (13) the right member tends to the right member of (13). Therefore $y_i(x)$ equals the right member of (13), or in other words equations (1.2) are satisfied. These are equivalent to the differential equations (1.1), and the theorem is proved.

It can also be shown that the solution thus obtained is unique. Suppose that there were two solutions of (1.1), which we denote by $y_i(x)$ and $Y_i(x)$, both having the values η_i at $x = \xi$. We assume that there is a point c in the interval from a to b at which they differ, and arrive at a contradiction. To be specific, we suppose c between a and ξ. Define

$$(14)\quad r(x) = \sum_{1}^{n} [Y_i(x) - y_i(x)]^2.$$

This vanishes at ξ but not at c. Let d denote the point between c and ξ which is the closest point to c at which r(x) vanishes. This may be ξ itself. Then r(x) is positive for $c \leq x < d$ and vanishes at $x = d$. Since both y_i and Y_i satisfy the differential equations (1.1), the derivative of r is

$$r'(x) = 2 \sum_{1}^{n} [Y_i(x) - y_i(x)][Y_i'(x) - y_i'(x)]$$

$$(15)\quad = 2 \sum_{1}^{n} [Y_i(x) - y_i(x)] \cdot [f_i(x, Y_1, \ldots, Y_n) - f_i(x, y_1, \ldots, y_n)].$$

If we replace the second factor by means of (3), we find

$$r'(x) \tag{16}$$
$$= 2 \sum_{i,j=1}^{n} (\partial f_i/\partial y_j)[Y_i(x) - y_i(x)][Y_j(x) - y_j(x)],$$

the partial derivatives being evaluated at some point on the line segment joining $(y_1, \ldots, y_n)$ to $(Y_1, \ldots, Y_n)$. The absolute values of the derivatives cannot exceed M, by hypothesis, and each of the other factors is at most equal to the square root of r, by (14), so from (16) we conclude that

$$|r'(x)| \leq 2Mn^2 r(x). \tag{17}$$

Next, for all x between c and d (not including d) we define

$$v = \log r(x). \tag{18}$$

Then by (17),

$$|v'| \leq 2Mn^2. \tag{19}$$

By the theorem of mean value, for each x between c and d

$$\begin{aligned} |v(x) - v(c)| &= |(x - c)v'(x^*)| \\ &\leq 2Mn^2(x - c), \end{aligned} \tag{20}$$

where x^* is some point between c and x. As x approaches d, we see by (20) that v(x) remains bounded. But as x approaches d we know that r approaches 0, so that v = log r cannot remain bounded. This is the contradiction that establishes the theorem.

If we try to apply the existence theorem just established to the normal equations of the trajectory, we find that the hypotheses are not satisfied. The right members of the equations are defined only for v below a certain number, the upper limit of the table of the drag function; and moreover the partial derivatives are unbounded because of the presence of the exponential H(y).

However, this is not really a serious matter. We can redefine H(y) at all points where it exceeds, say, 3, in such a way that it is bounded and has a bounded derivative with respect to y. And we can extrapolate the drag function beyond the upper limit of its experimentally determined range in such a way that both it and its derivative with respect to v remain bounded. The equations then satisfy the hypotheses of the existence theorem, and therefore have a solution. This solution will not, under anything like reasonable conditions, enter the region in which H exceeds 3. If it enters the region in which v is above the upper limit of the experimentally determined values, the solution has no physically trustworthy meaning. But this is unavoidable; the trouble lies not in the mathematics, but in the lack of physical information about the drag. In any ordinary application of ballistic theory, the solution lies within the region in which the drag function is known to some degree of accuracy; if this were not so, further experiments would be performed to increase our knowledge of the drag.

The choice of the first approximation $y_i(x;0)$ was quite arbitrary, except for the requirements that the functions be continuous and take on the values η_i at $x = \xi$. If we wish to obtain this first approximation by guessing the values of the functions f_i and integrating these guesses, it is our privilege to do so.

The procedure by which the existence theorem was proved furnishes a logically sound, though not very convenient, method of solving differential equations of the type (1.1) by numerical methods. Let us choose a sequence of equally spaced values of x, including among them the initial value ξ, the interval between successive tabular x's being small enough so that the integrations (2) can be performed by numerical quadrature with no more error than we have decided to allow. We could then make any sort of first estimate for the functions $y_i(x)$ or for the right members f_i, and begin the process of successive approximations specified by equations (2). Eventually the columns of values of $y_i(x)$ will begin to repeat themselves, showing no change (to the number of decimal places retained in the solution) from one approximation to the next. At this stage we have obtained a solution of the differential equations, accurate to within a certain error dependent on the number of decimal places retained in the process and the number of lines needed in the computation.

Suppose, to make the notation specific, that the initial value ξ of x is 0, and that the interval in x is ω. We wish to integrate the differential equations from 0 to some point b. The procedure of the preceding paragraph would call for an estimate of y_i all the way from 0 to b, this estimate being then replaced by a sequence of improvements converging to the solution desired. An obvious but important improvement in this procedure results from observing that the successive approximations to the y_i on each line $x = m\omega$ depend only on the earlier approximations on the lines from $x = 0$ to $x = m\omega$ and do not involve the values of the y_i on line $x = (m + 1)\omega$ and succeeding lines. This gives us the privilege of completing the approximation on each line before starting work on the next line. For instance, on lines $x = 0$ to $x = (m - 1)\omega$ we can calculate successive approximations until we reach final values. Now by examining the column of values of each y_i, corresponding to the values $x = 0, \omega, \ldots, (m - 1)\omega$, we can extrapolate to form a good first approximation to y_i on the line $x = m\omega$. Having a good first approximation, the later approximations will rapidly approach the final values, so that usually after one or two stages the final values are reached. This, in essence, is the whole of numerical integration of differential equations. There remains a certain amount of room for choice, in the manner in which the extrapolations are made and in the selection of quadrature formulas; but these are matters of detail rather than of basic principle.

5. Application to an example.

Although the methods of numerical integration that form the subject of this chapter are being discussed primarily because of their applications in ballistics, it will be easier to see the processes involved if we apply them to an example exhibiting the steps needed, but not requiring long computations. We shall discuss the equation

$$(1) \qquad y'' = -y,$$

the initial conditions being so chosen that we obtain the solution

$$(2) \qquad y = \sin x.$$

Thus we can check our results against a table of the natural sine with argument in radians.

The formulas of the preceding sections involved differences of order higher than the first, and these are not available at the beginning of the computation. So starting a solution is a special problem in itself, which we shall postpone for a while. Here we start in medias res, supposing that somehow or other we have managed to obtain several lines of the computation, and we shall show how to proceed to obtain more lines. The computations will be based on the four quadrature formulas of Section 3.

Let us then suppose that the computation has proceeded for a number of lines at interval $\omega = 0.1$, the last three lines being exhibited in the following table:

x	y	Δ^1	Δ^2	y'	y''	Δ^1	Δ^2	Δ^3
.5	.47943	9001	-389	.87758	-.47943	-9001	389	94
.6	.56464	8521	-480	.82534	-.56464	-8521	480	91
.7	.64422	7958	-563	.76484	-.64422	-7958	563	83

The value of y' for x = .6 was obtained by Simpson's rule, and only after the line x = .7 is complete can we check it by (3.3). According to this formula,

$$y'(.6) = .87758 + (0.1)[(- .47943 - .56464)/2 - (1/24)(.00480 + .00563)] \quad (3)$$
$$= .82533.$$

slightly "plus." This is an acceptable check. We leave the entry .82534 unaltered.

Next we start the process of successive approximations by making an extimate of y" for x = .8. This we do by extraprlation of the third difference by guesswork. We shall guess that the third difference on line x = .8 will be 73. (As is usual in entering differences we omit the decimal point, expressing them in terms of units of the last decimal place carried in the main column, which in this case is y".) This leads to the entries

(4)

y"	Δ^1	Δ^2	Δ^3
-.71744	-7322	636	73

(Here we have wasted some energy, since all we need at the moment is the last pair of entries, the third and second differences.) Now by (3.1) we compute

$$y'(.8) = .82534 + (0.1)[2(- .64422) + (1/3)(.00636)] \quad (5)$$
$$= .69671$$

Now by (3.12) we find

$$\Delta^1 y(.8) = (0.05)[.76484 + .69671 - (1/60)(- .07322)] \quad (6)$$
$$= .07314$$

(Ordinarily the decimal point would be omitted from before each five-figure entry in this equation.)

From this first difference and the entries on the preceding line we find that y(.8) is .71736, and its second difference is - 644, in units of the fifth decimal place. As a computational check we apply (3.9):

$$\text{(7)}\quad \begin{aligned} \Delta^2 y(.8) &= (0.01)[\,-\,.64422 + (1/12)(.00636)\,] \\ &= -\,.00644, \end{aligned}$$

agreeing to five decimal places with the other result.

The first phase of the process is now finished, and we go on to the second approximation. We have just obtained the approximation .71736 for y at x = .8, so the corresponding value of y" is the negative of this by (1), and we obtain the entries

	y''	Δ^1	Δ^2	Δ^3
(8)	-.71736	-7314	644	81

Our guess at the value 73 for the third difference was not a very good one, since this better approximation gives the third difference the value 81. We now repeat the whole sequence of operations, except for the verification of the entry for y' on line x = .6. The right member of (5) is amended this time by having .00644 in place of .00636; but to five decimals the value of y' is unchanged. The right member of (6) is amended by having - .07314 in place of - .07322; but to five decimal places this leaves y unaltered. So the second approximation yields the same values for y' and y as the first approximation did, and the process is complete. Line x = .8 now has its final values, and we are ready to go on to the next line.

It will be noticed that the first approximation did not look particularly accurate, since the extrapolation of y" was in error by eight units of the last decimal place. Nevertheless, it turned out that this was actually good enough. Re-computation did not alter the values of y' and y, and therefore was unnecessary.

But we had to go through the computation of the second approximation in order to find this out. In other words, the re-computation was unnecessary, but was nevertheless necessary in order that we could be sure that it was unnecessary! This is a rather undesirable state of affairs. Clearly it would be convenient to have some sort of reasonably simple test by which we could be sure that re-computation would not change the values of the functions without having to go through the whole process merely to verify that they do not change. This will be discussed further in Section 7.

The general type of second-order differential equation would have the form $y'' = f(x, y, y')$. The problem we have been discussing belongs to the special subclass in which the right member happens to be independent of y'. We did not choose to exploit this peculiarity in the preceding paragraphs, because we wished to exhibit the processes that would be used in solving a general second-order differential equation. However, now that that has been done, it is useful to point out that second-order equations in which the right member is independent of y' can be treated especially expeditiously. Let us return to the tabulation just before equation (3), and again guess that the third difference of y'' on line $x = .8$ will be 73. We again obtain (4), except that as remarked we have no need of the values of y' and its first difference, so they need not be computed. By (3.9) we compute $\Delta^2 y(.8)$; this is identical with (7). With this we compute the entries

(9)

y	Δ^1	Δ^2
.71736	7314	-644

This gives us (8) for the second approximation to y'', and as before we see that the second approximation to y is the same as the first, so that the line is finished. The amount of numerical work involved is considerably less than in the general process.

There is a disadvantage in that there is no check formula. But this can be repaired in part by carrying an extra column in which the running sum of the values of y" is entered and using on each line the following check, obtained from (3.9) by summing:

$$\Delta^1 y(m\,\omega) - \Delta^1 y(0)$$

$$(10) \qquad = \omega^2 \left\{ \sum_{j=0}^{m-1} y''(j\,\omega) + (1/12)\left[\Delta^1 y''(m\,\omega) - \Delta^1 y''(0)\right] \right\}.$$

For use in checking, this would be more convenient if rearranged into the form

$$\Delta^1 y(m\,\omega) = \left[\Delta^1 y(0) - (\omega^2/12)\,\Delta^1 y''(0)\right]$$

$$(11) \qquad + \omega^2 \left\{ \sum_{j=0}^{m-1} y''(j\,\omega) + (1/12)\,\Delta^1 y''(m\,\omega) \right\}.$$

The first term in the right member does not change with m.

6. The start of the solution.

When beginning the numerical integration of a differential equation we lack the differences that played an important role in the quadratures, and must use some special device as a substitute. The simplest possible device is to start with an interval ω so small that the trapezoidal rule is accurate enough for computing the first derivative by quadrature of the second derivative, assuming that we are still discussing second-order equations. There is much less chance of any trouble in obtaining the function itself from its derivatives, since if y' and y" are known with sufficient accuracy on two consecutive lines, equation (3.12) can be used to obtain an estimate of y whose error is a small multiple of the third difference of y".

The process can probably be more clearly understood from inspection of an example than from a discussion in general terms. So let us suppose that we are again trying to solve the differential equation (5.1), but that now we are to start from the initial conditions

(1) $y = .47943$ and $y' = .87758$ at $x = .5$.

We shall attempt to use $\omega = .05$ as interval at the start. By the differential equation (5.1) itself, we have $y'' = -.47943$ at $x = .5$. As a first approximation, we suppose that y'' is still equal to $-.47943$ at $x = .55$. By the trapezoidal rule ((3.3) with all differences omitted) we find that y' is .85361 at $x = .55$. By Morrey's rule (3.12), which since $\Delta^1 y'' = 0$ is merely the trapezoidal rule at this stage, we find $\Delta^1 y = .04328$, so that $y = .52271$ at $x = .55$. Now we start the second approximation. By the differential equation itself, we have the second approximation $y'' = -.52271$ at $x = .55$. Again we use the trapezoidal rule to obtain the second estimate $y' = .85253$ at $x = .55$. Now formula (3.12), which is no longer the trapezoidal rule because we have a first difference of y'' to use in it, gives us $y = .52269$. This differs little from the preceding approximation .52271, and we find, as we might expect, that the third approximation starting with $y'' = -.52269$ gives us $y = .52269$ back again, so that the process is closed, and we have reached final values. Whether we have reached correct values is quite another matter. We have used the trapezoidal rule in computing y' from y'', and thereby have disregarded the second-difference terms in (3.3). We will not be able to learn if we were justified in doing this until the computation has gone at least a line or two further, so that some second differences are available to furnish an estimate of the error that we have made.

The information so far attained is that in the following table.

x	y	Δ^1	Δ^2	y'	y"	Δ^1	Δ^2
.5	.47943			.87758	-.47943		
.55	.52269	4326		.85253	-.52269	-4326	

We can now abandon the trapezoidal rule. Lacking any information about the second difference of y", we assume as a first approximation that the first difference of y" is still equal to - 4326 on the line x = .60. Then by (3.1), we find y' = .82531, and by (3.12), y = .56465. We now proceed to the second approximation. By the differential equation, y" = - .56465 at x = .60, so that the first difference of y" is - 4196 and the second difference is 130. Again using (3.1) we find y' = .82533, and by (3.12) we obtain y = .56465 again. (The trigonometric table has .56464 instead, but our value is actually almost a half unit of the fifth decimal place less than .56465, so the agreement is good.)

From this point we may proceed in either of two ways. We may go on at once to the line x = .7. To do this we would extrapolate y" to the line x = .7, find y' by Simpson's rule using the values of y" on lines x = .5, .6 and .7, and then find y by (3.12), as first approximation; then use this first approximation with the differential equation y = - y" to obtain a second approximation to y", and so on. A difficulty with this procedure is that we are rather likely to make a poor extrapolation, lacking the higher order differences to guide us. Some help can be obtained from the second difference formed from the values of y" on lines x = .5, .55 and .6; we could assume as a first approximation that the second difference is constant and thus extrapolate y" to line x = .7. Nevertheless, it is still fairly likely that we would have to go through several approximations before the final values of y", y' and y are reached. As an alternative to this process, we could be less valorous and more discreet, computing two more lines at the shorter interval (that is, computing lines x = .65 and x = .7), then extrapolating y" to line x = .8 and proceeding from this point using only lines $x = .5$, $.6$, and $.7$, as in the preceding section. This procedure would call for computing one line, $x = .65$, which is ignored after it serves its purpose of guiding the extrapolation to $x = .7$. But in practice it often turns out to be a net saving to do this; if one re-computation of the line $x = .7$ is avoided, this extra line has paid for itself.

We now investigate a second method of starting the numerical integration which is quite advantageous whenever it happens to be easy to compute the third derivative y'''. In fact the advantage of this method is not at all bound up with the order of the equation; it is equally useful with a system of first-order equations like (1.1), provided that the derivatives y_i'' can be easily computed. By (2.8) we see that a knowledge of the first derivative of a function at $x = 0$ can furnish us with an estimate of the first difference of the function at $x = \omega$. But this is by no means all that can be done with the help of the knowledge of $f'(0)$. Let us imagine that $f(x)$ has been approximated by a polynomial $P(x)$, for which we choose to use formula (2.27), which is Stirling's formula. Then we can obtain an estimate of $f'(0)$ by differentiating the right member of (2.27), and setting $x = 0$. Recalling that $x = n\omega$, so that $dn/dx = 1/\omega$, we obtain

$$\begin{aligned} \omega f'(0) = {} & [\Delta^1 f(0) + \Delta^1 f(\omega)]/2 \\ & - [\Delta^3 f(\omega) + \Delta^3 f(2\omega)]/12 \\ & + [\Delta^5 f(2\omega) + \Delta^5 f(3\omega)]/60 + \dots, \end{aligned} \tag{2}$$

the succeeding terms involving means of differences of odd order greater than 5.

An immediate consequence of this equation is

(3) If differences of order 2 and higher are ignored,

$$\Delta^1 f(\omega) = \omega f'(0).$$

Next we replace $\Delta^1 f(0)$ by $\Delta^1 f(\omega) - \Delta^2 f(\omega)$, as we may by definition of the second difference. Solving (2) for the second difference yields

(4) If differences of order 3 and higher are ignored,

$$\Delta^2 f(\omega) = 2\,[\Delta^1 f(\omega) - \omega f'(0)].$$

If differences of order 4 and higher are to be ignored, we may consider that $\Delta^3 f(\omega)$ may be replaced by $\Delta^3 f(2\omega)$ in (2). Moreover, the relation

$$\Delta^1 f(0) = \Delta^1 f(\omega) - \Delta^2 f(2\omega) + \Delta^3 f(2\omega)$$

is an identity, by (1.5). Making these substitutions in (2) yields

(5) If differences of order 4 and higher are ignored,

$$\Delta^3 f(2\omega) = (3/2)\,\{\Delta^2 f(2\omega) - 2[\Delta^1 f(\omega) - \omega f'(0)]\}.$$

By use of (5) and the identity

$$\Delta^3 f(2\omega) = \Delta^2 f(2\omega) - \Delta^2 f(\omega)$$

we obtain

(6) If differences of order 4 and higher are ignored,

$$2[\Delta^1 f(\omega) - \omega f'(0)] = (2/3)\,\Delta^2 f(\omega) + (1/3)\,\Delta^2 f(2\omega).$$

It will be observed that (3) and (4) are immediate consequences of (6).

We shall now describe the application of these formulas to the start of the solution of a single second-order differential equation. The application to systems of second-order equations requires only the use of the procedure for each of the variables; and the application to systems of first-order equations requires only the omission of the steps needed to pass from y' to y. As soon as the first line is filled out, the knowledge of the value of y''' on this line enables us to estimate the first difference of y'' on the next line by (3);

hence we have a first approximation for y" on the second line. Now by the trapezoidal rule we compute y' on the second line, and by (3.12) we compute y on the second line. This gives us the second approximation to y" on the second line, and also furnishes us with a better estimate of the first difference of y". So (4) gives us a first estimate of the second difference of y" on the second line. In order to compute the second approximation to y' on the second line, we need no longer use the trapezoidal rule. The estimate of the second difference of y" will serve also as an approximation to the second difference of y" on the third line. So we can use (3.3) with these estimates for the second differences, obtaining a more accurate value of y' on the second line than the trapezoidal rule would furnish. We re-compute y by (3.12), and repeat the process until final values are reached.

Now we proceed to the third line. Since we have an accurate first difference and a good estimate for a second difference of y", we may hope to make a better extrapolation than if we knew the first difference alone. We make the extrapolation and compute y' on the third line by Simpson's rule, which for this purpose is conveniently expressed in the form (3.1). We next compute y on the third line by (3.12), checking the second difference by means of (3.9). This process is repeated until final values are reached. We now have the value of the second difference of y" on the third line. We also still have the value of the estimate (4) of the second difference of y" on the second line. We could estimate the third difference of y" on the third line by subtracting the latter of these from the former. But (5) tells us that we should not enter the result of this subtraction as the third difference of y". Instead, we should enter 3/2 of its value. Then for consistency we must revise the estimate of the second difference given by (4) and already used; it must be replaced by the result of subtracting the third difference just computed from the computed second difference of y" on the third line.

The most inaccurate step in this process is the first approximation to y' by use of the trapezoidal rule. But this is of no importance, since this estimate is later replaced by a more accurate one. Of the steps left in the final computation, the one most open to doubt is still the computation of y' on the second line by use of (3.3), in which the approximation (4) is used for the second difference on both the second and third lines. In the notation of (4), we have used the approximation

$$(7) \quad \int_0^{\omega} f(x)\,dx$$

$$= \omega\{ [f(0) + f(\omega)]/2 - (1/6)[\Delta^1 f(\omega) - \omega f'(0)]\}$$

as a substitute for (3.3). If fourth differences are ignored, the difference between (7) and (3.3) can easily be shown to be only

$$(8) \qquad (\omega/72)\,\Delta^3 f(2\omega).$$

If this amounts to less than half a unit of the last significant figure, the formula error committed by using (7) may be neglected. Otherwise we must carry out another improvement. After the third line is complete, we have an accurate second difference of y" on the third line and a good estimate on the second line. It is thus possible to re-compute y' on the second line by means of (3.3) without having to fall back on the approximate form (7). If the third difference of y" is too large to let us ignore the error (8), this revision should be made, and all subsequent revisions which it may imply.

The application of this method of starting the solution to the particular problem we have been using as an example will be sufficiently clearly shown by means of the following table. In this table the first, second and third approximations to the quantities on the second and third lines are all shown; in practice all but the final values would commonly be suppressed. All quantities other than the final values are underlined.

The subscripts show the order in which the numbers are computed. The quantities in square brackets in the last three columns are those furnished by the formulas (3), (4) and (5) respectively. The entry [508] in the final approximation to line x = .6 is that furnished by (4); the entry 480 below it is the result of correcting it with the help of (5), which yields the entry 85 as the third difference of y" at x = .7. Two of the entries have two subscripts. The value of y' for x = .6 was computed at the eighteenth step by formula (7) and verified at step 42 by (3.3). In fact, since the third difference of y" is only 84 we see that the error (8) is negligibly small. The second difference of y for x = .7 is computed at step 36 from the first difference given by (3.12) and verified immediately by (3.9). The initial conditions are those in (1), but we begin with interval .1 instead of with the interval of .05 that was used with the other method of starting.

x	y	Δ^1	$-\Delta^2$	y'	y"	Δ^1	Δ^2	Δ^3
.5	$.47943_{1}$			$.87758_{2}$	$-.47943_{3}$			
.6	$.\underline{56464}_{8}$	$\underline{8521}_{7}$		$.\underline{82525}_{6}$	$-.\underline{56719}_{5}$	$[-8776]_{4}$		
.6	$.\underline{56465}_{14}$	$\underline{8522}_{13}$		$.\underline{82533}_{12}$	$-.\underline{56464}_{9}$	$-\underline{8521}_{10}$	$[\underline{510}]_{11}$	
.6	$.56465_{20}$	8522_{19}		$.82533_{18,42}$	$-.56465_{15}$	-8522_{16}	$[508]_{17}$ 480_{32}	
.7	$.\underline{64422}_{26}$	$\underline{7957}_{25}$		$.\underline{76482}_{24}$	$-.\underline{64479}_{23}$	$-\underline{8014}_{22}$	$[508]_{21}$	
.7	$.\underline{64423}_{35}$	$\underline{7958}_{34}$	$-\underline{564}_{36,37}$	$.\underline{76484}_{33}$	$-.\underline{64422}_{27}$	$-\underline{7957}_{28}$	$\underline{565}_{29}$	$[\underline{85}]_{31}$
.7	$.64423_{45}$	7958_{44}	-564_{46}	$.76484_{43}$	$-.64423_{38}$	-7958_{39}	564_{40}	84_{41}

7. The tolerance.

It has already been remarked that there is a rather considerable amount of waste effort in the procedures outlined in preceding sections, because the last approximation merely duplicates the values found in the approximation before it, and serves no other purpose than to confirm that the preceding approximation had already furnished the final values. Clearly there would be an advantage in knowing at some stage that although the approximation completed did not repeat the values of the stage before it, another approximation would repeat the values just found. This would allow us to omit the last, merely confirmatory, approximation.

Let us return again to the example we have been investigating. Suppose that the difference between the values of y'' furnished by two consecutive approximations is ε. There is then the same difference between the first differences of y'' furnished by the two approximations, and likewise for the second differences. Then in computing y' by (1.1) we find a difference of $\omega\varepsilon/3$ between the values obtained in the two approximations. The corresponding difference between the two values of y, by (1.12), is

$$(\omega/2)[\omega\varepsilon/3 - (\omega/6)\varepsilon] = \omega^2\varepsilon/12.$$

It would not seem that this could be applied to the second line, since there we must use (6.7) if we adopt the starting process based on the derivative of y''; but it happens that (6.7) leads us to the same result. Thus we see that if the values of y'' at approximations $n - 1$ and n differ by ε, the values of y will differ by $\omega^2\varepsilon/12$, and so the values of y'' at approximations n and $n + 1$ will differ by $-\omega^2\varepsilon/12$.

No matter how small this quantity is, we can never guarantee that it will not affect the last significant figure of y'' when rounded off to some preassigned number of decimal places. For it may happen that the first omitted figure in y'' at stage n is very nearly a 5, so that a very small change would alter it from just below to just above 5, and thereby change the digit in the last exhibited decimal place by a unit. However, as long as we are rounding off all our entries to a certain number of decimal places (five, in the example) we are signifying our willingness to accept errors of half a unit of the last decimal place carried. Thus, it is not unreasonable to augment by a little the size of the error we are willing to accept, and agree that we shall stop the process of successive approximations as soon as we know that the next stage would not affect any entry by more than half a unit of the last decimal place carried. Then we are willing to stop the process as soon as the error $\omega^2\varepsilon/12$ of the next step does not exceed half a unit of the last decimal place carried. In our example, the permissible error is

$$\omega^2\varepsilon/12 = .5 \text{ units of the fifth decimal place,}$$

with $\omega = .1$. Hence

$$\varepsilon = 600 \text{ units of the last decimal place.}$$

In the calculations exhibited in the preceding sections, the discrepancy between first and second approximations was never as great as 600 units of the fifth decimal place. Hence in every instance the application of this criterion would have marked the values obtained in the second approximation as final, and we would never have gone on to a third approximation; whereas in every instance we did go on to a third approximation merely to confirm the fact that the second approximation gave final values. However, this is a slight overstatement. In the tabulation at the end of the preceding section it will be noticed that the first and second approximations both on line x = .6 and on line x = .7 differed from each other by less than the tolerance of 600 units just suggested. Yet in each case the second and third approximations to y" differed from each other by a unit of the fifth decimal place. This is particularly striking in the case of line x = .7, where the first and second approximations to y" differed only by 57 units of the fifth decimal place, much less than the permissible 600.

It happens that on both the lines the phenomenon mentioned in the preceding paragraph occurred. For instance, on line x = .7 the first difference, whose final value is recorded as 7958, is actually almost exactly halfway between 7957 and 7958, and the small change in y" from first to second approximation was enough to throw it over the halfway-mark. It happens that on both lines the final value of the first difference of y is just about half a unit of the fifth decimal place too large, so that in fact .64422 would be a more accurate final value for y(.7) than the final value recorded, .64423. (In fact, .64422 is the value shown in the table of natural functions.)

However, this same example of the table at the conclusion of Section 6 serves to show us that we are at the edge of a deeper problem than the one we have been investigating. Certainly there is an advantage, as we have said, in knowing when to stop the process of successive approximations, without having to go on until two successive approximations are duplicates. But, to state it somewhat superficially, it is hardly enough to know when the final values have been reached with error not over half a unit of the last decimal place carried. If the entire solution is to have error nowhere greater than some preassigned bound, we must also know at the least how many decimal places should be carried. Clearly it is not sufficient to carry five decimal places on each line if the whole table is to be accurate to five decimal places. In the table at the end of the preceding section, each of the first differences of y is less than half a unit of the fifth decimal place in error. Yet their sum is nearly a unit of the fifth decimal place in error, because the errors of the two first differences happen to be nearly half a unit each and are of the same sign. Thus an error has already occurred in y(.7) which will persist, unless by chance an equal and opposite error occurs later; and this is no more likely than that an equal error of the same sign will occur later. So we have before us an example of the effects of the accumulation of rounding errors.

The simplest, but not the most useful, way of attacking the problem of setting a bound on the error would be to fix a tolerance on each line so stringent that even if the greatest possible error occurred on each line and all were in the same direction the total effect would still be within the allowable amount. But it is evident that in a computation extending through a fairly large number of steps it is extremely unlikely that all the errors would be of maximum amount and of the same sign. Consequently we shall set ourselves the more difficult, but more realistic, problem of finding the limitations that we must set on the errors in each separate line in order that their cumulative effects shall remain under a preassigned bound, not with absolute certainty, but with a probability of, say, 99 per cent.

Let us suppose that an equation of the type

$$y'' = F(t, y, y')$$

has been solved by some selected process of numerical integration, the solution being computed for certain values $t_0, \ldots, t_n = T$ of the time t. On each line there will be some error in y'', due to two causes. First, the y'' written will be given to only a certain number of figures, being rounded off to a selected number of decimal places. From the values of the entries on preceding lines, by use of the formulas of the integration process being used, there would follow a value of y'' on the line $t = t_i$ which would ordinarily be an unending decimal. This is rounded off to the nearest multiple of 10^{-k}, where k is the number of decimal places carried. The change introduced by this rounding error will be called the rounding error in y'' on the line $t = t_i$, and will be denoted by the symbol $r''(t_i)$. Observe that this is not the difference between the number entered on the i-th line and the accurate value of $y''(t_i)$. It is the difference between the number entered and the number which would be accurately deduced from preceding entries, which themselves are already in error because of rounding.

The second kind of error in $y''(t_i)$ arises from the fact that the process of computation involves making a first approximation and improving it by successive approximations. No matter how far this process is carried, there will always remain some departure from the true value, depending on the error in the original extrapolation and on the number of stages of improvement. This remaining error will be called the extrapolation error. It is of course not the same as the error in the first extrapolation, but is the (usually much smaller) error in the final stage which is caused by the original error in the first extrapolation. We shall denote this extrapolation error in y'' on the line $t = t_i$ by $e''(t_i)$.

Each of these errors will produce some resulting error in the last computed value of y, that is y(T). The error in y(T) caused by, say, $r''(t_i)$ may be assumed proportional to $r''(t_i)$, since the error is small and the error in final value will be closely enough represented by its differential, which is linear. The quantity of prime interest to us is the sum of all these errors in the last value of y. We shall study this with the help of the Laplace-Liapounoff theorem (see Section 22 of Chapter I). But this theorem presupposes that the errors are independent. At first this might seem to be meaningless in the present situation. One objection is that it is meaningless to speak of the "probability" that the error in a specific computation shall be less than a certain amount; in one single computation, it is or it is not greater than this, and no probability is involved. This objection is superficial; if it really had any weight, it would indicate that no life insurance policy should be sold to any one man. However, it does raise a reasonable question. Probabilities should be discussed relative to some specified population. What is the population with which we are concerned here? A second objection is perhaps a bit deeper. Ignore for the moment the extrapolation error. With given initial conditions and function F, the rounding error on any one line is not necessarily forced upon us; we can select a different number of decimal places. But when this is done, the errors up to and including a given line determine the values of y", y' and y on that line, hence determine the values of y", y' and y on the next line which follow from these data, and thus finally determine the rounding error on the next line. In what sense can we then say that the rounding errors on successive lines are independent?

Let us denote by y^* and y'^* the initial values of y and y' used in computing the trajectory which we are studying. In this computation we may suppose that F was found from a table in which F was listed against closely spaced values of t, y and y'. We shall denote this tabulated function by $F^*(t, y, y')$. Given a positive number ϵ, we could consider the entire aggregate of solutions which could be obtained by replacing y^*, y'^* and the tabulated values of F^* by other numbers differing from the originals by not more than ϵ. This would give us an aggregate of solutions depending on a very large number of parameters (say N of them), each varying independently over an interval of length 2ϵ. Here we have a situation resembling that of the n roulette wheels discussed in Section 17 of Chapter I.

There is a "cube" in the space of N dimensions having each edge of length 2ϵ. Each point in this "cube" characterizes one solution of the differential equations. The probability measure of an interval lying in this cube will be taken to be the ratio of its N-dimensional volume to that of the whole cube, just as was done with the n roulette wheels. Now we have a specific population and a probability measure over it. To each individual member of the population corresponds a uniquely determined number which is the total effect of all rounding errors on the value of y(T). For each preassigned positive number e there will be a certain subcollection of the population on which this total error does not exceed e. The probability measure of this subcollection is the probability that the total rounding error does not exceed e. As always in probability theory, for any given solution we cannot be certain in advance that the rounding error is less than e. But if we have found that the probability is .99 that the total rounding error is less than e, then we may feel reasonably confident that if we compute ten thousand such solutions, about 9,900 of them will have errors less than e.

In order to discuss the independence of the individual rounding errors $r''(t_i)$, we assume that enough decimal places are carried so that as the parameters vary over the "cube" in N-dimensional space, each error $r''(t_i)$ goes many times through its cycle of values. Assuming that we are rounding to the nearest multiple of α, this cycle of values consists of going from $-\alpha/2$ to $+\alpha/2$. We are thus assuming that the surfaces of discontinuity of each $r''(y_i)$ in the "cube" are numerous. It is safe to assume that the correct values of the $y''(t_i)$ are linear functions of the parameters, since they may be replaced by differentials with respect to the parameters without serious error. For each i, we can find a parameter which affects the value of $y''(t_i)$, but does not affect $y''(t_j)$ for $j < i$; for instance, any parameter which changes $F(t, y, y')$ at a value of t between t_{i-1} and t_i produces this effect. So it is possible to replace the original parameters by new ones, linear combinations of the old ones, with the property that changing the parameter q_i changes $y''(t_i)$ but does not change any $y''(t_j)$ with $j < i$, for $i = 0, \ldots, n$. There will also be $N - (n + 1)$ other parameters $q_{n+1}, \ldots, q_{N-1}$ which may affect all the $y''(t_i)$. These new parameters will vary over the image of the original "cube" under the change of parameters. This image will be a parallelopiped Q. Since volumes are all multiplied by the same factor under a linear mapping, the probability measure of a set is still equal to the ratio of its volume to the volume of the whole set over which the parameters range, namely, vol Q.

If we write the N parameters in the order

(1) $$q_n, q_{n-1}, \ldots, q_o, q_{n+1}, \ldots, q_{N-1}.$$

the interior of this parallelopiped is defined by a sequence of inequalities

(2) $$a_k < q_k < b_k,$$

where each a_k and b_k are functions of all the parameters which follow q_k in the arrangement (1). If $f(q)$ is a function integrable over the parallelopiped, its mathematical expectation is (1/vol Q) times its integral, which can be evaluated by integrating first with respect to q_n between a_n and b_n, all other variables being fixed, then integrating the result of this with respect to q_{n-1} between a_{n-1} and b_{n-1}, and so on.

Consider now a sequence of inequalities

(3) $$h_i \leq r''(t_i) < k_i \qquad (i = 0, \ldots, n),$$

where $-\alpha/2 \leq h_i < k_i \leq \alpha/2$ for each i. Let S_i be the set of values of the parameters q on which the i-th of the inequalities (3) is satisfied, and let $K_i(q)$ be its characteristic function, that is, the function which is equal to 1 at all points q in S_i and is equal to 0 elsewhere. For each i, the parameters q_j preceding q_i in the arrangement (1) have no effect on the entries on lines $t = t_o, \ldots, t_i$ of the solution, and therefore have no effect on $r''(t_i)$. Consequently $K_i(q)$ is independent of the parameters preceding q_i in the arrangement (1). Let S be the set of parameters q at which all the inequalities (3) are satisfied, and let $K(q)$ be its characteristic function. Then $K(q)$ is equal to 1 if and only if all inequalities (3) are satisfied, which is true if and only if each $K_i(q)$ has the value 1. It follows that $K(q)$ is the product of the $K_i(q)$ for $i = 0, \ldots, n$. The probability measure of the set S is (1/vol Q) times the integral of $K(q)$ over the parallelopiped Q.

Since K is the product of the K_i and K_i is independent of the parameters preceding q_i in (1), this probability measure is

$$(4)\quad p(S) = \frac{1}{\text{vol } Q} \int_{a_{N-1}}^{b_{N-1}} \cdots \int_{a_{n+1}}^{b_{n+1}} \int_{a_0}^{b_0} K_0(q) \int_{a_1}^{b_1} K_1(q) \cdots \int_{a_n}^{b_n} K_n(q)\, dq_n \ldots dq_0 dq_{n+1} \ldots dq_{N-1}.$$

In the innermost integral, as the variable of integration q_n varies from a_n to b_n the function $y''(t_n)$ varies linearly between certain limits, and except for a small set of values of the q_i with $i \neq n$ this function will run through a range considerably longer than α. Correspondingly the difference between $y''(t_n)$ and the nearest multiple of α, which is $r''(t_n)$, will run many times through its full cycle of values. In each cycle, the fraction of the total length of the cycle during which (3) is satisfied is $(k_n - h_n)/\alpha$. So if $y''(t_n)$ ran through an integral multiple of α, $K_n(q)$ would have the value 1 on a subset of the interval

$$a_n < q_n < b_n$$

with total length $(k_n - h_n)/\alpha$ times the length $b_n - a_n$ of the interval. On the rest of the interval it has value 0. In general, $y''(t_n)$ will not run through a range whose length is an integral multiple of α, but if we ignore the fractional part left over we shall not be greatly in error in our estimate of the ratio of the part of the interval on which $K_n = 1$ to the length of the whole interval. Thus with small error we may replace K_n by 1 in (4) if we simultaneously write a factor $(k_n - h_n)/\alpha$ in front of the integral.

This being done, we can rearrange the order of the integrations in (4) so that the innermost integration is with respect to q_{n-1}. The same argument can be repeated to show that there is small error if we replace K_{n-1} by 1, at the same time writing a factor $(k_{n-1} - h_{n-1})/\alpha$

in front of the integral. This process can be repeated until all the K_i have been replaced by 1. But then the integral is the integral of 1 over the parallelopiped Q, and its value is vol Q. Hence (4) reduces to

$$(5) \qquad p(S) = (k_1 - h_1)\ldots(k_n - h_n)/\alpha^n.$$

In particular, if for some i we choose h_i and k_i to be arbitrary numbers satisfying $-\alpha/2 \leq h_i < k_i \leq \alpha/2$ but for all other numbers j from 1 to n we choose $h_j = -\alpha/2$ and $k_j = \alpha/2$, the set S_j on which these last inequalities hold is the whole population. So the common part of the sets $S_1, \ldots, S_n$ is simply S_i itself, and (5) reduces to

$$(6) \qquad p(S_i) = (k_i - h_i)/\alpha.$$

Comparison of (5) and (6) shows that

$$(7) \qquad p(S) = p(S_1)\ldots p(S_n),$$

apart from the small error made by ignoring fractions of cycles of rounding, and so the functions

$$r''(t_1), \ldots, r''(t_n)$$

may be regarded as independent.

The independence of the extrapolation errors $e''(t_i)$ is a rather different matter. Here we have no chance of constructing a proof like that in the preceding paragraphs, because the extrapolation error is the residual effect of the computer's inability to guess without error at the value of y" on the next line. In some computation processes a formula is used to guide the extrapolation from each completed line to the next line. If this is done, the extrapolation errors cannot be considered independent. There will, for example, be long stretches of consecutive lines on each of which the formula will give too large an estimate. But experience indicates that a computer of any ability, faced with this situation, will show sufficient independence of action to refuse to accept the results of the formula, and will introduce an ad hoc correction so as to make the extrapolated value more nearly equal to the final value. As soon as the computer introduces this element of guesswork, the rigidly predictable nature of the extrapolation has been done away with. In the process described in preceding sections the computer is expected to make an extrapolation according to his best judgment from the values and differences available. In this case there is no question of predictability of extrapolation on one line from that on the preceding line. However, it may still be argued that the extrapolation errors are not independent, since a computer who has made an error in excess on one line may be inclined to make an error in defect on the next line in overcompensation. This cannot be readily refuted. An examination of some trajectory computation sheets does not seem to indicate that there is such a tendency, but a long and careful analysis would be needed to settle the point. However, it hardly seems to be worth the trouble to make such an analysis. For it is very unlikely that the effect of an erroneous guess would persist beyond the next line, so that the sums of extrapolation errors on consecutive triples of lines would presumably be independent, or very nearly so. The possible interrelation between consecutive lines might conceivably affect the total error somewhat, but it is unlikely that this effect would be large enough to disturb the following discussion to any serious extent.

Having now decided that the errors $r''(t_i)$ and $e''(t_i)$ may be treated as independent, we need to find the error in $y(T)$ produced by either of these errors on line $t = t_i$. Here the example we have been considering is somewhat deceptive. In this example it happens to be very easy to compute y'' from y, since the equation is $y'' = -y$. Ordinarily, and in particular in trajectory computations, this is the hardest part of the computation; in order to find x'' and y'' from x', y and y'

it is necessary to find v^2 and use both the G-table and the air density table. So it is usually the case that the last part of the computation is not the finding of a value of x" or y", but is rather the computation of y' and y from the final values of the second derivatives. We shall assume that this is the case henceforth. We shall suppose, to be specific, that the computation process is that presented in the preceding sections of this chapter. A similar analysis would apply to any other method, and in fact the numerical results would be only slightly different for any other method in use.

Let us suppose that the computation is first carried out without error, and is then repeated, but the second time an error ε is made in the value of $y''(t_i)$. This error affects the values of y' and y on this and subsequent lines, and thereby affects the values of y" on the subsequent lines. However, in practice the process of successive approximations must be rapidly convergent if the computation is to be feasible, and as a result a small error in $y''(t_i)$ can produce only a very small error in $y''(t_{i+1})$ and $y''(t_{i+2})$. Hence we may suppose that on the two or three lines following the line $t = t_i$ there is no appreciable error in y". The application of the quadrature formulas to the erroneous column of values of y" is straightforward. If we suppose that at each step the values of y' given by the two formulas (3.2) and (3.3) are averaged, we find that apart from a very small error, the values of y' computed on line $t = t_{i+2}$ and the next few lines are in excess by $\omega\varepsilon$, while the values of y on lines $t = t_{i+2}$, $t = t_{i+3}$, etc., are in excess by approximately $\omega^2\varepsilon$, $2\omega^2\varepsilon$, etc. Thus the effect of the error ε in $y''(t_i)$ is almost exactly the same as though the computation were started again at $t = t_i$ with the same value for $y(t_i)$ but with a new initial value for $y'(t_i)$ which is $\omega\varepsilon$ too great.

If we imagine the solution to be re-computed with initial value t_i for the independent variable, the same initial value for y at time t_i as on the original solution, but with various values of $y'(t_i)$, the value found for $y(T)$ will be a function of $y'(t_i)$. Let us denote the derivative of this function $y(T)$ with respect to $y'(t_i)$ by the symbol

$$(8) \qquad \left. \frac{\delta y}{\delta y'(t_i)} \right|_{t = T} ,$$

which may be safely abbreviated to $\delta y(T)/\delta y'(t_i)$ as long as it is understood that the value of y at the particular value $t = T$ is being investigated as a function of $y'(t_i)$. Then the effect of a small change $\omega \varepsilon$ in $y'(t_i)$ will produce a change

$$(9) \qquad [\delta y(T)/\delta y'(t_i)]\omega\varepsilon$$

in $y(T)$, except for an error of the second order of small quantities. Thus if the rounding error in y'' on line $t = t_i$ is $r''(t_i)$ and the extrapolation error is $e''(t_i)$, the total effect on $y(T)$ produced by all these errors is

$$(10) \quad \Delta y(T) = \sum_{i=0}^{n} [\delta y(T)/\delta y'(t_i)][r''(t_i) + e''(t_i)]\,\omega,$$

since we have already shown that the effect of the error $r''(t_i) + e''(t_i)$ is almost exactly the same as the effect of a change $[r''(t_i) + e''(t_i)]\,\omega$ in $y'(t_i)$. This formula remains valid even if ω is permitted to vary during the computation, provided that $r''(t_i) + e''(t_i)$ is multiplied by the interval ω being used just after the i-th line, that is, $t_{i+1} - t_i$; the truth of this statement is evident from the discussion in the preceding paragraph.

The quantities $r''(t_i)$ and $e''(t_i)$ are independent, and so are their constant multiples appearing in (10). As stated immediately after (I.21.10), the variance of the sum of these independent functions is the sum of their variances, so that

$$V[\Delta y(T)]$$

(11)
$$= \sum_{i=0}^{n} \omega^2[\delta y(T)/\delta y'(t_i)]^2\{V[r''(t_i)] + V[e''(t_i)]\},$$

where V [ ] denotes the variance of the expression in the square bracket.

Let us suppose that a number ε_p has been selected as a "permissible error," in the sense that we are willing to accept a probability of not over .01 that the error in y(T) will exceed ε_p. From the central limiting theorem of probability theory (see Section 22 of Chapter I), if n is large we may assume that the error $\Delta y(T)$ is approximately normally distributed; its variance as given by this theorem is the same as that already computed by (11). In order that the probability of having error greater than ε_p shall not exceed .01, it is necessary that ε_p shall be at least 2.585 times the standard deviation of $\Delta y(T)$; that is,

(12)
$$V[\Delta y(T)] \leq (\varepsilon_p/2.585)^2.$$

But in (11) the left member of (12) is expressed as the sum of two sets of terms, one set involving the rounding errors $r''(t_i)$ and the other involving the extrapolation errors $e''(t_i)$. If we ignore all other rounding errors (this will be discussed later) we can insure that (12) is satisfied by selecting two numbers c_1 and c_2, the sum of whose squares is 1, and then requiring that

(13)
$$\sum_{i=0}^{n} \omega^2[\delta y(T)/\delta y'(t_i)]^2 V[r''(t_i)] \leq (c_1\varepsilon_p/2.585)^2,$$

(14)
$$\sum_{i=0}^{n} \omega^2[\delta y(T)/\delta y'(t_i)]^2 V[e''(t_i)] \leq (c_2\varepsilon_p/2.585)^2.$$

The function $V[r''(t_i)]$ changes only slowly from line to line, except at the places (relatively few in any one computation) at which the number of decimal places is changed. Similarly $V[e''(t_i)]$ changes only slowly from line to line, except where the tolerance set on the agreement of last and second-last lines is suddenly changed. We may therefore regard $V[r''(t_i)]$ as the value at $t = t_i$ of a function $V[r''(t)]$ which has a few discontinuities and is slowly changing between them; and likewise for $V[e''(t_i)]$. Also, the tabular interval ω is constant except at a few places where the interval is changed, and therefore may be regarded as a function $\omega(t)$ which is constant except at a few jumps. If in the left member of (13) one of the factors ω is replaced by its value $t_{i+1} - t_i$ and the other factor ω is replaced by $\omega(t)$, the left member of (13) takes the form of a sum such as is used in defining a definite integral, by means of a passage to the limit. Therefore we see that there will be little error in replacing this sum by the integral of the slowly varying function which forms the coefficients, so that (13) is only slightly different from

$$(15) \quad \int_0^T \omega[\delta y(T)/\delta y'(t)]^2 V[r''(t)]\, dt \leq (c_1 \epsilon_p/2.585)^2.$$

This will be satisfied if the variance of r" satisfies the condition

$$(16) \qquad V[r''(t)] \leq (c_1 \epsilon_p/2.585)^2/J\omega(t),$$

where for brevity

$$(17) \qquad J = \int_0^T [\delta y(T)/\delta y'(t)]^2\, dt.$$

In the same way (14) is nearly equivalent to an expression similar to (15), with r" replaced by e" in the left member and c_1 by c_2 in the right member. This is satisfied if

$$(18) \qquad V[e''(t)] \leq (c_2 \epsilon_p/2.585)^2/J\omega(t).$$

Since the standard deviation $\sigma[r''(t)]$ of $r''(t)$ is the square root of the variance $V[r''(t)]$, (16) is equivalent to

$$(19) \qquad \sigma[r''(t)] \leq c_1\varepsilon_p/2.585\sqrt{J\omega(t)},$$

while (18) is equivalent to

$$(20) \qquad \sigma[e''(t)] \leq c_2\varepsilon_p/2.585\sqrt{J\omega(t)}.$$

In order to make proper use of (19) we must observe that the hitherto neglected rounding error in $y'(t_1)$ is in fact replaceable by an equivalent rounding error in $y''(t_1)$. For we have already shown that the effect of an error of ε in $y''(t_1)$ is essentially the same as the effect of an error of $\omega\varepsilon$ in $y'(t_1)$. Hence a rounding error of ε' in $y'(t_1)$ can be replaced by a rounding error of ε'/ω in $y''(t_1)$. Furthermore, no serious trouble arises from the fact that the rounding errors in y' are usually not uniformly distributed between their least and greatest values. The use of Simpson's rule with its denominator 3 will cause all rounding errors in y' to belong to a finite set, which in the extreme case will consist of three values, namely, 1/3, 0 and - 1/3 units of the last decimal place. (For instance, in applying Simpson's rule to a column of figures each written to the nearest unit and with tabular interval 1, each entry in the column of the integral will end in .0, .3333... or .6666..., and each rounding error will be 0, 1/3 or - 1/3 after the integral is rounded to the nearest unit.) But the variance of this distribution is easily shown to be $2\alpha^2/27$ if figures are rounded to the nearest multiple of α, as compared with the variance $\alpha^2/12$ for the distribution in which all values between $-\alpha/2$ and $\alpha/2$ are equally likely. So for the present purpose we may safely treat the rounding error in y' as though all values between the least and the greatest were equally likely, and replace the error in y' by an equivalent error in y''. Thus if y'' is rounded to the nearest multiple of α'' and y' is rounded to the nearest multiple of α', we may consider that y'' is subjected to two independent rounding errors with the respective variances $\alpha''^2/12$ and $\alpha'^2/12\omega^2$, so that the variance in y'' due to rounding may be taken to be

$$(21) \qquad V[r''(t_i)] = (\alpha''^2 + \alpha'^2/\omega^2)/12.$$

By this artifice we avoid having to make a special allowance for the rounding error in y'.

As an example, suppose that we wish to construct a table of the natural sine with argument in radians from 0 to 2π, the tabular interval being 0.1, and that we wish to have no more than one chance in a hundred that the error in the last entry will exceed one unit of the fifth decimal place. In order to avoid having to consider extrapolation error, we suppose for the present that each line is re-computed until no further change is found, so that the extrapolation error may be supposed to be zero. The solution of the equation $y'' = -y$ with value $y(t_i)$ and derivative $y'(t_i)$ at $t = t_i$ is $y(t_i) \cos (t - t_i) + y'(t_i) \sin (t - t_i)$, so

$$\delta y(T)/\delta y'(t_i) = \sin (T - t_i).$$

With $T = 2\pi$, (17) yields $J = \pi$. Since we have conducted the computation in such a way that the extrapolation error vanishes, we may take $c_1 = 1$ and $c_2 = 0$. Now (21) and (16) yield

$$(\alpha''^2 + 100\alpha'^2)/12 \leq \varepsilon_p^2/(2.585)^2(.31416),$$

or

$$\alpha''^2 + 100\alpha'^2 \leq (5.72)10^{-10}.$$

We can satisfy this by taking $\alpha'' = 10^{-5}$ and $\alpha' = 10^{-6}$. So we see that in solving this equation in Section 5, if we wished to have a probability of .99 that the error would not exceed .00001 at $t = 2\pi$, we should have carried six decimal places rather than five in the column of values of y'.

If in integrating this equation we had chosen to exploit the absence of y' from the differential equation, using the process described at the end of Section 5, there would have been no column of values of y'. But in this method the first differences of y are in effect the integrals of y' between successive tabular values of t, and thus are nearly the same as $\omega y'$. Thus the requirement of six decimal places in y' would be replaced by the requirement of seven decimal places in $\Delta^1 y$.

The discussion of the requirement (18) or (20) requires a closer inspection of the peculiarities of the equation or equations being integrated, and so it will be postponed until we are ready to apply it to the differential equations of the trajectory.

8. Application to the normal equations.

The method of numerical integration demonstrated in the example in Section 5 could be applied with very little change to the normal equations of motion of a projectile. The formula for deducing x'' and y'' from y, x' and y' is more complicated, and there are two variables x and y to solve for instead of one, but the alterations of procedure needed to take care of these changes are quite evident. However, the normal equations have two individual features that permit us to make other small amendments in the computation procedure and thereby save some work. First, x'' and y'' are independent of x, so it is possible to leave the computation of x to the last, after the whole column of values of x' is completed. Second, the dependence of x'' and y'' on y is less sensitive than their dependence on x' and y'. An error in extrapolation of y'' will affect both y and v, and thereby affect both the factor $H(y)$ and the factor $G(v)$ in the formulas for x'' and y''. But ordinarily the per cent of change in H is very much less than the per cent of change in G. As a result, we can afford to permit the extrapolation errors in y to accumulate for two lines, verifying $y(t_1)$ only after $x'(t_{i+1})$ and $y'(t_{i+1})$ have already reached their final values.

The equations to be integrated are

$$(1)\qquad \begin{aligned}\ddot{x} &= -E\dot{x}\\ \ddot{y} &= -E\dot{y} - g\end{aligned}$$

if the y-axis is positive upward, and are

$$(2)\qquad \begin{aligned}\ddot{x} &= -E\dot{x},\\ \ddot{y} &= -E\dot{y} + g\end{aligned}$$

if the y-axis is positive downward, as shown in equations (IV.1.24, 25) and (IV.2.7). For artillery trajectories it is usual to make use of (1), with

$$(3)\quad E = H(y)a(y)G(v/a(y))/C = \gamma H(y)a(y)G(v/a(y)).$$

(See (IV.1.24)). For bomb trajectories it is usual, for reasons explained in Section 2 of Chapter IV, to choose the y-axis positive downward and to assume constant temperature and exponential density, in which case E is given in (IV.2.7):

$$(4)\qquad E = \gamma_s e^{hy} G(v).$$

The columns of the computation are to be headed

t, x, Δ^1, Δ^2, y, Δ^1, Δ^2, $\dot{x}$, $\dot{y}$, $H(y)/C$, a, a^2, $v^2/100a^2$, G, E, $\ddot{x}$, Δ^1, Δ^2, $\Delta^2\ddot{x}$ extrapolated, $\omega\overset{\ast}{x}_o$, $\ddot{y}$, Δ^1, Δ^2, $\Delta^2\ddot{y}$ extrapolated, $\omega\overset{\ast}{y}_o$, $\dot{x}$ tentative, $\dot{y}$ tentative, tolerance,

provided that (1) and (3) are to be integrated. If (2) and (4) are to be integrated, the columns a and a^2 are omitted, $v^2/100a^2$ is replaced by $v^2/100$, and $H(y)/C$ is replaced by γ_s exp (hy).

Suppose that lines $t = t_{i-3}, t_{i-2}, t_{i-1}$ are complete, and that line $t = t_i$ is also complete except for the final values of x, y, and $\dot{x}$ and $\dot{y}$. It is desired to proceed to the computation of line $t = t_{i+1}$. The first stage is to extrapolate the second differences of $\ddot{x}$ and $\ddot{y}$ to line $i + 1$. These estimated values are entered in the columns headed $\Delta^2\ddot{x}$ extrapolated and $\Delta^2\ddot{y}$ extrapolated, respectively. Next we compute a tentative value of $\Delta^2 y(t_{i+1})$ by means of (3.9), which in the present notation has the form

$$(5) \quad \Delta^2 y(t_{i+1}) = \omega^2[\ddot{y}(t_i) + (1/12)\Delta^2\ddot{y}(t_{i+1})].$$

A similar tentative value of $\Delta^2 y(t_i)$ is already on the line $t = t_i$, left there from the preceding stage of the computation; and on line $t = t_{i-1}$ there are verified values of y and $\Delta^1 y$. So we can form an estimate of $y(t_{i+1})$ by using the equation

$$(6) \quad y(t_{i+1}) = y(t_{i-1}) + 2\Delta^1 y(t_{i-1}) + 2\Delta^2 y(t_i) + \Delta^2 y(t_{i+1}),$$

which is a simple consequence of the definitions of the first and second differences. With this estimate of $y(t_{i+1})$ we can find $H(y)$ or $\exp(hy)$ according as we are integrating (1) or (2), and appropriately we can enter $H(y)/C$ or $\gamma_s \exp(hy)$. Also, with this same estimate of $y(t_{i+1})$ we can find a and a^2 if we are integrating equations (1); this is unnecessary if we are assuming constant temperature.

The extrapolated second differences of $\ddot{x}$ and $\ddot{y}$ are also used to find the tentative values of $\dot{x}$ and $\dot{y}$ on line $t = t_{i+1}$. The formula used is (3.1), which in the present notation takes the form

$$(7) \quad \dot{y}(t_{i+1}) = \dot{y}(t_{i-1}) + \omega[2\ddot{y}(t_i) + (1/3)\Delta^2\ddot{y}(t_{i+1})],$$

together with a similar equation for $\dot{x}$. These tentative values of the first derivatives are entered in the appropriately headed columns, and from them we compute the value of

$$(8) \quad v^2/100 = (\dot{x}/10)^2 + (\dot{y}/10)^2.$$

If we are integrating equations (2), this is used to enter the G-table to find G(v). If we are integrating (1), it is first divided by a^2 and the quotient used to enter the G-table. Now E is computed by (3) or (4). This permits us to compute $\ddot{x}$ and $\ddot{y}$ by (1) or (2), whichever is applicable; and we can then form the differences $\Delta^1\ddot{x}$, $\Delta^2\ddot{x}$, $\Delta^1\ddot{y}$ and $\Delta^2\ddot{y}$.

At this stage we make use of the tolerance. We compute very roughly the discrepancy between the second differences just found and the extrapolated values with which the computation was begun; that is, we make a rough estimate of the quantity

$$(9)\quad \{[\Delta^2\ddot{y}(t_{i+1}) - (\Delta^2\ddot{y}(t_{i+1})\text{ extrap.})]^2 + [\Delta^2\ddot{x}(t_{i+1}) - (\Delta^2\ddot{x}(t_{i+1})\text{ extrap.})]^2\}^{\frac{1}{2}}.$$

If this exceeds the tolerance listed in the last column, the line should be re-computed, using the values of the second differences just computed as replacements for the extrapolated values; and so on until computed values of the second differences are reached which differ from the preceding estimates by an amount which does not exceed the tolerance. The question of finding the tolerance has been discussed to some extent in the preceding section, and will be further investigated in Section 10.

Suppose now that the computation, or if necessary, the re-computation, has proceeded as far as indicated, so that the final values of $\ddot{x}$ and $\ddot{y}$ have been found. With these final values of $\ddot{y}$ and $\Delta^2\ddot{y}$ on line $t = t_{i+1}$ the value of $\dot{y}(t_i)$ is computed by (3.3), which in the present notation has the form

$$(10)\quad \dot{y}(t_i) = \dot{y}(t_{i-1}) + \omega\{[\ddot{y}(t_{i-1}) + \ddot{y}(t_i)]/2 - [\Delta^2\ddot{y}(t_i) + \Delta^2\ddot{y}(t_{i+1})]/24\}.$$

This value is not yet entered. Instead, as a check the left member of (10) is also computed by (7) with $i + 1$ and i replaced by i and $i - 1$ respectively, and of course with the final value of the second difference instead of the extrapolated value. If the results of these two computations of $\dot{y}(t_i)$ differ by more than four units of the last decimal place, it is extremely likely that a computational error has been made. Otherwise, as remarked in the discussion after (3.3), the average of the two computed values should be entered as the final value of $\dot{y}(t_i)$, the odd half unit of the last decimal place being thrown toward (7) if the results of (7) and (10) differ by one or three units.

By a similar process we compute, verify and enter the final value of $\dot{x}(t_i)$.

Having the final value of $\dot{y}$ for $t = t_i$, we can use (3.12) to obtain the final value of $y(t_i)$. In the present notation, this formula is

$$\text{(11)} \qquad \begin{aligned} \Delta^1 y(t_i) = {} & (\omega/2)[\dot{y}(t_{i-1}) + \dot{y}(t_i)] \\ & - (\omega^2/12)\,\Delta^1 \ddot{y}\,(t_i). \end{aligned}$$

From this first difference we readily find $y(t_i)$ and the second difference $\Delta^2 y(t_i)$. This last is verified by (5) with i and $i + 1$ replaced by $i - 1$ and i respectively.

Finally, after the last line has been reached, we compute the values of x by the analogue of (11), with x in place of y, verifying each entry by the analogue of (5).

An example of a trajectory computation (integration of equations (2)) by this process will be found at the end of Section 10.

9. The start of a trajectory.

In Section 6 we explained a method of beginning the integration of a differential equation which is a considerable time-saver when it happens to be easy to compute the derivatives of order greater by 1 than those involved in the differential equation. These derivatives would be the third derivatives of x and y at $t = 0$ in the case of the computation of a trajectory. Accordingly, we shall now find the expressions for these third derivatives. To be specific, we shall first consider the equations in the form (8.1).

If we differentiate the second of equations (8.1) and then replace $\ddot{y}$ by using (8.1) again, we find

$$(1) \qquad \begin{aligned} \dddot{y} &= -\dot{y}(dE/dt) - E\ddot{y} \\ &= Eg + \dot{y}(E^2 - dE/dt). \end{aligned}$$

Similarly, from the first of equations (8.1) we obtain

$$(2) \qquad \dddot{x} = \dot{x}(E^2 - dE/dt).$$

In order to carry out the differentiations, it is convenient to transform E from the form (8.3) to the form involving B, by means of (IV.1.16). Since we are assuming that $H(y) = e^{-hy}$, we have

$$(3) \qquad E = \gamma e^{-hy} vB(v/a(y)).$$

Hence

$$(4) \quad \partial E/\partial y = -hE - \gamma [a(y)]^{-2} (da/dy) e^{-hy} v^2 B'(v/a(y)),$$

$$(5) \quad \partial E/\partial v = \gamma e^{-hy} B(v/a(y)) + \gamma e^{-hy} vB'(v/a(y)).$$

From (IV.1.16) we readily deduce that

$$vG'(v)/G(v) = 1 + vB'/B.$$

It is customary to add 1 to this common value and call the result the "Mayevski n" for the given drag function. The reason is that the drag $vG(v)$ for standard temperature was represented by Mayevski in an approximate form, by splitting the interval of velocities into parts on each of which the drag was assumed to have the form Av^n. This would give $G = Av^{n-1}$, whence $vG'/G = n - 1$. So, in accordance with tradition we shall define

$$(6) \quad n(v) = 1 + vG'(v)/G(v) = 2 + vB'(v)/B(v).$$

In particular, for the argument $v/a(y)$ this yields

$$(7) \qquad n(v/a(y)) - 2 = \frac{v\, B'(v/a(y))}{a(y)\, B(v/a(y))} .$$

If we substitute this in (4) and (5), and write a' for da/dy, we obtain

$$(8) \quad \partial E/\partial y = - E\{h + [n(v/a(y)) - 2]a'(y)/a(y)\},$$

$$(9) \quad \partial E/\partial v = [n(v/a(y)) - 1] E/v.$$

Also we have

$$(10) \quad dv/dt = (\dot{x}\ddot{x} + \dot{y}\ddot{y})/v = - E\, v - g\dot{y}/v.$$

From this with (8) and (9),

$$\begin{aligned} dE/dt &= (\partial E/\partial y)(dy/dt) + (\partial E/\partial v)(dv/dt) \\ (11) \qquad &= - E\Big(\dot{y}\{h + [n(v/a(y)) - 2]a'(y)/a(y)\} \\ &\qquad + [n(v/a(y)) - 1](E + g\dot{y}/v^2)\Big) . \end{aligned}$$

The function $h + (n - 1)g/v^2$ has been tabulated, at least for the Gâvre drag function, under the name $f(v)$:

$$(12) \qquad f(v) = h + [n(v) - 1]g/v^2.$$

By substituting this in (11) and rearranging terms we obtain

$$\begin{aligned} &E^2 - dE/dt \\ (13) \quad &= E\Big(En(v/a(y)) \\ &\qquad + \dot{y}\{f(v/a(y)) + [n(v/a(y)) - 2]a'/a(y)\}\Big) . \end{aligned}$$

Henceforth we shall assume that

$$a(y) = \exp(-a_1 y), \tag{14}$$

so that $a'/a = -a_1$. If this and (13) are substituted in (1) and (2), these reduce to

$$\dddot{x} = -\ddot{x}\Big(En(v/a(y)) + \dot{y}\{f(v/a(y)) - a_1[n(v/a(y)) - 2]\}\Big),$$

$$\dddot{y} = E\Big[g + \dot{y}\Big(En(v/a(y)) + \dot{y}\{f(v/a(y)) - a_1[n(v/a(y)) - 2]\}\Big)\Big]. \tag{15}$$

These apply only to the case (8.1) in which the y-axis is positive upward. If the positive y-axis is downward, all the equations are altered by having $-a_1$, $-h$ and $-g$ in place of a_1, h and g respectively.

It is debatable whether there is any gain in using these formulas to start the trajectory computation. Even if tables of $n(v)$ and $f(v)$ are available, the effort of computing the quantities (15) may be greater than the saving effected by their use in the method of Section 6. However, there is one important case in which equations (15) simplify considerably. If the computation is started from the summit, so that $\dot{y} = 0$ at $t = 0$, equations (15) take the form

$$\dddot{x} = -nE\ddot{x}, \quad \dddot{y} = Eg, \tag{16}$$

when the y-axis is positive upward; if it is positive downward, these are replaced by

$$\dddot{x} = -nE\ddot{x}, \quad \dddot{y} = -Eg. \tag{17}$$

If a table of the function n(v) is available, these are quite easily computed, since E and $\ddot{x}$ have to be calculated in any case.

It has already been explained in Section 6 how these derivatives can be utilized to obtain an estimate of the first differences of $\ddot{x}$ and $\ddot{y}$ before the second line is computed, an estimate of the second differences as soon as the second line is complete, and an estimate of the third differences as soon as the third line is complete. The technique of Section 6 applies here with no change, and the explanation need not be repeated.

10. The tolerance in trajectory computations.

In Section 7 it was pointed out that the final error in the value of the solution was the combined effect of the accumulated rounding errors and the accumulated extrapolation errors. The effect of the accumulated rounding errors was treated fairly completely in Section 7, and the results can be taken over and applied to trajectory computations with little change. If we denote the rounding errors in x" and y" on the line corresponding to time t by the symbols $r_x''(t)$ and $r_y''(t)$ respectively, and denote the extrapolation errors in x" and y" on the same lines by $e_x''(t)$ and $e_y''(t)$ respectively, we see readily that (7.11) is replaced by

$$V[\Delta x(T)]$$

$$(1) = \sum_{i=0}^{n} \omega^2 \{ (\delta x(T)/\delta \dot{x}(t_i))^2 (V[r_x''(t_i)] + V[e_x''(t_i)]) + (\delta x(T)/\delta \dot{y}(t_i))^2 (V[r_y''(t_i)] + V[e_y''(t_i)]) \},$$

with an analogous expression for the variance $V[\Delta y(T)]$ in the y-coordinate at time T. However, it is usually true that the range at altitude Y, or y(T), is of considerably more ballistic interest than the time at the same altitude. This is clearly true for gunfire at stationary or slowly moving targets; and even in anti-aircraft gunfire and bombing an investigation will show that computation errors in time of flight are less likely to be significant than those in range. If this point of view is taken, it follows that the quantity of primary interest is the error $\Delta x_{y=Y}$ in the range at altitude Y. This will be abbreviated to $\Delta x(Y)$. The rate of change of this quantity with respect to change of $\dot{x}(t)$ will be denoted by

$$(2) \qquad \left.\frac{\delta x}{\delta \dot{x}(t)}\right|_{y=Y} ,$$

which may be abbreviated to

$$(3) \qquad \delta x(Y)/\delta \dot{x}(t).$$

Analogously, the rate of change of range x corresponding to altitude Y with respect to change of velocity component $\dot{y}$ at time t will be abbreviated to $\delta x(Y)/\delta \dot{y}(t)$, or written in full by an expression analogous to (2), with $\dot{y}$ in place of $\dot{x}$. The equation (7.11) or equation (1) above, will be replaced by

$$V[\Delta x(Y)]$$

$$(4) = \sum_{i=0}^{n} \omega^2 \{(\delta x(Y)/\delta \dot{x}(t_i))^2 (V[r_x''(t_i)] + V[e_x''(t_i)])$$

$$+ (\delta x(Y)/\delta \dot{y}(t_i))^2 (V[r_y''(t_i)] + V[e_y''(t_i)])\}.$$

As in Section 7, we select a "permissible error" ε_p, "permissible" in the sense that we are willing to accept a probability of not over .01 that the error in x at altitude Y shall exceed ε_p. Then, as in (7.12), we must require that the variance in the error of range at altitude Y shall satisfy

$$(5) \qquad V[\Delta x(Y)] \leq (\varepsilon_p/2.585)^2.$$

We allocate part of the permissible error to rounding error and part to extrapolation error, by selecting two numbers c_1 and c_2 the sum of whose squares is 1 and requiring that

$$(6)\quad \sum_{i=0}^{n} \omega^2\{(\delta x(Y)/\delta\dot{x}(t_i))^2 V[r_x''(t_i)] + (\delta x(Y)/\delta\dot{y}(t_i))^2 V[r_y''(t_i)]\} \leq (c_1\varepsilon_p/2.585)^2,$$

$$(7)\quad \sum_{i=0}^{n} \omega^2\{(\delta x(Y)/\delta\dot{x}(t_i))^2 V[e_x''(t_i)] + (\delta x(Y)/\delta\dot{y}(t_i))^2 V[e_y''(t_i)]\} \leq (c_2\varepsilon_p/2.585)^2.$$

In Section 7 we saw that the effect of rounding errors in $\dot{x}$ and $\dot{y}$ can be accounted for by regarding them as equivalent to certain errors in the second derivatives. If $\ddot{x}$ and $\ddot{y}$ are both rounded to the nearest multiple of α'', while $\dot{x}$ and $\dot{y}$ are both rounded to the nearest multiple of α', by (7.21) we see that the variances in the rounding errors are to be taken as

$$(8)\quad V[r_x''(t_i)] = V[r_y''(t_i)] = (\alpha''^2 + \alpha'^2/\omega^2)/12.$$

As in Section 7, the sum in (6) can be approximated by an integral; if we define

$$(9)\quad J = \int_0^T [(\delta x(Y)/\delta\dot{x}(t))^2 + (\delta x(Y)/\delta\dot{y}(t))^2]\, dt,$$

we see as in (7.16) that inequality (6) holds if

$$(10)\quad (\alpha''^2 + \alpha'^2/\omega^2)/12 \leq (c_1\varepsilon_p/2.585)^2/J\omega.$$

If the permissible error ε_p and the portion of it allotted to rounding error have been decided on, and some estimate of the integral J is available, this inequality furnishes an estimate of the number of decimal places needed in the first and second derivatives.

Next we investigate the extrapolation error. The sum in the left member of (7) is certainly not decreased if we replace each of $(\delta x(Y)/\delta \dot{x}(t))^2$ and $(\delta x(Y)/\delta \dot{y}(t))^2$ by the greater of the two; this then becomes a common factor, the other factor being the sum of the variances. As before, the sum can be approximated by an integral. We find that in order to insure that (7) be satisfied, it is sufficient to require that

$$(11)\quad V[e_x''(t)] + V[e_y''(t)] \leq (c_2\varepsilon_p/2.585)^2/J_1\omega,$$

where

$$(12)\quad J_1 = \int_0^T \max\left[(\delta x(Y)/\delta \dot{x}(t))^2,\ (\delta x(Y)/\delta \dot{y}(t))^2\right] dt.$$

In order to make use of this we need estimates for the left members of these inequalities. We shall suppose that the process of successive approximations is continued until a stage is reached at which the sum of the squares of the changes in $\ddot{x}$ and $\ddot{y}$ from the values in the preceding stage is less than some selected value R^2; this number R is the "tolerance." As a first step in estimating the left members of (11) and (12), we wish to find a bound for the error remaining in this last stage of the approximation.

Let us denote the errors in $\ddot{x}(t)$ and $\ddot{y}(t)$ at one stage of the approximation by the letters q, r respectively, and their values at the next stage by q', r' respectively. The computation of this next stage begins with an estimate of $y(t)$ by (8.5), wherein i is replaced by i - 1 as it will be in the other formulas we shall refer to in this connection. The resulting error in $y(t_i)$ is $r\omega^2/12$. This causes an error in the density ratio H whose amount is, to first-order terms,

$$e^{-h(y + r\omega^2/12)} - e^{-hy} = e^{-hy}(-hr\omega^2/12),$$

or the negative of this if the y-axis is positive downward. Next the components of velocity are computed by (8.7) and its analogue for $\dot{x}$. The resulting errors in $\dot{x}$ and $\dot{y}$ are $q\omega/3$ and $r\omega/3$ respectively. To terms of first order in q and r, the resulting error in the velocity is

$$\sqrt{(\dot{x} + q\omega/3)^2 + (\dot{y} + r\omega/3)^2} - \sqrt{\dot{x}^2 + \dot{y}^2} = (1/v)(q\dot{x} + r\dot{y})\omega/3.$$

Therefore the resulting error in G(v) is

$$[G'(v)/v](q\dot{x} + r\dot{y})\omega/3.$$

If temperature changes are being ignored, the error q' in the re-computed value of $\ddot{x}$ is given by

$$\begin{aligned} q' &= -C^{-1}e^{-hy}(1 - hr\omega^2/12) \\ &\quad \cdot [G(v) + G'(v)(q\dot{x} + r\dot{y})\omega/3v](\dot{x} + q\omega/3) \\ &\quad + C^{-1}e^{-hy}G(v)\,\dot{x}, \end{aligned} \tag{13}$$

wherein h is to be replaced by − h if the y-axis is positive downward. We expand the product in the right member of this equation and discard all terms of degree higher than 1 in the small error terms. At the same time we replace $\dot{x}$ and $\dot{y}$ by $v \cos \theta$ and $v \sin \theta$, where θ is the angle made by the tangent to the trajectory with the positive x-axis; and we also replace G' by means of (9.6). The result is

$$\begin{aligned} q' &= C^{-1}e^{-hy}G(v)[\dot{x}hr\omega^2/12 - q\omega/3 \\ &\quad - (n - 1)(q \cos^2\theta + r \cos\theta \sin\theta)\omega/3]. \end{aligned} \tag{14}$$

In the same way, the error r' in the re-computed value of $\ddot{y}$ is

$$r' = C^{-1}e^{-hy}G(v)[\dot{y}hr\omega^2/12 - r\omega/3 - (n-1)(q\cos\theta\sin\theta + r\sin^2\theta)\omega/3]. \tag{15}$$

As before, if the y-axis is positive downward we should replace h by − h.

Equations (14) and (15) assume simpler forms if we resolve the vector (q, r) into its components (τ, π) respectively parallel and perpendicular to the line tangent to the trajectory, and likewise resolve (q', r') into components (τ', π'). The equations of transformation are

$$\begin{aligned} \tau &= q\cos\theta + r\sin\theta; \\ \pi &= -q\sin\theta + r\cos\theta, \end{aligned} \tag{16}$$

with similar equations for τ' and π'. With this notation, (14) and (15) yield

$$\begin{aligned} \tau' &= (-n\omega E/3)\tau + (h\omega^2 vE/12)r, \\ \pi' &= (-\omega E/3)\pi, \end{aligned} \tag{17}$$

where as usual $E = C^{-1}e^{-hy}G(v)$, and h is to be replaced by − h if the y-axis is positive downward.

However, instead of using (17) we shall replace τ' and π' by the still simpler approximations τ^* and π^* defined by

$$\begin{aligned} \tau^* &= -(n\omega E/3)\tau, \\ \pi^* &= -(\omega E/3)\pi. \end{aligned} \tag{18}$$

In order to estimate the error caused by using the estimates (18) in place of (17), it is convenient to introduce a symbol for the length of a vector. If (a, b) is any vector, we shall denote its length by $|(a, b)|$. Since the transformation (16) is a rotation it leaves lengths unaltered, so that

$$|(q, r)| = |(\tau, \pi)|, \tag{19}$$

and likewise for all the other vectors. The difference between the vectors (τ', π') and (τ^*, π^*) is the vector $(hr\omega^2 vE/12, 0)$ by (17) and (18), so by the triangle inequality the difference in the lengths of these two vectors cannot be greater than the length of their difference vector. Thus the error in using $|(\tau^*, \pi^*)|$ in place of $|(\tau', \pi')|$ is at most

$$\begin{aligned} &|hr\omega^2 vE/12| \\ &\leq h\omega^2 vE|(q, r)|/12 = h\omega^2 vE|(\tau, \pi)|/12. \end{aligned} \tag{20}$$

Since the Mayevski n is always greater than 1, it is clear from (18) that

$$|(\tau^*, \pi^*)| \geq (\omega E/3)|(\tau, \pi)|.$$

Combining the two preceding inequalities, we see that

$$\text{error}/|(\tau^*, \pi^*)| \leq h\omega v/4.$$

Since h = .0000316, this shows that for v < 3160 feet/second and $\omega \leq 8$ seconds the ratio of the error to the length $|(\tau^*, \pi^*)|$ is less than 0.2. Usually the velocities and intervals are considerably smaller than these values, so that it is safe to use τ^* and π^* as estimates for τ' and π' in studying the convergence of the successive approximations.

If we had been taking temperature changes into account, there would have been another small term in the analysis, but the effect of an error in altitude through resulting error in temperature is smaller than the effect produced through error in density, and this latter we have just shown to be small enough to be ignored safely, so the temperature effect can also be ignored.

At any one stage of the computation the errors q, r, etc., are not available; the quantities that we know are the amounts by which the estimates of the second derivatives have changed from one approximation to the next. Thus q, r, q' and r' are not known to us, but the change in $\ddot{x}$ can be seen by comparing the re-computed value with the preceding estimate, and this change is the same as q' - q. Likewise the value of r' - r is easily obtained being the amount by which the re-computed value of $\ddot{y}$ differs from the value obtained at the preceding approximation. We therefore wish to find an upper bound for the error $|(q', r')|$ in terms of the easily obtained quantities q - q' and r - r'. From (18) we easily compute that

$$\tau^* = \frac{n\omega E}{3 + n\omega E}(\tau^* - \tau),$$

(21)

$$\pi^* = \frac{\omega E}{3 + \omega E}(\pi^* - \pi).$$

Since the Mayevski n is greater than 1, these equations imply that

$$(22) \quad \begin{aligned} |(\tau^*, \pi^*)| &< (n\omega E/3)|(\tau^* - \tau, \pi^* - \pi)| \\ &= (n\omega E/3)\sqrt{(\tau^* - \tau)^2 + (\pi^* - \pi)^2}. \end{aligned}$$

Returning to the original x- and y-axes, we find from (22) that

$$(23) \quad |(q', r')| < (n\omega E/3)\sqrt{(q' - q)^2 + (r' - r)^2}.$$

Let q_0 and r_0 be the errors in the original extrapolation of $\ddot{x}$ and $\ddot{y}$ respectively. After a certain number of successive approximations (possibly at the very start) it will be found that the re-computed values differ from the preceding by less than the tolerance R, and these are then the final values. Their errors are, as above, q' and r' respectively. If many slightly differing trajectories are computed, the original errors (q_0, r_0) will have a certain probability distribution over the plane. Concerning this probability density we assume only that along any line through the origin, small errors are more probable than large ones. This can be symbolized as follows. Let us introduce polar coordinates s and ϕ in the (q, r)-plane, and let $f_0(s, \phi)$ be the probability density of the original error (q_0, r_0). We assume that for fixed angle ϕ, the function $f_0(s, \phi)$ decreases or remains constant as s increases. It can be shown (but we omit the proof) that in any small sector lying between angles ϕ and $\phi + d\phi$ and between two values of s, and for any given value of the integral of f_0 over this sector, the greatest value of the integral of $s^2 f_0$ for all functions of the type described is assumed when f_0 is constant.

We shall suppose that the errors at any stage are deduced from those of the preceding stage by (18) (which is slightly in error because of the ignoring of the density effect) and that the line is re-computed if the error $|(q, r)|$ exceeds R; this is slightly pessimistic, since (21) shows that there is a small chance that the line will be re-computed even though the error is a little less than R. Consider the distribution of the errors in a sector between ϕ and $\phi + d\phi$. Those for which $|(q_0, r_0)|$ does not exceed R are not re-computed; for these the final values have an error $|(q', r')|$ which cannot exceed $(n\omega E/3)R$. The second moment of the distribution about the origin cannot exceed that of a distribution with the same total probability and with uniform distribution over a sector reaching out to distance $(n\omega E/3)R$. This in turn is less than the second moment of a uniform distribution, with same total probability, in a sector between two circles with radii $(n\omega E/3)^2R$ and $(n\omega E/3)R$. Next, the part of the original distribution which needs two stages of approximation is first mapped on a sector reaching out as far as R from the origin, and then the final errors lie in a sector between two circles whose radii are at most $(n\omega E/3)^2R$ and $(n\omega E/3)R$ respectively.

The second moment of this distribution cannot exceed that of a uniform distribution in the same sector, with the same total probability. Continuing the process, we see that the second moment of the final distribution of errors, taken about the origin, cannot exceed that of a distribution lying between the circles of radii $(n\omega E/3)^2R$ and $(n\omega E/3)R$ respectively and having density independent of distance s from the origin, along each radius. Thus the distribution of the final errors (q', r') is described by a probability density whose second moment cannot exceed that of the following distribution: the probability density is a function $f(\phi)$ of angle alone, when s lies between $(n\omega E/3)^2R$ and $(n\omega E/3)R$, and is zero elsewhere. Since the integral of the probability density must equal 1, we have

$$1 = \int_{(n\omega E/3)^2R}^{(n\omega E/3)R} \int_0^{2\pi} f(\phi)s \, d\phi \, ds$$

$$= \frac{1}{2}\left\{\int_0^{2\pi} f(\phi)\, d\phi\right\}\{R^2(n\omega E/3)^2 - R^2(n\omega E/3)^4\}.$$

The second moment of the distribution of (q', r') about the origin satisfies the inequality

(25) Second moment of (q', r')

$$\leq \int_{(n\omega E/3)^2R}^{(n\omega E/3)R} \int_0^{2\pi} f(\phi)\, s^3 \, d\phi \, ds$$

$$= \frac{1}{4}\left\{\int_0^{2\pi} f(\phi)\, d\phi\right\}\{R^4(n\omega E/3)^4 - R^4(n\omega E/3)^8\}.$$

If f' is the probability density of the distribution of (q', r'), the variance of q' is the integral of $(q')^2f'$ and that of r' is the integral of $(r')^2f'$.

Their sum is the integral of $s^2 f'$, which is the second moment of the distribution of (q', r'). By this, (24) and (25), we find

$$(26)\quad V[q'] + V[r'] \leq R^2(n\omega E/3)^2[1 + (n\omega E/3)^2]/2.$$

If the process of successive approximations is to work at all successfully, the convergence factor $n\omega E/3$ must be fairly small, say less than 0.2. If this is the case, we may omit the term $(n\omega E/3)^2$ in comparison with the 1 in the square bracket. Furthermore, the final extrapolation errors, here denoted by q' and r', are the same quantities as were previously denoted by $e_x''(t)$ and $e_y''(t)$ respectively. So (26) can be written in the form

$$(27)\quad V[e_x''(t)] + V[e_y''(t)] \leq R^2(n\omega E/3)^2/2.$$

If we compare this with (11), we see that the latter is surely satisfied if

$$(28)\quad R^2(n\omega E/3)^2/2 \leq (c_2\varepsilon_p/2.585)^2/J_1\omega,$$

whence

$$(29)\quad R \leq (1.641)c_2\varepsilon_p/nE\sqrt{J_1\omega^3}.$$

To recapitulate the meanings of these symbols, ε_p is the permissible error, in the sense that we are willing to accept a chance of not over 0.01 that the accumulated error will exceed ε_p; $c_2\varepsilon_p$ is the portion of this error allotted to extrapolation errors; n is the Mayevski n, defined in (9.6); E is the function which appears in the computation of the normal trajectory; ω is the interval at which the computation is being made; and J_1 is the quantity defined in (12).

The most serious difficulty in using (29) is the estimation of the integral J_1. If we have already computed a number of trajectories without use of the tolerance, it may be possible to use these to form a crude estimate of J_1 with little computation. Suppose, for example, that it has been found that the use of two decimal places in the first and second derivatives is not quite adequate; we would like to have only half the probable error in the results that we obtained with two decimal places. We shall suppose that through most of the computations the interval ω was 2. Since the probable error due to rounding was about twice as great as desired, the variance was four times as great as desired, and instead of (10) we had, roughly,

$$[(.01)^2 + (.01)^2/4]/12 = 4(\varepsilon_p/2.585)^2/J\omega.$$

Here we set $c_1 = 1$ and $c_2 = 0$, since all the error was rounding error. If we now decide to carry three decimal places, the rounding error will diminish to a tenth of what it was, or a fifth of what we can allow. Thus in the new system we shall have $c_1 = 1/5$, so that $c_2 = \sqrt{24}/5 = .98$. The difference between this and 1 is negligible. Since J_1 cannot exceed J, the right member of (28) is at least $(\varepsilon_p/2.585)^2/J_1\omega$, and by the preceding inequality this is about .000002. We therefore wish to have $R^2(n\omega E/3)/2 \leq .000002$, or

$$R \leq .006/n\omega E.$$

If the accuracy of the collection of trajectories had been just acceptable, instead of the error being twice as great as desirable, we could have used twice this tolerance, carrying three decimals. If the accuracy had been quite satisfactory, we could have been satisfied that the rounding error was well within the permissible amount, and used the tolerance $.012/n\omega E$ without increasing the number of decimal places. However, this increase in the number of decimal places is not nearly so troublesome as might be anticipated. Ordinarily, an increase in the number of decimal places from two to three would demand that on each line the successive approximations be carried on until ten times as great an accuracy is reached, since ordinarily the successive approximations are continued until the last line agrees with the second last to the number of decimal places carried. But when a tolerance is used, the successive approximations are continued only until the agreement between consecutive approximations is within the tolerance. If the tolerance is, say, .071, the agreement must be to within seven units of the last decimal place if we are carrying two decimal places, but to within seventy-one such units if we are carrying three decimal places. So the additional labor involved in carrying the extra decimal place amounts to not much more than the trouble of pressing an extra key on the computing machine and writing one more digit on the computing sheet.

In general, it would seem that the use of the tolerance is precluded by the fact that R cannot be found without some estimate for J_1, and this in turn requires some knowledge of the data of the trajectory. However, if even a few trajectories have been completed these can be used to find some values of J_1, which in turn will furnish estimates for use in computing later trajectories. Suppose then that a trajectory has been computed; we wish to find, a posteriori, what J_1 is for this trajectory. This can be done with great accuracy by the methods of differential corrections, to be explained in the following chapters. But great accuracy is not needed in estimating J_1. The formula (29) for the tolerance is a cautious one, and a fifty per cent error in J_1 would probably be harmless; certainly an error in excess could do no harm other than making the tolerance somewhat smaller than necessary. So it is profitable for us to consider simple ways of estimating J_1. In the case of gunfire, for example, it might well be profitable to use the Siacci approximations in estimating J_1, even when they are not nearly good enough to use on the trajectory itself. The case of bomb trajectories for level bombing we shall now consider in some detail.

Consider a trajectory to have been completed. Along this trajectory the coordinate x satisfies the differential equation

$$\ddot{x} = -E\dot{x}. \tag{30}$$

Suppose that at some instant t^* the x-component of velocity is increased by an amount e, the y-component of velocity and the position being unaltered. Then at each subsequent instant x will be increased by an amount $\xi(t)$ and will satisfy a new differential equation

$$(31) \qquad \ddot{x} + \ddot{\xi} = -(E + \Delta E)(\dot{x} + \dot{\xi}).$$

The initial value of ξ is $\xi(t^*) = 0$ and of its derivative is $\dot{\xi}(t^*) = e$. It is easily seen that if e is positive, $\xi(t)$ will be positive for all $t > t^*$. The change ΔE arises from two causes. First, the x-component of velocity is increased, so v is increased, and therefore G is increased. Second, this increase in G produces a small secondary effect, since the added resistance slows down the fall of the projectile and thereby decreases the drop y at times $t > t^*$, which in turn causes a decrease in air density at time t. But this latter effect is a small one, and we shall ignore it. That is, we shall assume that the effect of a positive change e in the x-component of velocity at time t^* is to increase E at all subsequent times. Then by (31)

$$\ddot{x} + \ddot{\xi} \leq -E(\dot{x} + \dot{\xi}),$$

and so by (30)

$$(32) \qquad \ddot{\xi} \leq -E\dot{\xi}.$$

Let us divide by $\dot{\xi}$ and integrate from t^* to t. If we recall that $\dot{\xi}(t^*) = e$, we obtain

$$(33) \qquad \dot{\xi}/e \leq \exp\left\{-\int_{t^*}^{t} E\,dt\right\}.$$

Treating (30) similarly yields

$$(34) \qquad \dot{x}(t)/\dot{x}(t^*) = \exp\left\{-\int_{t^*}^{t} E\,dt\right\}.$$

Hence

$$(35) \qquad \dot{\xi}/e \leq \dot{x}(t)/\dot{x}(t^*).$$

Integration from t^* to the time of impact T yields

$$(36) \qquad \xi(T)/e \leq [x(T) - x(t^*)]/\dot{x}(t^*).$$

But the left member of this inequality is the ratio of the change of range at height Y to the change in $\dot{x}(t^*)$ causing it, and its limit as e tends to zero is by definition the quantity $\delta x(Y)/\delta \dot{x}(t^*)$. So if we let e approach 0 in (36) we obtain

$$(37) \qquad \delta x(Y)/\delta \dot{x}(t^*) \leq [x(T) - x(t^*)]/\dot{x}(t^*).$$

The quantity in the right member of this inequality is easily computed from the trajectory sheet. Incidentally, it will be seen in Chapter IX that this same quantity is needed in order to compute the effects of cross winds on the trajectory.

If we can be sure that $\delta x(Y)/\delta \dot{y}(t^*)$ does not exceed the right member of (37), we can form a safe estimate of J_1 by squaring and integrating the right member of (37). After the trajectory has become steep, some seconds after release, we may feel confident of this, for when the angle of descent is considerable a change in the y-component of velocity has little effect on the point of impact, while a change in x-component alters the direction of motion and thereby causes a more noteworthy change in the range. So, without making a really rigorous investigation, we may feel satisfied that the right member of (37) is an overestimate for $\delta x(Y)/\delta \dot{y}(t^*)$ along the whole trajectory if it can be shown to be an overestimate at the beginning. But at the beginning of the trajectory it is easy to find the value of $\delta x(Y)/\delta \dot{y}(t^*)$. For at the beginning $\dot{y}$ has the value 0. If this is increased by an amount e, since $\ddot{y} = g$ the velocity is the same (to first-order terms) as though release had occurred with $\ddot{y} = 0$ at time $-e/g$. To first-order terms, this leaves the altitude of release unaltered, while the x-coordinate of release is altered from 0 to $-v_0 e/g$, so that the range is increased by this amount. To first-order terms, the ratio of the change of range to the change in $\dot{y}(0)$ which causes it is $-v_0/g$, and by definition this is $\delta x(Y)/\delta\dot{y}(0)$. Therefore perhaps we may proceed on the assumption that $\delta x(Y)/\delta\dot{y}(t^*)$ is numerically less than the right member of (37) whenever the inequality

$$(38) \qquad v_0/g \leq x(T)/\dot{x}(0)$$

is satisfied. This can be written in the somewhat more convenient form

$$(39) \qquad X \geq v_0^2/g,$$

where as usual X denotes the total range, that is, x(T). This condition is rather easily satisfied. For bombs with γ near zero, it holds as long as the trajectory is long enough so that the angle of descent exceeds 45°. Even with γ as large as 2 and v_0 as great as 500 miles per hour, it is still satisfied if Y exceeds 20,000 feet.

By (37) we can estimate J_1, but we still need an estimate for J, as defined in (9). For safety, we wish to have an overestimate of J. But by comparing (9) and (12), we see at once that J cannot exceed $2J_1$. So, summarizing the results attained, we have the following:

(40) <u>For trajectories with $\dot{y}(0) = 0$ in which (39) is satisfied, we may safely use the estimates</u>

$$J_1 = \int_0^T \{[X - x(t)]/\dot{x}(t)\}^2 \, dt.$$

$$J = 2J_1.$$

From (40) we can deduce an even simpler and even more cautious estimate, which is close to (40) for projectiles with small γ. Since $\dot{x}(t)$ diminishes as t increases, the horizontal distance $X - x(t)$ travelled between time t and time T cannot exceed the product of the time elapsed T - t by the horizontal component of velocity $\dot{x}(t)$ at the beginning of the time interval. Thus the integrand in the estimate (40) for J_1 cannot exceed $(T - t)^2$, and from (40) we obtain the corollary:

(41) *For trajectories with $\dot{y}(0) = 0$ in which (39) is satisfied, we may safely use the estimates*

$$J_1 = T^3/3,$$

$$J = 2T^3/3.$$

Finally, we shall suggest an estimate less cautious than (41), and as yet almost untried in practice, which nevertheless seems to be serviceable for bombs of low ballistic coefficient. If the range X were known in advance, we would know the ratio at time t = 0 of the integrand in (40) to that in the integral basic to (41); it is $X^2/v_o^2T^2$. These integrands will not have a constant ratio, but in the early part of the trajectory we may presumably anticipate that the ratios of the integrands will not change very rapidly; and it is this early part of the trajectory which contributes most heavily to the integrals. So we may expect that a rough estimate to the integral in (40) will be found by multiplying the estimate in (41) by the ratio $X^2/v_o^2T^2$. Thus it is probably safe to use the following estimate:

(42) *For trajectories with $\dot{y}(0) = 0$ in which (39) is satisfied, we may use the estimates*

$$J_1 = X^2T/3v_o^2,$$

$$J = 2J_1,$$

where X and T are estimates (for safety's sake, overestimates) of the time of flight and range of the projectile.

For bombs with small γ, estimates (40), (41) and (42) are nearly identical.

If we combine estimates (42) with (10) and (29), we find that for trajectories with $\dot{y}(0) = 0$ on which (39) is satisfied the number of decimal places should be selected according to the inequality

$$(43)\quad \sqrt{\alpha''^2 + \alpha'^2/\omega^2} \leq (1.65c_1\varepsilon_p v_o/X\sqrt{T})/\sqrt{\omega},$$

where α'' is the value of a unit of the last decimal place kept in $\ddot{x}$ and $\ddot{y}$ (for instance, $\alpha'' = .01$ if the second derivatives are rounded to two decimal places) and α' is the value of a unit of the last decimal place in $\dot{x}$ and $\dot{y}$. The tolerance R is furnished by the formula

$$(44)\qquad R = \frac{2.85c_2\varepsilon_p v_o/X\sqrt{T}}{nE\sqrt{\omega^3}}.$$

Since the computer usually has more trouble in getting an adequate extrapolation than in merely writing an extra figure, it is reasonable to allot more of the total error to extrapolation than to rounding. In particular, the values $c_1 = .6$ and $c_2 = .8$ would seem to be satisfactory all-round choices.

We now illustrate the use of the methods of this and preceding sections by means of a numerical example. We wish to compute the trajectory of a bomb dropped horizontally from 30,000 feet at 240 miles per hour (= 107.290 meters per second). The summital ballistic coefficient of the bomb, with respect to the Gâvre drag function, is .39811. We shall suppose that because we have already had some experience in such computations we feel confident that the time of flight will not exceed 90 seconds, and that the range will not exceed 3000 meters. The permissible error ε_p will be chosen to be 2 meters, and as suggested above we shall choose $c_1 = .6$, $c_2 = .8$. Then by (43) we have

$$\sqrt{\alpha''^2 + \alpha'^2/\omega^2} \leq .0070/\sqrt{\omega}.$$

This is amply satisfied if we use three decimal places in the first and second derivatives. The quantity

$\ddot{y}$	Δ^1	Δ^2	$\Delta^2\ddot{y}$ (extrap.)	ωy_0	$\dot{x}$ tentative	$\dot{y}$ tentative	tolerance
9.300				-.1950			
9.610	-190	10			105.193	4.851	.68
9.428	-182	8	8		103.164	9.611	
	-372	36		-.3900			
9.034	-344	28	32		99.290	18.867	
	-716	158		-.7800			
8.435	-649	67	65		92.091	36.379	
7.786	-649	0	0		85.384	52.608	.09
7.106	-680	-31	-30		78.974	67.505	
6.386	-720	-40	-40		72.750	81.002	
5.642	-744	-24	-30		66.700	93.027	
4.885	-757	-13	-15		60.836	103.558	
4.141	-744	13	2		55.171	112.572	.07
3.425	-716	28	25		49.752	120.137	
2.757	-668	48	50		44.625	126.311	
2.142	-615	53	44		39.807	131.194	
1.589	-553	62	60		35.331	134.919	
1.103	-486	67	70		31.211	137.603	
.684	-419	67	75		27.439	139.382	
.329	-355	64	70		24.027	140.384	
.032	-297	58	60		20.945	140.733	
-.211	-243	54	50		18.204	140.545	
-.407	-196	47	45		15.747	139.918	.05
-.561	-154	42	40		13.593	138.943	
-.678	-117	37	38		11.687	137.699	
-.765	-87	30	32		10.027	136.252	
-.827	-62	25	25		8.567	134.655	
-.875	-48	14	16		7.312	132.954	
-.900	-25	23	15		6.214	131.169	
-.917	-17	8	10		5.274	129.358	
-.920	-3	14	9		4.462	127.513	
-.917	3	6	-4		3.767	125.675	
-.910	7	4	9		3.173	123.855	.04
-.898	12	5	6		2.665	122.043	
-.877	21	9	5		2.232	120.263	
-.859	18	-3	0		1.869	118.536	
-.839	20	2	5		1.557	116.830	
-.821	18	-2	4		1.296	115.178	
-.798	23	5	2		1.078	113.550	
-.775	23	0	3		.889	111.984	
-.755	20	-3	3		.738	110.456	
-.735	20	0	3		.609	108.963	
-.712	23	3	2		.500	107.512	.04
-.695	17	-6	3		.411	106.115	
-.670	25	8	4		.335	104.736	
-.652	18	-7	-6		.277	103.422	
-.630	22	4	-4		.223	102.135	
-.624	6	-16	-2		.182	100.898	

$2.85 c_2 \epsilon_p v_0 / X\sqrt{T}$ in (44) has the value .017, so that the tolerance is

$$R = .017/nE\sqrt{\omega^3}.$$

Now we use the starting process of Section 9 and the computation method of Section 8 to obtain the results here presented.

After the trajectory is completed, we can compute the estimate (40) to see if our tolerance was well chosen. The integral in (40) can be computed easily by Simpson's rule, using values every ten seconds. It turns out that R is about 2.5 times as great as necessary, or, stated in another way, there is less than a chance in a hundred that the error is as great as .8 meters. Even this is pessimistic, since the remaining errors are mostly much less than the tolerance. Nevertheless, in this particular computation little would have been gained by increasing the tolerance, since only one or two lines had to be re-computed. The tolerance served its purpose admirably, since without it many of the lines would have had to be re-computed.

11. Morrey's method of integration of the normal equations.

A method of trajectory computation has been devised by C. B. Morrey which is capable of furnishing quick solutions of the equations of motion for artillery projectiles. The central feature of the method is the use of the Siacci S-function to determine the values of v_x as a function of x, from which the slope and altitude and time are obtainable by quadratures.

If in equation (V.1.13) we choose c = 1, equations (V.1.15) furnish us with

(1) $$dx/dv_x = -1/E.$$

On substituting the definition (IV.1.24) of E and rearranging factors, this becomes

(2) $$\gamma\, a(y)H(y)\, dx = -dv_x/G(v/a).$$

Let us define

$$(3) \qquad X(x) = \int_0^x a(y(x))H(y(x))\gamma \, dx.$$

If at horizontal ranges x_1 and x_2 the horizontal component of velocity has the respective values v_{x1} and v_{x2}, we find by integrating both members of (2) that

$$(4) \qquad X(x_2) - X(x_1) = \int_{v_{x2}}^{v_{x1}} \frac{dv_x}{G(v/a)} .$$

As in the Siacci method, we wish to replace the argument v/a in the denominator of the integrand by a "pseudo-velocity" $u = cv_x$ in such a way that the right member of (4) is only slightly changed in value. Let ε denote the change in the value of the right member caused by replacing v/a by cv_x; then by definition

$$(5) \qquad \varepsilon = \int_{v_{x2}}^{v_{x1}} \left[\frac{1}{G(cv_x)} - \frac{1}{G(v/a)} \right] dv_x.$$

Since cv_x will remain near v/a along the arc from x_1 to x_2, we can use Taylor's development to linear terms to obtain

$$(6) \qquad \begin{aligned} \frac{1}{G(cv_x)} - \frac{1}{G(v/a)} &= - \frac{G'(cv_x)[cv_x - v/a]}{G(cv_x)G(v/a)} \\ &= - \frac{[n(cv_x) - 1](cv_x - v/a)}{cv_x G(v/a)} . \end{aligned}$$

We substitute this in equation (5) and change variable of integration to time t, by means of the equation $dv_x = - Ev_x \, dt$; we thus obtain

$$\varepsilon = - \int_{t_1}^{t_2} (cv_x - v/a)(1/c) \{\gamma Ha[n(cv_x) - 1)]\} \, dt,$$

where t_1 and t_2 are the times corresponding to x_1 and x_2 respectively. If we replace v by $v_x \sec \theta$, where as usual θ is the inclination of the tangent to the trajectory, we find

$$(7)\quad c\varepsilon/\gamma = -\int_{x_1}^{x_2} [c - (\sec \theta)/a] Ha[n(cv_x) - 1]\, dx.$$

In order to make this error vanish, we should choose c so that

$$(8)\qquad c = \frac{\int_{x_1}^{x_2} [(\sec \theta)/a] Ha[n(cv_x) - 1]\, dx}{\int_{x_1}^{x_2} Ha[n(cv_x) - 1]\, dx}.$$

This determines c in an exact, but useless, manner; for in order to compute the integrals in (8) exactly we would have to have the trajectory completed. What we wish is a simple expression which will approximate the quantity c given in (8). Since H, a and n - 1 are always positive, by the theorem of the mean for integrals the value of c is between the greatest and the least values of $(\sec \theta)/a$. Morrey selects the arithmetic mean of these two values, so that

$$(9)\qquad c = \tfrac{1}{2}[(\sec \theta_1)/a(y_1) + (\sec \theta_2)/a(y_2)].$$

Once c is chosen (by this formula or any other), equation (4) is replaced by the approximate form

$$(10)\qquad X(x_2) - X(x_1) = \int_{v_{x1}}^{v_{x2}} \frac{dv_x}{G(cv_x)} = (1/c)[S(u_2) - S(u_1)],$$

where u has been defined as

$$(11)\qquad u = cv_x$$

and S(u) is the Siacci function defined in (V.2.8).

Equation (10) will be used in the form

$$S(u_2) = S(u_1) + c\,\Delta X. \tag{12}$$

The other ballistic formulas needed in the process are obtained from (11) and (V.1.8). They are

$$t' = c/u, \qquad u = c/t', \tag{13}$$

$$m' = -\,gt'^2, \tag{14}$$

$$y' = m, \tag{15}$$

where the prime denotes differentiation with respect to x. The integration of X'(x) will be accomplished by use of the quadrature formula

$$\int_0^{\omega} f(x)\ dx$$

$$= \tfrac{1}{2}\omega[f(0) + f(\omega) - (1/6)\,\Delta^2 f(\omega) - (1/12)\Delta^3 f(\omega) - (1/19)\Delta^4 f(\omega) + \ldots], \tag{16}$$

which is obtained from (3.3) by replacing $\Delta^2 f(2\omega)$ by $\Delta^2 f(\omega) + \Delta^3 f(2\omega)$ and so on.

After several lines have been completed, the procedure for obtaining a given line $x = x_{i+1}$ from the preceding lines is as follows. Each line (or more properly stage) of the computation occupies two ruled lines of the computing sheet. The first of these we shall call the "upper line"; it contains among other things the extrapolated values. We first estimate the slope m_{i+1} corresponding to x_{i+1} by extrapolating the second difference of $m' = -\,gt'^2$ and using (3.1). (Morrey suggests the formula

$$m_{i+1} = 5m_{i-1} - 4m_i + 2\omega m_{i-1}' + 4\omega m_i', \tag{17}$$

which gives the same result as assuming the second difference of m' to be the same on line i + 1 as on line i.) Next y_{i+1} is computed by Simpson's rule, (3.2), and entered on the upper line. With the help of the tables, this y determines H and a for $x = x_{i+1}$. These are entered in the appropriate columns, as final values on the lower line. From m_{i+1} the values of $1 + m_{i+1}^2$ and its square root are computed; the latter is sec θ_1. These are also entered on the lower line. From H and a the value of $X_{i+1}' = \gamma H(y_{i+1})a(y_{i+1})$ is computed and entered on the second line. Now the right member of (16), with X' in place of f, gives $\Delta^1 X_{i+1}$, which is entered on the upper line. By (9), c is computed and entered on the upper line. By (13), u_i is computed and entered on the upper line, and likewise the corresponding value of S. Next $S(u_{i+1})$ is computed by (12) and entered on the lower line. The corresponding u_{i+1} is found from the tables and entered on the lower line. Now t_{i+1}' is computed by (13) and entered on the lower line. From this m_{i+1}' is computed by (14) and entered on the lower line; by (16), m_{i+1} is also computed and entered on the lower line. Finally, y_{i+1} is computed by (3.12) and entered on the lower line. The quadrature of t' can be left until last, and effected by any good quadrature formula.

As usual in numerical integrations, the first step requires special methods. It is of course possible to use several steps at such short intervals that quadratures can be performed by the trapezoidal rule. However, by differentiating the third of equations (V.1.8) we obtain

(18) $$m'' = -2gEt'^3.$$

Thus the value of m"(0) can be obtained at the cost of computing E_o, which is not so great since

(19) $$E_o = X_o'G(v_o/a(0)),$$

and X' has to be computed in any case and some S-tables also provide a column of values of 1/G. From (18) we obtain the approximation

(20) $$m_1' = m_o' + \omega m_o''.$$

Now m_1 is computed by the trapezoidal rule, and y_1 by

(3.12). Likewise, recalling that H and a are respectively the exponentials of $-hy$ and $-a_1y$, we obtain

$$X_0'' = -(h + a_1)\, m_0 X_0'. \tag{21}$$

This furnishes an estimate of X_1', by (20) with X in place of m, and by the trapezoidal rule we compute X_1. From this point on the computation proceeds as already described, with the minor exception that $\Delta^1 X_2$ is best obtained by computing X_2 with the help of Simpson's rule.

12. A modification of Morrey's method.

We now develop a modified form of Morrey's integration method which seems to have some advantages over the original, presented in the preceding section. The first modification concerns the choice of the constant c. In order to keep the notation reasonably simple, we shall denote the values of H, a, n, θ corresponding to a point x^* by the symbols H^*, a^*, n^*, θ^* respectively. By the theorem of the mean for integrals, there is a point x^* between x_1 and x_2 such that

$$\int_{x_1}^{x_2} H(n-1)\,dx = \int_{x_1}^{x_2} H^*(n^*-1)\,dx. \tag{1}$$

From this we deduce

$$\int_{x_1}^{x_2} \sec\theta\, H(n-1)\,dx$$

$$= \int_{x_1}^{x_2} \sec\theta\, H^*(n^*-1)\,dx + \int_{x_1}^{x_2} (\sec\theta - \sec\theta^*)[H(n-1) - H^*(n^*-1)]\,dx. \tag{2}$$

If either $\sec\theta$ or $H(n-1)$ were constant, the second integral in the right member would vanish. We now show that unless sec θ and H(n - 1) both undergo considerable changes, the second integral in the right member is much smaller than the first. On an arc short enough to use in numerical integration, we can form an adequate estimate of the second integral by approximating sec θ and H(n - 1) by linear functions. If we define

$$\Delta \sec \theta = \sec \theta_2 - \sec \theta_1,$$

(3) $$\Delta[H(n-1)] = H(y_2)[n(cv_{x2}) - 1] - H(y_1)[n(cv_{x1}) - 1],$$

the approximations

$$\sec \theta = \sec \theta \;\; + (x - x^*) \, \Delta \sec \theta/(x_2 - x_1),$$

$$H(n-1) = H^*(n^* - 1) + (x - x^*) \cdot \Delta[H(n-1)]/(x_2 - x_1)$$

will be accurate enough to give a rough value for the last integral in (2). We thus find that this integral is approximately equal to

$$(1/3)\,\Delta \sec \theta \cdot \Delta[H(n-1)]\,[(x_2 - x^*)^3 + (x^* - x_1)^3]/(x_2 - x_1)^2.$$

But if H(n - 1) is roughly linear, x^* will be roughly halfway between x_1 and x_2, and the integral will be roughly

$$(1/12)(x_2 - x_1)\,\Delta \sec \theta \,\Delta[H(n-1)].$$

The first integral in the right member of (2) is roughly equal to

$$H^*(n^* - 1) \sec \theta^* \,(x_2 - x_1),$$

so the ratio of the second integral to the first is about

$$(1/12)\,(\Delta \sec \theta/\sec \theta^*)\{\Delta[H(n-1)]/H^*(n^* - 1)\}.$$

So if we restrict ourselves to arcs along which sec θ changes by less than ten per cent and H(n - 1) also changes by less than ten per cent, the second integral in the right member of (2) will be less than a thousandth of the first.

A similar analysis can be carried through with a in place of sec θ. Here the result is even more extreme, since the variation of a along an arc small enough to use in numerical integration is unlikely to be as much as one per cent. We thus find that to a high order of accuracy

$$(4) \quad \int_{x_1}^{x_2} aH(n - 1)\,dx = \int_{x_1}^{x_2} aH^*(n^* - 1)\,dx.$$

A further approximation is permissible here. Since a is nearly constant, and even more nearly linear, we may safely integrate the right member of (4) by the trapezoidal rule, obtaining

$$(5) \int_{x_1}^{x_2} aH(n - 1)\,dx = \tfrac{1}{2}(a_2 + a_1)\,H^*(n^* - 1)\,(x_2 - x_1).$$

Here too we may safely anticipate that the error will be well under one part in a thousand.

According to (11.8) the value of c which we seek is the ratio of the integral in (2) to that in (5). If in (2) we discard the last integral and replace sec θ by ds/dx, where s is arc length along the trajectory, we find that (11.8) becomes

$$(6) \quad \begin{aligned} c &= \left\{\int_{x_1}^{x_2} (ds/dx)\,dx\right\} \left\{\frac{1}{([a_2 + a_1]/2)(x_2 - x_1)}\right\} \\ &= \frac{1}{(a_2 + a_1)/2}\;\frac{(\text{length of arc from } x_1 \text{ to } x_2)}{x_2 - x_1}. \end{aligned}$$

This formula is very accurate. It would seem possible to base an improved form of Siacci approximation on it, using for example a parabolic approximation to the trajectory to obtain estimates of the length of arc and of the integral X(x). But we shall not investigate this possibility. Instead, we replace (6) by a somewhat less accurate but more convenient expression, substituting the length of the chord from (x_1, y_1) to (x_2, y_2) in place of the length of the arc. If the arc curves not more than eight degrees, so that the angle between tangent and chord does not exceed about four degrees, this approximation is in error by less than one part in a thousand. We thus obtain the estimate

$$(7) \qquad c = \frac{1}{(a_1 + a_2)/2} \sqrt{1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2} .$$

The proposed modification of Morrey's method uses two lines for each stage of the computation as his does, and also uses the same column headings; however, after the second line nothing is entered in the column headed m. If the slope is desired for any reason, it can be obtained by quadrature of m' after the end of the trajectory is reached. If it is not desired it can be simply omitted.

The start of the computation can be essentially the same as in the preceding section. The quantities m_0'' and X_0'' are computed by (11.18) and (11.21) respectively, and are used to estimate m_1' and X_1' respectively by means of (11.20) and its analogue for X'. Now m_1 is computed by the trapezoidal rule and entered on the upper line, and y_1 is computed by (3.12) and entered on the upper line. This value of y_1 is used to compute H and y, which are entered as final values on the lower line. Also it is used to compute c by (7). Having c and $t_0' = \sec \theta_0/v_0$, we find u_0 by (11.13), determine the corresponding $S(u_0)$ (these are entered on the upper line), determine $S(u_1)$ by (11.12) and look up the corresponding u_1 (these are entered on the lower line), and by (11.13) we find t_1' and enter it on the lower line. With this we compute m_1' by (11.14) and enter it on the lower line.

At this point we depart somewhat from the procedure of the preceding section. We now have an accurate value for $\Delta^1 m_1'$. We also had already found the estimate $\omega m_0''$

for this same quantity. By (6.4), we can enter double the difference $\Delta^1 m_1' - \omega m_0''$ as an estimate for $\Delta^2 m_1'$. This permits us to obtain m_1 by (3.3) with $\Delta^2 m_2'$ taken equal to the same estimated $\Delta^2 m_1'$; as we showed in (6.8), the resulting integral is in error only by the small amount $\omega \Delta^3 m_2'/72$. The final value of y_1 is now computed by (3.12) and entered on the lower line.

The first step being completed, the third, fourth and subsequent steps are obtained by the procedure now to be described; so is the second step, except for two minor features which will be pointed out at the proper time. First, the second difference of m' is extrapolated to the new line $x = x_{i+1}$ whose computation is just beginning. From this the first estimate of $\Delta^2 y_{i+1}$ is computed by (3.9), and $\Delta^1 y_{i+1}$ and y_{i+1} are also computed and entered on the upper line. As in the step already described, these are used to compute the final values of H, a, X' and c; with this c and the value of t_i' we compute u_i by (11.13), find $S(u_i)$ and enter it on the upper line, add $c \Delta^1 X$ to find $S(u_{i+1})$ and enter this and the corresponding u_{i+1} on the lower line. Now we compute t_{i+1}' by (11.13) and m_{i+1}' by (11.14), and we difference m'. We now compute the final $\Delta^2 y_{i+1}$ by (3.12), using this time the computed second difference of m' instead of the extrapolated second difference.

The two distinctive features of the second step are, first, that it is best to compute X_2 by Simpson's rule instead of by (11.16), since the differences are not yet available to sufficiently high orders to use the latter formula; and, second, as soon as m_2' is computed the third difference $\Delta^3 m_2'$ can be estimated by (6.5). As was previously pointed out, the easiest way to use this last formula is to compute $\Delta^2 m_2'$ as usual, subtract the estimate for $\Delta^2 m_1'$ which is already entered on the preceding line, and write 3/2 the difference as an estimate for $\Delta^3 m_2'$. The previous estimate for $\Delta^2 m_1'$ should be revised to be consistent with this newly estimated third difference.

Chapter VII

DIFFERENTIAL CORRECTIONS TO TRAJECTORIES COMPUTED BY THE METHOD OF SIACCI

1. Differential corrections.

The two preceding chapters have been concerned with methods for integrating the normal equations of the trajectory, and thus have dealt with trajectories along which all conditions were assumed to be standard. But this is only a part of what is needed. We have already seen, in the beginning of Chapter IV, that it is also essential to have methods by which the effects of departures from standard conditions can be found. For example, the main part of a firing table for a certain combination of gun and projectile will consist of a table showing the range as a function of the elevation under standard conditions. But in the field the conditions will not be standard, and the artillery officer must be provided with some means by which he can make the necessary allowances for the effects of the various departures from standard conditions. Moreover, unless the organization constructing the firing table has some rapid calculating device permitting the rapid computation of trajectories for various ballistic coefficients under the non-standard conditions prevailing at the time of the experimental firings on which the table is based, it will also be necessary to have some method of allowing for the effects of those departures from standard conditions in order to find what the range and time of flight would have been in the experimental firings had the conditions been standard. Thus some method of making corrections for departures from standard conditions may be needed in the preparation of the firing table.

The simplest deviations from standard conditions are those which are capable of being expressed by a single number, such as a change in initial velocity or in ballistic coefficient, a constant percentage of deviation from standard density or normal temperature, or a range wind (following wind) which is the same at all points of the trajectory. The more complicated type of deviation from standard conditions requires the specification of a function. For example, we may need to find the effect of a departure from standard density which is different at different altitudes, or of a wind whose strength varies with the altitude. These more complicated departures will be studied in Chapters VIII and IX. In this chapter we shall consider only the simpler type, which can be expressed by a single number. Also, except in this section we shall consider only trajectories which have been computed by the Hitchcock-Kent modification of the Siacci method (Section 2 of Chapter V). When methods of Siacci type have been used to compute the trajectory, only the simpler departures from standard conditions can be treated with any facility.

Suppose then that we have selected some independent variable, for example t, and by using the appropriate system of equations we have computed two trajectories. One of these is a normal trajectory, with certain initial conditions and with standard density, temperature and ballistic coefficient and with no wind. The other trajectory is a "disturbed" trajectory, in which there is a constant wind with components w_x, w_y, w_z, the ballistic coefficient is increased from C_s to $C_s + \Delta C_s$, the relative air density is increased from H to $(1 + \kappa)H$, the relative velocity of sound from a to $(1 + \alpha)a$, the coordinates at time 0 are increased by $\Delta x(0)$, $\Delta y(0)$ and $\Delta z(0)$ respectively, and the components of velocity at time

0 are increased by $\Delta v_x(0)$, $\Delta v_y(0)$ and $\Delta v_z(0)$ respectively. Then, for example, the x-coordinate at time t on the disturbed trajectory will depend on these disturbances, and thus will be a function

$$x(t, w_x, w_y, w_z, C_s, \kappa, \alpha, \Delta x(0), \Delta y(0), \Delta z(0), \Delta v_x(0), \Delta v_y(0), \Delta v_z(0))$$

of all these variables. Along the normal trajectory the x-coordinate will also be a function of time t, and may be denoted by the same symbol as above with the last twelve of the thirteen arguments set equal to zero (denoting that there is no departure from standard conditions). Similar functional symbols will stand for the values of y, z, v_x, etc.

An obvious question to ask is this: How much has the value of x been changed because of the presence of the disturbances w_x, etc.? An obvious answer is that it has been changed by the amount

$$(1) \quad \Delta x = x(t, w_x, \ldots, \Delta v_z(0)) - x(t, 0, \ldots, 0).$$

But this is quite superficial. If we had used a different independent variable, say v_x, we would have given a similar "obvious" answer, but it would not have been the same number as in (1). So we must reflect a bit. When we ask how much x has been changed, we are asking for the difference between the x of some point on the disturbed trajectory and the x of a corresponding point on the normal trajectory. But this is meaningless until we decide what the corresponding points are. In giving the "obvious" answer (1), we tacitly assumed that "corresponding" points on the two trajectories are those which correspond to equal values of t. Had we used v_x as independent variable, the "obvious" way of defining corresponding points would be to say that they are points which correspond to equal values of v_x. In field artillery, neither of these is an important as the correspondence set up by saying that corresponding points are those with equal values of y. For the x corresponding to y = 0 is the range of the projectile, and what the artillery officer wants to know is the effect on range produced by the various disturbances.

Thus it is clear that the notation for the changes in x, etc., produced by the disturbances should indicate the manner in which the correspondence between points of disturbed and normal trajectories has been set up. It should also indicate in some way the disturbances that have produced the effect. Accordingly we introduce the symbol

$$(2) \quad \Delta x([w_x, \ldots, \Delta v_z(0)] | t) = x(t, w_x, \ldots, v_z(0)) - x(t, 0, \ldots, 0).$$

The first symbol in the parenthesis will designate the departure from standard conditions. The letter after the vertical bar will designate the variable used in setting up the correspondence between the two trajectories. The left member of (2) could be read "the change in x at points of equal time t produced by disturbances $w_x, \ldots, \Delta v_z(0)$." Analogous symbols will be used for the other variables and other methods of setting up correspondence also. For example, $\Delta v_x(w_x | y)$ will be the difference between the x-components of velocity at the point of the disturbed trajectory with y-coordinate y and the point of the normal trajectory with the same y-coordinate, the disturbance being a range wind w_x.

If we are interested in some point of a normal trajectory and the nearby points of a disturbed trajectory, we always have the privilege of using any one of several different variables as independent variable on an arc containing the given point. Any one of these same variables can also be used to set up the correspondence between the two trajectories. For by using it as independent variable we automatically set up such a correspondence. On the other hand, at certain points some of the variables may not be permissible choices as independent variable. For example, if the point in question is the summit of the trajectory, we cannot use y as independent variable. Such variables are also not capable of being used to set up the correspondence between the trajectories. For example, if near the summit we try to set points of equal y in correspondence, we find that for each number y between the maximum ordinates of the normal and the disturbed trajectories there are two corresponding points on one trajectory and none at all on the other.

In order to define the concept of a differential effect, it is convenient to introduce some sort of measure of the "amount" or "norm," of a disturbance. This can be done in any one of many ways. For example, we might define it to be the sum of the absolute values of w_x, ..., $v_z(0)$, or we might define it to be the square root of the sum of their squares. In any case, this "norm" is to have the following properties. The norm of the zero disturbance is zero; all other disturbances have positive norms. If a disturbance is multiplied by a number k, its norm is multiplied by $|k|$. The norm of the sum of two disturbances is not more than the sum of their norms. (Thus the norm is a generalization of the length of a vector.) The norm of a disturbance $[w_x, \ldots, \Delta v_{zo}]$ will be denoted by the symbol $N[w_x, \ldots, \Delta v_{zo}]$; or, in case there is no danger of misunderstanding, we may abbreviate this to N.

Each of the functions $x(t, w_x, \ldots, \Delta v_{zo})$, etc., is for fixed t a function of the disturbing variables $w_x, \ldots, \Delta v_{zo}$. According to the definition standard in the calculus, the function $x(t, w_x, \ldots, \Delta v_{zo})$ has, for fixed t, a differential with respect to the remaining variables at (0, ..., 0) provided that there is a linear function

$$A_1 w_x + \ldots + A_{12} \Delta v_{zo} \tag{3}$$

which approximates the right member of (1) to within an error which tends to zero more rapidly than first order in N as the norm N tends to zero. That is, the ratio

$$\text{(4)} \qquad \frac{\Delta x([w_x, \ldots, v_{zo}] \mid t) - [A_1 w_x + \ldots + A_{12} v_{zo}]}{N}$$

approaches 0 when N approaches 0. Whenever this differential exists, we shall designate it by a symbol analogoùs to the left member of (2), namely

$$dx([w_x, \ldots, \Delta v_{zo}] \mid t).$$

Because of the assumptions we have made concerning the differentiability of G(v), H(y), etc., it follows from standard theorems on differential equations that $x(t, w_x, \ldots, \Delta v_{zo})$ has continuous partial derivatives of first and second (and in fact higher) orders with respect to all the variables. By the theorem of mean value,

$$\begin{aligned} &x(t, w_x, \ldots, \Delta v_z(0)) - x(t, 0, \ldots, 0) \\ &\quad = [\partial x/\partial w_x] w_x + \ldots + [\partial x/\partial \Delta v_z(0)] \Delta v_z(0) \\ (5) \qquad &\quad + \tfrac{1}{2} \{ [\partial^2 x/\partial w_x^2]^* w_x \\ &\qquad + [\partial^2 x/\partial w_x \partial w_y]^* w_x w_y + \ldots \\ &\qquad + [\partial^2 x/\partial \Delta v_z(0)^2]^* (\Delta v_z(0))^2 \}, \end{aligned}$$

wherein the first-order partial derivatives are evaluated at (t, 0, ..., 0), and the asterisk on the second-order partial derivatives indicates that they are to be evaluated at a point $(t, w_x^*, \ldots, \Delta v_z(0)^*)$ on the line segment joining the point (t, 0, ..., 0) to the point $(t, w_x, \ldots, \Delta v_z(0))$. But if we confine our attention to any finite part of the normal trajectory and to bounded disturbances, these second-order partial derivatives all remain below some finite bound.

Since the number of terms involving second-order partials is also finite (there are 78 of them) and each one also has two other factors each at most equal to N, the sum of all the terms involving second-order partial derivatives is at most equal to some finite multiple, KN^2, of the square of N. This shows, first, that the linear function constituting the first terms in the right member of (5) approximates the left member with an error of second order in N, and is by definition the differential of x with respect to $(w_x, \ldots, v_z(0))$. Second, it gives us the specific form of this differential;

$$(6) \quad \begin{aligned} & dx([w_x, \ldots, \Delta v_z(0)] \mid t) \\ & = [\partial x/\partial w_x] w_x + \ldots + [\partial x/\partial \Delta v_z(0)] \Delta v_z(0). \end{aligned}$$

(Here, however, we insert a word of caution. Equation (6) is valid only because t was used as independent variable on the trajectories. If, for example, the Siacci pseudo-velocity p were the independent variable, the x-coordinate along the disturbed trajectory would be a function $x(p, w_x, \ldots, \Delta v_z(0))$, and the linear expression formed out of its partial derivatives with respect to the variables w_x, etc., as in the right member of (6), would be the differential $dx([w_x, \ldots, v_z(0)] \mid p)$.) Third, from (5) and (6) we see that

$$(7) \quad \begin{aligned} & |dx([w_x, \ldots, v_z(0)] \mid t) - \Delta x([w_x, \ldots, v_z(0)] \mid t)| \\ & \leq KN^2. \end{aligned}$$

This is stronger than the requirement that the ratio (4) should tend to zero with N, as required in the definition of a differential.

It would be ambiguous to denote the value of, say, $dx(w_x|t)$ at t = 0 by the symbol $dx(w_x| 0)$. We could not distinguish this from the result of setting y = 0 in the differential $dx(w_x| y)$, which is quite a different thing. Hence the result of setting t = 0 in

$dx(w_x|t)$ will be written more explicitly in the form $dx(w_x|t = 0)$.

It is the accuracy of the approximation (7) that permits us to use the differential effect dx as a substitute for the actual effect Δx. For by (7), by making N small enough we can make the difference between dx and Δx less than any preassigned multiple of N, however small. The virtue of the differential effect, as compared with Δx itself, lies chiefly in the fact that it is usually much easier to compute. An additional virtue is that differential effects are superposable. Thus, for instance, the differential effect of a wind w_x and a change of density ΔH acting simultaneously is the same as the sum of their differential effects when acting separately, as is evident from (6). This permits us to compute the differential effects of the disturbances one by one and then to find their accumulated differential effect by simply adding the separate results.

Usually it will happen that the differential effects are desired for equal values of some variable, for example, y, whereas it is much easier to use some other variable as independent variable in computing the trajectory. Therefore it is important to have a method by which differential effects at equal values of one variable can be deduced from differential effects at equal values of some other variable. Here we have a multitude of different quantities; there is the variable used as independent variable in computing the trajectories, which may be any one of several; there is the variable at equal values of which we are asked to find the differential effects; there are the quantities whose changes are sought; and there are the various disturbances causing the changes. Therefore we shall introduce a sort of generic notation to cover all these possibilities. Let P be a point of a normal trajectory. Let A and B be any two variables which can be used as independent variables on an arc of trajectory including P. (Thus A and B may each be any one of the variables x, t, the Siacci pseudo-velocity p, the slope m, the inclination θ, y unless P is the summit, and v unless dv/dt is zero at P.) Let C be any of the quantities determined along the trajectory; it is not necessary that C should be capable of use as independent variable. Let q be any disturbance. Then, as we are about to prove, the differential effect on C at equal values of A and the differential effect on C at equal values of B are related to each other by the equation

(8) $dC(q|B = B_0) = dC(q|A = A_0) - (dC/dB)dB(q|A = A_0)$,

where the derivative dC/dB is to be evaluated at the point P of the normal trajectory.

We will now prove this statement. Let A_0, B_0, C_0 be the values of A, B, C respectively, at the point P of the normal trajectory. If the norm N of the disturbance is sufficiently small, there is a point Q of the disturbed trajectory at which A has this same value A_0. Then, according to the definition of the Δ symbol introduced in (2), we have

$$\begin{aligned} A &= A_0, \\ B &= B_0 + \Delta B(q|A = A_0), \qquad (9) \\ C &= C_0 + \Delta C(q|A = A_0), \end{aligned}$$

at this point Q. Likewise, if N is small enough there is a point R of the disturbed trajectory at which B has the value B_0. At this point R we have

$$\begin{aligned} A &= A_0 + \Delta A(q|B = B_0), \\ B &= B_0, \qquad (10) \\ C &= C_0 + \Delta C(q|B = B_0). \end{aligned}$$

If we consider B as the independent variable we observe that on the disturbed trajectory, from beginning to end of the arc QR, the value of B changes by

$$- \Delta B(q|A = A_0),$$

while the value of C changes by

$$\Delta C(q|B = B_o) - \Delta C(q|A = A_o).$$

By the theorem of mean value, there is a point B^* between B_o and $B_o + \Delta B(q|A = A_o)$ such that

$$(11) \qquad \frac{\Delta C(q|B = B_o) - \Delta C(q|A = A_o)}{-\Delta B(q|A = A_o)} = (dC/dB)^*,$$

the derivative in the right member being evaluated at $B = B^*$.

From (5) we readily see that as long as N remains under a finite bound, Δx will not exceed a finite multiple of N. For the linear terms each consist of a bounded expression times a factor at most N, and the remaining terms do not exceed (KN)N, which is a bounded multiple of N, if N remains bounded. This applies equally well to ΔB and to $\Delta(dC/dB)$. In particular, the values of dC/dB at the points of normal and disturbed trajectories with $B = B^*$ differ by at most a bounded multiple of N. But B^* differs from B_o by at most a bounded multiple of N, so the values of dC/dB at the points of the normal trajectory corresponding to $B = B^*$ and to $B = B_o$ also differ by at most a bounded multiple of N. Hence

$$(12) \qquad (dC/dB)^* = (dC/dB)_o + c_1 N,$$

where the subscript o indicates that dC/dB is evaluated on the normal trajectory at the point P, and all that we need to know about c_1 is that it stays under some finite bound. From (11) and (12) it follows that

$$(13) \qquad \begin{aligned} \Delta C(q|B = B_o) = {} & \Delta C(q|A = A_o) \\ & - (dC/dB)_o \Delta B(q|A = A_o) \\ & - c_1 N \Delta B(q|A = A_o). \end{aligned}$$

According to (7),

$$(14)\qquad \begin{aligned} \Delta C(q|A = A_0) &= dC(q|A = A_0) + \varepsilon_2 N^2, \\ \Delta B(q|A = A_0) &= dB(q|A = A_0) + \varepsilon_3 N^2, \end{aligned}$$

where ε_2 and ε_3 remain under some finite bound. If we substitute these expressions in (13), and recall that $\Delta B(q|A = A_0)$ itself remains under some finite multiple of N, we see that (13) yields

$$(15)\qquad \begin{aligned} \Delta C(q|B = B_0) = dC(q|A = A_0) \\ - (dC/dB)_0 dB(q|A = A_0) + \varepsilon_4 N^2, \end{aligned}$$

where ε_4 remains under some finite bound. But now the first two terms in the right member of (15) are linear in the disturbance q, and this linear function approximates $\Delta C(q|B = B_0)$ to within an error which is of second order in N. Hence by definition this linear expression is the differential $dC(q|B = B_0)$. But this statement is exactly the equation (8) which we were to prove, and the proof is complete.

2. Application to Siacci trajectories.

The methods discussed in the preceding section will now be applied to trajectories computed by the Hitchcock-Kent modification of the Siacci method. However, we shall not consider exactly the disturbances listed in the preceding section. For one thing, if the x and y of the muzzle are changed, everything else remaining fixed, the trajectory will merely be translated by the same amount as the muzzle. So changes in the initial values of x and y will be ignored, and we shall always assume that x and y both are zero at t = 0. For another, we shall postpone consideration of effects of winds until Section 4, because by considering wind effects separately we can arrange the work so that much of it will be useful later, not merely for Siacci trajectories. Finally, we shall also consider the effects of change in the value of g.

Equations (V.2.9, 10, 11, 12) give the approximate solutions of the equations of motion furnished by the Hitchcock-Kent modification of the Siacci method. We change the form of these equations slightly by replacing p_0 by v_0 (which is equal to it) and by substituting (V.2.10) in (V.2.12); the result is

$$
\begin{aligned}
t &= t(p, C_s, v_0, \theta_0, a, g) \\
&= C_s\ [T(p/a) - T(v_0/a)]/a, \\
x &= x(p, C_s, v_0, \theta_0, a, g) \\
&= C_s \cos\theta_0 [S(p/a) - S(v_0/a)], \\
(1)\quad m &= m(p, C_s, v_0, \theta_0, a, g) \\
&= \tan\theta_0 - C_s \sec\theta_0 [I(p/a) - I(v_0/a)]/2a^2, \\
y &= y(p, C_s, v_0, \theta_0, a, g) \\
&= C_s[\sin\theta_0 + C_s I(v_0/a)/2a^2][S(p/a - S(v_0/a)] \\
&\quad - C_s^2 [A(p/a) - A(v_0/a)]/2a^2.
\end{aligned}
$$

According to (1.6) and the remarks following it,

$$
\begin{aligned}
&dx([\Delta C_s, \Delta v_0, \Delta\theta_0, \Delta a, \Delta g] | p) \\
(2)\qquad &= [\partial x/\partial C_s]\Delta C_s + [\partial x/\partial v_0]\Delta v_0 \\
&\quad + [\partial x/\partial\theta_0]\Delta\theta_0 + [\partial x/\partial a]\Delta a \\
&\quad + [\partial x/\partial g]\Delta g,
\end{aligned}
$$

with similar equations for the differential effects on t and y. In computing the last term it must be recalled that g, although apparently absent from (1), is actually present as a factor in I and A. We can now compute the fifteen partial derivatives which appear in (2) and its analogues for t and y (differential effects on m could be computed too, but are not particularly important), and simplify them by use of (1); the results are the following, using (1), (2) and (V.2.8).

(3) Differential effects of change ΔC_s in ballistic coefficient at muzzle:

$dt(\Delta C_s | p) = t(\Delta C_s/C_s)$,

$dx(\Delta C_s | p) = x(\Delta C_s/C_s)$,

$dy(\Delta C_s | p) = (2y - x \tan \theta_0)(\Delta C_s/C_s)$.

(4) Differential effects of change Δv_0 in initial velocity:

$dt(\Delta v_0 | p) = [C_s/v_0 aG(v_0/a)]\Delta v_0$,

$dx(\Delta v_0 | p) = [C_s \cos \theta_0/aG(v_0/a)]\Delta v_0$,

$dy(\Delta v_0 | p)$

$= [C_s/aG(v_0/a)][(gx \sec \theta_0/v_0^2) + \sin \theta_0]\Delta v_0$.

(5) Differential effects of change $\Delta \theta_0$ in angle of departure:

$dt(\Delta \theta_0 | p) = 0$,

$dx(\Delta \theta_0 | p) = -(x \tan \theta_0)\Delta \theta_0$,

$dy(\Delta \theta_0 | p) = x\Delta \theta_0$.

(6) Differential effects of change Δa in relative velocity of sound:

$dt(\Delta a | p)$

$= \Big(-t + C_s\{[1/aG(p/a)] - [1/aG(v_0/a)]\}\Big)(\Delta a/a)$,

$dx(\Delta a | p)$

$= \{[p/aG(p/a)] - [v_0/aG(v_0/a)]\} C_s \cos \theta_0(\Delta a/a)$,

$dy(\Delta a | p)$

$= \{2(x \tan \theta_0 - y)$

$+ C_s[(gx \sec \theta_0 - v_0^2 \sin \theta_0)/v_0 aG(v_0/a)]$

$+ C_s[(p \cos \theta_0 \tan \theta)/aG(p/a)]\} (\Delta a/a)$.

(7) Differential effects of change Δg in acceleration due to gravity:

$$dt(\Delta g|p) = 0,$$

$$dx(\Delta g|p) = 0,$$

$$dy(\Delta g|p) = (y - x \tan \theta_0)(\Delta g/g).$$

Comparison of effects at equal values of p is of course not very useful in itself. In field artillery the effects should be for equal values of y, specifically for y = 0. In forward fire from airplanes the natural independent variable is slant range, so that the effects should be given at equal values of x. In anti-aircraft fire it is possible that effects at equal values of t might be desired. All of these can be deduced from the effects (3) to (7) at equal values of p with the help of (1.8). In this formula we take A to be the pseudo-velocity p, B to be any one of the variables x, y or t, and C to be any other of the variables x, y or t. In each case dC/dB is one of the six derivatives

$$\begin{aligned} &dx/dt = p \cos \theta_0, \quad dy/dt = p \cos \theta_0 \tan \theta, \\ (8) \quad &dx/dy = \cot \theta, \quad dt/dy = \cot \theta \sec \theta_0/p, \\ &dt/dx = \sec \theta_0/p, \quad dy/dx = \tan \theta. \end{aligned}$$

Thus if q is any one of the disturbances whose effects have been evaluated for equal values of p, we transform to equal values of t, x or y by the equations

$$\begin{aligned} &dx(q|t) = dx(q|p) - p \cos \theta_0 \, dt(q|p), \\ &dy(q|t) = dy(q|p) - p \cos \theta_0 \tan \theta \, dt(q|p), \\ (9) \quad &dx(q|y) = dx(q|p) - \cot \theta \, dy(q|p), \\ &dt(q|y) = dt(q|p) - (1/p) \cot \theta \sec \theta_0 \, dy(q|p), \\ &dt(q|x) = dt(q|p) - (1/p) \sec \theta_0 \, dx(q|p), \\ &dy(q|x) = dy(q|p) - \tan \theta \, dx(q|p). \end{aligned}$$

All that remains is to replace q by ΔC_s, Δv_0, $\Delta \theta_0$, Δa and Δg successively and substitute for the differential effects in the right members by means of equation (3) to (6). The results are as follows.

(10) Differential effects of change ΔC_s in ballistic coefficient at muzzle:

$$dx(\Delta C_s|t) = (x - pt\cos\theta_o)(\Delta C_s/C_s),$$

$$dy(\Delta C_s|t)$$
$$= (2y - x\tan\theta_o - pt\cos\theta_o\tan\theta)(\Delta C_s/C_s),$$

$$dx(\Delta C_s|y)$$
$$= [x(1 + \tan\theta_o\cot\theta) - 2y\cot\theta](\Delta C_s/C_s),$$

$$dt(\Delta C_s|y)$$
$$= [t + (x\tan\theta_o - 2y)\cot\theta\sec\theta_o/p](\Delta C_s/C_s),$$

$$dt(\Delta C_s|x) = [t - x\sec\theta_o/p](\Delta C_s/C_s),$$

$$dy(\Delta C_s|x) = [2y - x(\tan\theta + \tan\theta_o)](\Delta C_s/C_s).$$

(11) Differential effects of change Δv_o in initial velocity:

$$dx(\Delta v_o|t) = C_s[(v_o - p)\cos\theta_o/v_o aG(v_o/a)]\Delta v_o,$$

$$dy(\Delta v_o|t)$$
$$= C_s\{[(v_o^2\sin\theta_o - gx\sec\theta_o)/v_o^2 aG(v_o/a)]$$
$$- [p\cos\theta_o\tan\theta/v_o aG(v_o/a)]\}\Delta v_o,$$

$$dx(\Delta v_o|y)$$
$$= C_s\{[(\cos\theta_o - \cot\theta\sin\theta_o)/aG(v_o/a)]$$
$$+ [gx\sec\theta_o\cot\theta/v_o^2 aG(v_o/a)]\}\Delta v_o,$$

$$dt(\Delta v_o|y)$$
$$= C_s\{[1/av_oG(v_o/a)]$$
$$+ [gx\sec^2\theta_o\cot\theta/av_o^2 p\,G(v_o/a)]$$
$$- [\tan\theta_o\cot\theta/apG(v_o/a)]\}\Delta v_o,$$

$$dt(\Delta v_o|x) = -C_s[(v_o - p)/av_o p\,G(v_o/a)]\Delta v_o,$$

$$dy(\Delta v_o|x)$$
$$= C_s\{[(\sin\theta_o - \tan\theta\cos\theta_o)/aG(v_o/a)]$$
$$- [gx\sec\theta_o/v_o^2 aG(v_o/a)]\}\Delta v_o.$$

(12) Differential effects of change $\Delta\theta_o$ in angle of departure:

$$dx(\Delta\theta_o|t) = -x\tan\theta_o\,\Delta\theta_o,$$

$$dy(\Delta\theta_o|t) = x\,\Delta\theta_o,$$

$$dx(\Delta\theta_o|y) = -x(\tan\theta_o + \cot\theta)\Delta\theta_o,$$

$$dt(\Delta\theta_o|y) = -(x/p)\cot\theta\sec\theta_o\,\Delta\theta_o,$$

$$dt(\Delta\theta_o|x) = (x/p)\tan\theta_o\sec\theta_o\,\Delta\theta_o,$$

$$dy(\Delta\theta_o|x) = x(1 + \tan\theta\tan\theta_o)\Delta\theta_o.$$

(13) Differential effects of change Δg in acceleration due to gravity:

$$dx(\Delta g|t) = 0,$$

$$dy(\Delta g|t) = (y - x\tan\theta_o)(\Delta g/g),$$

$$dx(\Delta g|y) = \cot\theta\,(x\tan\theta_o - y)(\Delta g/g),$$

$$dt(\Delta g|y) = (1/p)\cot\theta\sec\theta_o\,(x\tan\theta_o - y)(\Delta g/g),$$

$$dt(\Delta g|x) = 0,$$

$$dy(\Delta g|x) = (y - x\tan\theta_o)(\Delta g/g).$$

The relative velocity of sound is not directly measured, but is inferred from the temperature. If Θ denotes the absolute temperature, it is assumed that a is proportional to the square root of Θ. Then

$$\Delta a/a = \Delta\Theta/2\Theta$$

except for an error of higher order than the first. Consequently, the differential effects of a change $\Delta\Theta$ in absolute temperature can be found from the differential effects of a change Δa by merely replacing $\Delta a/a$ by $\Delta\Theta/2\Theta$. This we now do.

(14) Differential effects of a change $\Delta\Theta$ in absolute temperature:

$$dx(\Delta\Theta|t)$$

$$= \cos\theta_o\,\{pt + C_s[(p - v_o)/aG(v_o/a)]\}\Delta\Theta/2\Theta,$$

$dy(\Delta\Theta|t)$

$$= \{pt \cos\theta_0 \tan\theta + 2(x\tan\theta_0 - y) + C_s[p\cos\theta_0 \tan\theta/aG(v_0/a)] - C_s[v_0 \sin\theta_0/aG(v_0/a)] + C_s[gx \sec\theta_0/v_0 aG(v_0/a)]\}(\Delta\Theta/2\Theta),$$

$dx(\Delta\Theta|y)$

$$= \{-2(x\tan\theta_0 - y)\cot\theta - C_s[gx\sec\theta_0 \cot\theta/v_0 aG(v_0/a)] + C_s[v_0 \sin\theta_0 \cot\theta/aG(v_0/a)] - C_s[v_0\cos\theta_0/aG(v_0/a)]\}(\Delta\Theta/2\Theta),$$

$dt(\Delta\Theta|y)$

$$= \{-(2/p)(x\tan\theta_0 - y)\sec\theta_0 \cot\theta - t + C_s[v_0\tan\theta_0\cot\theta/apG(p/a)] - C_s[gx\sec^2\theta_0\cot\theta/av_0pG(p/a)] - C_s[1/aG(v_0/a)]\}(\Delta\Theta/2\Theta),$$

$dt(\Delta\Theta|x)$

$$= \{-t + C_s[(v_0 - p)/apG(v_0/a)]\}(\Delta\Theta/2\Theta),$$

$dy(\Delta\Theta|x)$

$$= \{2(x\tan\theta_0 - y) + C_s[v_0(\tan\theta\cos\theta_0 - \sin\theta_0)/aG(v_0/a)] + C_s[gx\sec\theta_0/v_0aG(v_0/a)]\}(\Delta\Theta/2\Theta).$$

In order to apply these formulas to field artillery trajectories with standard ground impact, $y = 0$, it is only necessary to set $y = 0$ in the formulas for $dx(q|y)$ and $dt(q|y)$. However, one small notational change is customary in this case. Since with standard ground impact θ is always negative at impact, it is usual to define the striking angle ω to be the negative of the value of θ at impact. Moreover, the values of x and t at impact are usually designated either by X and T respectively, or else by x_ω and t_ω. With a similar convention for θ we would have $\omega = -\theta_\omega$. The resulting changes in the foregoing formulas are quite obvious.

3. Identical relationships between effects of certain disturbances.

The five types of disturbances considered in the preceding section do not produce independent effects. In fact, there are three sets of identical relationships satisfied by them, so that from two properly selected sets of effects we could have deduced the other three, with the help of the identities. The identities we shall establish are satisfied by any system of exact solutions of the normal equations, and do not depend on the Siacci equations. Thus in a sense this is not the appropriate place to establish them. However, they will be established here in order that we may have them to check the results already obtained.

The first identical relationship is the following:

(1) Let "trajectory 1" be computed for a projectile of ballistic coefficient C, with initial position $x = x_0$ and $y = y_0$ at $t = 0$, initial velocity v_0 and angle of departure θ_0, gravity constant g, relative sound velocity a(y) and relative air density H(y). Let "trajectory 2" be computed for a projectile of the same ballistic coefficient C, the same initial position $x = x_0$ and $y = y_0$ at $t = 0$, the same angle of departure θ_0 and the same relative air density H(y), but

with initial velocity kv_0, *gravity constant* k^2g *and relative sound velocity* $ka(y)$, *where* k *is a positive constant. Then every point on trajectory* 1 *is on trajectory* 2, *and vice versa, and at coincident points the two trajectories have equal slopes. The time taken to reach a given point on trajectory* 2 *is* (1/k) *times the time taken to reach the same point on trajectory* 1, *and the components of velocity at this point are* k *times as great on trajectory* 2 *as on trajectory* 1.

It is convenient to use slope as independent variable in proving this statement. The equations of motion are then (V.1.10). However, we shall use the single letter r to designate the x-component of velocity, and we shall also write an alternative form of the last equation, changing back to K_D by (IV.1.16). Thus the equations are

$$dt/dm = -r/g, \quad dx/dm = -r^2/g, \quad dy/dm = -mr^2/g,$$

$$(2) \quad dr/dm = a(y)\,H(y)\,G(r\sqrt{1+m^2}\,/\,a(y))r^2/gC$$

$$= [\rho^* H(y) K_D(r\sqrt{1+m^2}/u_s(0)a(y))\sqrt{1+m^2}\,r^3/gC].$$

Given any initial position $x = x_0$, $y = y_0$, any ballistic coefficient C, any initial velocity v_0, any initial time $t = t_0$ at $m = m_0$, any gravity constant g, any relative sound velocity law $a(y)$ and any relative air density law $H(y)$, we can find exactly one set of solutions of these equations. These solutions we designate by the symbols

$$
(3)\quad
\begin{aligned}
t &= t(m, C, x_o, y_o, v_o, t_o, g, a(\ ), H(\ )),\\
x &= x(m, C, x_o, y_o, v_o, t_o, g, a(\ ), H(\ )),\\
y &= y(m, C, x_o, y_o, v_o, t_o, g, a(\ ), H(\ )),\\
r &= r(m, C, x_o, y_o, v_o, t_o, g, a(\ ), H(\ )).
\end{aligned}
$$

The empty parentheses following the letters a and H are intended to serve as reminders that at a given m, the values of t, etc., do not depend merely on the values of a and H at that specific point, but are dependent on the entire aggregate of values of the functions a(y) and H(y) along all the preceding portion of the trajectory. With this notation, "trajectory 1" is defined by the functions as written in (3), which we shall also call by the simpler names

$$t_1(m),\ x_1(m),\ y_1(m),\ r_1(m).$$

"Trajectory 2" is defined by the functions

$$
\begin{aligned}
t &= t_2(m)\\
&= t(C, x_o, y_o, kv_o, 0, k^2g, ka(\ ), H(\ )), \text{ etc.}
\end{aligned}
$$

The statement (1) which we wish to prove takes the form of the system of equations

(4)

$$
\begin{aligned}
&t(m, C, x_o, y_o, v_o, 0, g, a(\ ), H(\ ))\\
&\qquad = kt(m, C, x_o, y_o, kv_o, 0, k^2g, ka(\ ), H(\ )),\\
&x(m, C, x_o, y_o, v_o, 0, g, a(\ ), H(\ ))\\
&\qquad = x(m, C, x_o, y_o, kv_o, 0, k^2g, ka(\ ), H(\ )),\\
&y(m, C, x_o, y_o, v_o, 0, g, a(\ ), H(\ ))\\
&\qquad = y(m, C, x_o, y_o, kv_o, 0, k^2g, ka(\ ), H(\ )),\\
&r(m, C, x_o, y_o, v_o, 0, g, a(\ ), H(\ ))\\
&\qquad = (1/k)\, r(m, C, x_o, y_o, kv_o, 0, k^2g, ka(\ ), H(\ )).
\end{aligned}
$$

These can be written in the briefer forms

$$(5)\qquad \begin{aligned} t_1(m) &= kt_2(m), & x_1(m) &= x_2(m), \\ y_1(m) &= y_2(m), & r_1(m) &= r_2(m)/k. \end{aligned}$$

It is interesting to observe that for exact solutions of the normal equations, the identities (5) can be established by a dimension-theoretic argument. On trajectory 1, regarded as an existing physical entity, let us change from the original unit of time to a new unit k times as great. The trajectory is unchanged, to each point on the trajectory correspond the same x, y, m as before. But the new time t_2 is (1/k) times the old, the new velocity r_2 is k times the old, and the new gravitational acceleration is k^2g. H, C, ρ^*, K_D and its argument, having dimension 0 in time, are unaffected. There is a slight, but not insuperable, difficulty in extending this argument to trajectories computed by the Siacci method. Instead of modifying the proof, we present another.

By the definition of "trajectory 2," the equations

$$(6)\qquad \begin{aligned} dt_2/dm &= -\, r_2/(k^2g), \\ dx_2/dm &= -\, r_2{}^2/(k^2g), \\ dy_2/dm &= -\, mr_2{}^2/(k^2g), \\ dr_2/dm &= \{\, \rho^* H(y) K_D(r_2\sqrt{1+m^2}/u_s(0)ka(y)) \,\} \\ &\qquad\qquad \{\, \sqrt{1+m^2}\; r_2{}^3/k^2gC_o \,\} \end{aligned}$$

are satisfied, with initial conditions

$$(7)\qquad \begin{aligned} t_2(m_o) &= 0, & x_2(m_o) &= x_o, \\ y_2(m_o) &= y_o, & r_2(m_o) &= kv_{xo}. \end{aligned}$$

It follows at once that

$$d(kt_2)/dm = -(r_2/k)/g,$$

$$dx_2/dm = -(r_2/k)^2/g,$$

$$dy_2/dm = -m(r_2/k)^2/g,$$

(8) $$d(r_2/k)/dm = \{\rho^* H(y) K_D([r_2/k]\sqrt{1+m^2}/u_s(0)a(y)\} \{\sqrt{1+m^2}(r_2/k)^3/gC\},$$

$$k_2t_2(m_o) = 0, \quad x_2(m_o) = x_o,$$

$$y_2(m_o) = y_o, \quad (r_2(m_o)/k) = v_{xo}.$$

But by definition of "trajectory 1,"

$$dt_1/dm = -r_1/g,$$

$$dx_1/dm = -r_1{}^2/g,$$

$$dy_1/dm = -mr_1{}^2/g,$$

(9) $$dr_1/dm = \{\rho^* H(y) K_D(r_1\sqrt{1+m^2}/u_s(0)a(y)\} \cdot \{\sqrt{1+m^2}\, r_1{}^3/gC\},$$

$$t_1(m_o) = 0, \quad x_1(m_o) = x_o,$$

$$y_1(m_o) = y_o, \quad r_1(m_o) = v_{xo}.$$

By (8), the right members of equations (5) satisfy certain differential equations and initial conditions. By (9), the left members of (5) satisfy the same differential equations with the same initial conditions. Therefore equations (5) are identities, and our statement is established.

It should be observed that (1) is a statement about the exact solutions of the normal equations. But it is also valid for the Siacci approximations. For the basic Siacci approximation consists in the substitution of $v_x \sec \vartheta_o$ for v in the argument of G, which is equivalent to replacing m by m_o in the argument of K_D in (2), hence in (6), (8), and (9) also. The rest of the proof needs no change.

In order to obtain a relation between differential effects from statement (1), it is desirable to introduce a new symbol, which has the same relation to the symbol say, $dx(q|y)$, as a derivative has to a differential. If a disturbance q is expressible by means of a single number, we define $[\delta x/\delta q]_y$ to be the ratio $dx(q|y)/q$. As pointed out after (1.6), if y is the independent variable this ratio is the same as $\partial x/\partial q$. But if y is not the independent variable this last symbol will have some other meaning, whereas the symbol $[\delta x/\delta q]_y$ will continue to have the same meaning irrespective of the choice of independent variable. More generally, if q is any disturbance that can be expressed by a single number, and A is any variable that could be used as independent variable along an arc of the trajectory, and B is any other variable along the trajectory, then

$$(10) \qquad [\delta B/\delta q]_A = dB(q|A)/q.$$

In this terminology, for example, (1.6) would take the form

$$(11) \qquad dx([w_x, \ldots, \Delta v_z(0)]|t) = [\delta x/\delta w_x]_t w_x + \ldots + [\delta x/\delta\Delta v_z(0)]_t \Delta v_z(0).$$

The distinction between this equation and (1.6) is that the latter is valid only if t is the independent variable, whereas (11) is valid whatever the independent variable may be.

From (1.8) we deduce that whenever the disturbance is capable of being expressed by a single number q, the relation

$$(12) \qquad [\delta C/\delta q]_B = [\delta C/\delta q]_A - (dC/dB)[\delta B/\delta q]_A$$

is satisfied, subject to the requirements that A and B are both capable of serving as independent variables near the point at which the differential effect is being computed. The proof of (12) is trivial; all that is needed is to divide both members of (1.8) by q. Of course it should be kept in mind that (12) is a less general formulation than (1.8); the latter applies to any disturbance we have considered and in fact will continue to apply to the more complicated disturbances to be considered in the next two chapters, while (12) is valid only when the disturbance can be expressed by a single number. In fact, it is this same restricted application that diminishes the usefulness of the symbolism (10). Nevertheless, when it can be applied it is frequently convenient.

In general, the methods we have been discussing would not apply to equations (4), because in them a function is being modified; the function $a(y)$ is being replaced by $ka(y)$, the change depending on y. We could modify the preceding discussions so as to apply to this case too, since in reality the disturbance is expressed with the help of a single real number k. But it hardly seems profitable to make any ad hoc modification, since the general case will be treated in the next chapter. For the purposes of this chapter, the relative sound velocity function is assumed constant along the trajectory, and the relative air density likewise. So we may at present regard a and H as constants, as we did in the preceding section. Now we can write such symbols as $\partial t/\partial a$, as before.

In the first three of equations (4), let us differentiate with respect to k and then set $k = 1$. Since m is the independent variable, these partial derivatives are the same as the symbols (10) with subscript m; thus, for example, the partial derivative of the right member of the first of equations (4) with respect to a is the same as $[\delta x/\delta a]_m$. The results may then be written

$$(13)\qquad \begin{aligned} 0 &= t + v_0[\delta t/\delta v_0]_m + 2g[\delta t/\delta g]_m + a[\delta t/\delta a]_m, \\ 0 &= \phantom{t + {}} v_0[\delta x/\delta v_0]_m + 2g[\delta x/\delta g]_m + a[\delta x/\delta a]_m, \\ 0 &= \phantom{t + {}} v_0[\delta y/\delta v_0]_m + 2g[\delta y/\delta g]_m + a[\delta y/\delta a]_m. \end{aligned}$$

ver, these are not really what we want. Differential effects at equal values of m are not as interesting as those at equal values of y, or of x, or of t. By (12) with A = m we can deduce these more interesting quantities from (13). Thus, for example, to get the identity connecting the differential effects at equal values of y, we take C = x and B = y in (12) and thus see that the desired identity follows by multiplying the last of equations (13) by

$$- dx/dy = - \cot \theta$$

and adding to the second of equations (13). We thus obtain six identities

$$(14)\quad \begin{aligned} - v_x t + v_o[\delta x/\delta v_o]_t + 2g[\delta x/\delta g]_t + a[\delta x/\delta a]_t &= 0, \\ - v_y t + v_o[\delta y/\delta v_o]_t + 2g[\delta y/\delta g]_t + a[\delta y/\delta a]_t &= 0, \\ t + v_o[\delta t/\delta v_o]_x + 2g[\delta t/\delta g]_x + a[\delta t/\delta a]_x &= 0, \\ v_o[\delta y/\delta v_o]_x + 2g[\delta y/\delta g]_x + a[\delta y/\delta a]_x &= 0, \\ t + v_o[\delta t/\delta v_o]_y + 2g[\delta t/\delta g]_y + a[\delta t/\delta a]_y &= 0, \\ v_o[\delta x/\delta v_o]_y + 2g[\delta x/\delta g]_y + a[\delta x/\delta a]_y &= 0. \end{aligned}$$

It will be found that the differential effects listed in the preceding section satisfy these equations.

In order to state our next set of identities it is convenient to introduce the symbols

$$(15)\qquad a_k(y) = a(ky), \quad H_k(y) = H(ky),$$

where k is any positive number. We can then state and prove the following identity.

(16) Let "trajectory 1" be computed for a projectile of ballistic coefficient C, with initial position $x = 0$ and $y = 0$ at time $t = 0$, initial velocity v_o and angle of departure θ_o, gravity constant g, relative

sound velocity function $a(y)$ and relative air density $H(y)$. Let "trajectory 2" be computed with the same initial position and velocity and the same angle of departure, but with ballistic coefficient C/k, gravity constant kg, relative sound velocity function $a_k(y)$ and relative air density $H_k(y)$, where k is any positive number. Then at points of equal slope the coordinates and time of flight on trajectory 1 are k times as great as on trajectory 2, velocity components being unaltered.

We again use the slope as independent variable, and designate the general solution by the symbols (3). Trajectory 1 is defined by the functions (3) as written; these can be abbreviated, as before

$$t_1(m),\ x_1(m),\ y_1(m),\ r_1(m).$$

Trajectory 2 is defined by the functions

(17)
$$t_2(m) = t(m,\ C/k,\ 0,\ 0,\ v_o,\ 0,\ kg,\ a_k(\ ),\ H_k(\ )),\ \text{etc.}$$

Statement (16) will be established if we prove

(18)

$$t_1(m) = kt_2(m)$$
$$= kt_2(m,\ C/k,\ 0,\ 0,\ v_o,\ 0,\ kg,\ a_k(\ ),\ H_k(\ )),$$
$$x_1(m) = kx_2(m)$$
$$= kx_2(m,\ C/k,\ 0,\ 0,\ v_o,\ 0,\ kg,\ a_k(\ ),\ H_k(\ )),$$

restricted in their appli-
position is $x_0 = y_0 = 0$. How-
this is no real restriction;
initial conditions into this
introducing "ballistic coeffi-
titude," or "summital ballistic
ith this small change, they can
the formulas of the preceding

of identities is one that has been
years. It is obtained by the sim-
fting the origin to a new point on
We shall assume that both relative
nd relative air density are exponen-

$$= e^{-a_1 y}, \quad H(y) = e^{-hy}.$$

enient to use the time as the independ-
Let the solution of the equations of
itial position $x = x_0$ and $y = y_0$ at time
al velocity v_0, angle of departure θ_0,
efficient C, gravity constant g, relative
ty function a(y) and relative air density
ignated by the symbols

$$(t, C, x_0, y_0, v_0, \theta_0, g, a(\ ), H(\ )),$$

$$y(t, C, x_0, y_0, v_0, \theta_0, g, a(\ ), H(\ )).$$

first trajectory we take the conditions as
xcept for the one specialization that at $t = 0$
jectile is at the origin, so that $x_0 = y_0 = 0$.
other trajectory (or rather our other way of
g the same trajectory) we let k be any number,
hange the origin of time from $t = 0$ to $t = k$.
is, we make the transformation

$$t^* = t - k. \qquad)$$

x_k, y_k, v_k, θ_k be the coordinates, the velocity
the inclination at the time $t = k$, which is the

these equations, we obtain the differential equations and initial conditions of trajectory 1. Thus

$$(21)\quad kt_2 = t_1, \quad kx_2 = x_1, \quad ky_2 = y_1, \quad r_2 = r_1$$

identically in m, and our statement is established. It could also have been established by a dimension-theoretic argument, by changing length and time units in the ratio 1:k.

As before, the proof is unchanged if we replace m by m_0 in the argument of K_D. This converts the exact equations (2) into the Siacci approximations, so (16) is valid for the Siacci approximations as well as for the exact solutions of the normal equations.

In the Siacci method of the preceding section the functions a(y) and H(y) were taken to be constants. Hence a_k and H_k are identical with a and H, and do not depend on k. The left members of equations (18) do not depend on k. We differentiate with respect to k and set k equal to 1; the results are the following:

$$(22)\quad \begin{aligned} 0 &= t - C[\delta t/\delta C]_m + g[\delta t/\delta g]_m, \\ 0 &= x - C[\delta x/\delta C]_m + g[\delta x/\delta g]_m, \\ 0 &= y - C[\delta y/\delta C]_m + g[\delta y/\delta g]_m. \end{aligned}$$

In the same way that we deduced equations (14) from (13), we transform (22) into the system of equations

$$(23)\quad \begin{aligned} x - v_x t - C[\delta x/\delta C]_t + g[\delta x/\delta g]_t &= 0, \\ y - v_y t - C[\delta y/\delta C]_t + g[\delta y/\delta g]_t &= 0, \\ y - x \tan\theta - C[\delta y/\delta C]_x + g[\delta y/\delta g]_x &= 0, \\ t - x/v_x - C[\delta t/\delta C]_x + g[\delta t/\delta g]_x &= 0, \\ t - y/v_y - C[\delta t/\delta C]_y + g[\delta t/\delta g]_y &= 0, \\ x - y \cot\theta - C[\delta x/\delta C]_y + g[\delta x/\delta g]_y &= 0. \end{aligned}$$

These formulas seem to be restricted in their application, since the initial position is $x_0 = y_0 = 0$. However, we already know that this is no real restriction; we can always bring the initial conditions into this form by the device of introducing "ballistic coefficient corrected for altitude," or "summital ballistic coefficient," C_s. With this small change, they can be used as checks on the formulas of the preceding section.

The final group of identities is one that has been well known for some years. It is obtained by the simple process of shifting the origin to a new point on the trajectory. We shall assume that both relative sound velocity and relative air density are exponential functions,

$$(24) \qquad a(y) = e^{-a_1 y}, \quad H(y) = e^{-hy}.$$

It is now convenient to use the time as the independent variable. Let the solution of the equations of motion with initial position $x = x_0$ and $y = y_0$ at time $t = 0$, initial velocity v_0, angle of departure θ_0, ballistic coefficient C, gravity constant g, relative sound velocity function a(y) and relative air density H(y) be designated by the symbols

$$(25) \qquad \begin{aligned} x &= x(t, C, x_0, y_0, v_0, \theta_0, g, a(\ ), H(\ )), \\ y &= y(t, C, x_0, y_0, v_0, \theta_0, g, a(\ ), H(\ )). \end{aligned}$$

For our first trajectory we take the conditions as listed, except for the one specialization that at $t = 0$ the projectile is at the origin, so that $x_0 = y_0 = 0$. For our other trajectory (or rather our other way of writing the same trajectory) we let k be any number, and change the origin of time from $t = 0$ to $t = k$. That is, we make the transformation

$$(26) \qquad t^* = t - k.$$

Let x_k, y_k, v_k, θ_k be the coordinates, the velocity and the inclination at the time $t = k$, which is the same as the time $t^* = 0$. We translate the axes by a transformation

(27) $$x^* = x - x_k, \quad y^* = y - y_k.$$

Now the differential equations satisfied by the particle are

(28)
$$d^2x^*/dt^{*2}$$
$$= -\{H(y^*)H(y_k)/C\}$$
$$\cdot\{a(y^*)a(y_k)G(v/a(y_k)a(y^*))dx^*/dt\},$$
$$d^2y^*/dt^{*2}$$
$$= -g - \{H(y^*)H(y_k)/C\}$$
$$\cdot\{a(y^*)a(y_k)G(v/a(y_k)a(y^*))dy^*/dt^*\},$$

wherein we have used the fact that a and H are exponential functions, so that $a(y_k + y^*) = a(y_k)a(y^*)$ and similarly for H. The initial conditions are $x^* = y^* = 0$ at time $t^* = 0$, velocity $= v_k$ and inclination $= \theta_k$ at time $t^* = 0$. But then by the definition of the symbols (25) the solutions of (28) are the functions

(29)
$$x^*(t^*) = x(t^*, C/H(y_k), 0, 0, v_k,$$
$$\theta_k, g, a(y_k)a(\ ), H(\ )),$$
$$y^*(t^*) = y(t^*, C/H(y_k), 0, 0, v_k,$$
$$\theta_k, g, a(y_k)a(\ ), H(\ )).$$

The differential equations (28) are those satisfied by the original trajectory, only the notation being altered, and the initial conditions on the new trajectory are the same as the coordinates and velocity components at the point $t = k$ of the original trajectory. So the solution (29) is the same as the original trajectory, only the notation being changed. If we write the statement that the two trajectories are the same, and change back from x^*, y^* to x, y by (27), we obtain the identities

$$x(t^*, C/H(y_k), 0, 0, v_k, \theta_k, g, a(y_k)a(\ ), H(\ ))$$
$$+ y_k = y(t, C, 0, 0, v_o, \theta_o, g, a(\ ), H(\ )),$$
(30)
$$y(t^*, C/H(y_k), 0, 0, v_k, \theta_k, g, a(y_k)a(\ ), H(\ ))$$
$$= y(t, C, 0, 0, v_o, \theta_o, g, a(\ ), H(\ )),$$
where t and t^* are related to each other by (26).

From this point on we shall assume that a(y) is actually independent of y, which is equivalent to assuming temperature the same at all altitudes. We shall differentiate both members of (30) with respect to k and then set k = 0. Since $v^2 = v_x{}^2 + v_y{}^2$, with the help of (V.1.1) with the disturbances set equal to zero we find

(31)
$$\begin{aligned} dv/dt &= (v_x\dot{v}_k + v_y\dot{v}_y)/v \\ &= \dot{v}_x \cos\theta + \dot{v}_y \sin\theta \\ &= -Ev - g\sin\theta. \end{aligned}$$

From (24) we find

(32)
$$dH/dt = -hH(y)\dot{y},$$

while the time derivative of θ can be found from (V.1.11). The result of the differentiation is then

(33)
$$\begin{aligned} &-v_x + hv_o \sin\theta_o\, C[\delta x/\delta C]_t \\ &\quad -(Ev_o + g\sin\theta_o)[\delta x/\delta v_o]_t \\ &\quad -(g\cos\theta_o/v_o)[\delta x/\delta\theta_o]_t \\ &\quad + v_o\cos\theta_o = 0, \\ &-v_y + hv_o\sin\theta_o\, C[\delta y/\delta C]_t \\ &\quad -(Ev_o + g\sin\theta_o)[\delta y/\delta v_o]_t \\ &\quad -(g\cos\theta_o/v_o)[\delta y/\delta\theta_o]_t \\ &\quad + v_o\sin\theta_o = 0. \end{aligned}$$

With the help of (12) this can be transformed into the set

$$
\begin{aligned}
&hv_o \sin \theta_o\, C[\delta y/\delta C]_x \\
&\qquad - (Ev_o + g \sin \theta_o)[\delta y/\delta v_o]_x \\
&\qquad - (g \cos \theta_o/v_o)[\delta y/\delta \theta_o]_x \\
&\qquad + v_o(\sin \theta_o - \tan \theta \cos \theta_o) = 0, \\
&1 + hv_o \sin \theta_o\, C[\delta t/\delta C]_x \\
&\qquad - (Ev_o + g \sin \theta_o)[\delta t/\delta v_o]_x \\
&\qquad - (g \cos \theta_o/v_o)[\delta t/\delta \theta_o]_x \\
(34) \qquad &\qquad - (v_o/v_x) \cos \theta_o = 0, \\
&1 + hv_o \sin \theta_o\, C[\delta t/\delta C]_y \\
&\qquad - (Ev_o + g \sin \theta_o)[\delta t/\delta v_o]_y \\
&\qquad - (g \cos \theta_o/v_o)[\delta t/\delta \theta_o]_y \\
&\qquad - (v_o/v_y) \sin \theta_o = 0, \\
&hv_o \sin \theta_o\, C[\delta x/\delta C]_y \\
&\qquad - (Ev_o + g \sin \theta_o)[\delta x/\delta v_o]_y \\
&\qquad - (g \cos \theta_o/v_o)[\delta x/\delta \theta_o]_y \\
&\qquad + v_o(\cos \theta_o - \cot \theta \sin \theta_o) = 0
\end{aligned}
$$

By setting $\theta_o = 0$ we obtain a special case of the last pair of equations which is frequently useful in range bombing reductions. The standard trajectory has $\theta_o = v_{yo} = 0$. Ordinarily there is a small error, and the initial v_y is not exactly zero. Since to first-order terms, $\Delta\theta_0 = (1/v_0)\Delta v_{y0}$, the last two of equations (34) simplify to

$$(35)\quad \begin{aligned} [\delta t/\delta v_{y0}]_y &= (1/g)(1 - Ev_0[\delta t/\delta v_0]_y), \\ [\delta x/\delta v_{y0}]_y &= (v_0/g)(1 - E[\delta x/\delta v_0]_y). \end{aligned}$$

The right members can easily be estimated from the ballistic tables. Usually it will be found that the term involving E is considerably smaller than the other, so that with sufficient accuracy the right members of these equations can be simplified to $1/g$ and v_0/g respectively.

If the differential effects found in the preceding section are substituted in equations (33) and (34), it will be found that these latter are not satisfied. The reason is that (33) and (34) were derived on the basis that if the position and velocity of the projectile at any point of a trajectory are taken as initial values, the trajectory re-computed with these new initial values will be identical with the original. This is true of the exact solutions, but is is not true of the Siacci approximations. In particular, the Siacci approximations fail to satisfy (31). However, it is still possible to apply (33) and (34) to the Siacci method by means of an indirect procedure. If θ_0 is 0, the error in the Siacci method is a second-order infinitesimal near the beginning, and (31) holds, and the process of re-computing the trajectory by the Siacci method starting with a point near (0, 0) as initial point will produce an error of order higher than the first. So (33) and (34) should apply to the Siacci trajectories in case $\theta_0 = 0$; and in fact they do, as we see by substitution. If we wished to derive the differential effects of the preceding section by means of these identities, we could apply them to the special case $\theta_0 = 0$, transform to the (L, D)-system with $\theta_0 = 0$, and make use of the property of parallelogram rigidity to obtain the differential effects for arbitrary values of θ_0.

4. Differential effects of constant winds.

It has already been remarked that it is difficult to find the effects of a varying wind when the Siacci method is used. However, even without the Siacci approximations it is not hard to find the differential effects of a constant wind, with the same speed and direction at all altitudes, and the results can be used in connection with the Siacci method if desired. Since the differential effects of various disturbances are superposable, it is permissible to consider range winds and cross winds separately and finally to add their effects. We shall consider range winds first. It is convenient to use time as independent variable.

Suppose that the wind is in the direction of the x-axis, having components $(w_x, 0, 0)$. The coordinates at time t of a projectile of ballistic coefficient C, having coordinates x_o, y_o and velocity components v_{xo}, v_{yo} at time $t = t_o$, will depend on all these quantities and also on the gravity constant g and the relative sound velocity function a(y) and the relative air density function H(y). However, the quantities g, C, a(y), H(y), x_o, y_o will not be varied in this discussion, so we may safely omit them from the notation, and write the coordinates of the projectile as

$$x = x(t, w_x, v_{xo}, v_{yo}),$$
(1)
$$y = y(t, w_x, v_{xo}, v_{yo}).$$

Now let us construct a new coordinate system, denoted by (x^*, y^*, z^*), which at time t_o coincides with the original axis system but is fixed relative to the air mass. If at time t a point has coordinates (x, y, z) with respect to the original axes, its coordinates in the new system will be

$$x^* = x - w_x(t - t_o),$$
(2)
$$y^* = y,$$
$$z^* = z.$$

At time t_o the coordinates of the projectile are the same in the new system as in the original, namely (x_o, y_o). The components of velocity are

$$(3) \qquad dx^*/dt = dx/dt - w_x, \quad dy^*/dt = dy/dt,$$

so the initial velocity has components $v_{xo} - w_x$, v_{yo}. With respect to the new axes the wind is zero, so the solutions of the equations of motion are

$$(4) \qquad \begin{aligned} x^* &= x(t, 0, v_{xo} - w_x, v_{yo}), \\ y^* &= y(t, 0, v_{xo} - w_x, v_{yo}). \end{aligned}$$

From (1), (2) and (4) we have the identity

$$(5) \qquad \begin{aligned} x(t, w_x, v_{xo}, v_{yo}) &= x(t, 0, v_{xo} - w_x, v_{yo}) \\ &\quad + w_x(t - t_o), \\ y(t, w_x, v_{xo}, v_{yo}) &= y(t, 0, v_{xo} - w_x, v_{yo}). \end{aligned}$$

If we differentiate with respect to w_x and then set $w_x = 0$, we obtain

$$(6) \qquad \begin{aligned} [\delta x/\delta w_x]_t &= -[\delta x/\delta v_{xo}]_t + (t - t_o), \\ [\delta y/\delta w_x]_t &= -[\delta y/\delta v_{xo}]_t. \end{aligned}$$

With the help of (3.12), these can be transformed into

$$(7) \qquad \begin{aligned} [\delta y/\delta w_x]_x &= -[\delta y/\delta v_{xo}]_x - (t - t_o) \tan \theta, \\ [\delta t/\delta w_x]_x &= -[\delta t/\delta v_{xo}]_x - (t - t_o)/v_x, \\ [\delta t/\delta w_x]_y &= -[\delta t/\delta v_{xo}]_y, \\ [\delta x/\delta w_x]_y &= -[\delta x/\delta v_{xo}]_y + t - t_o. \end{aligned}$$

In the notation of differential effects, the last two can be written

$$(8) \qquad \begin{aligned} dt(w_x|y) &= -[\delta t/\delta v_{xo}]_y w_x, \\ dt(w_x|y) &= \{t - t_o - [\delta x/\delta v_{xo}]_y\} w_x. \end{aligned}$$

If we differentiate both members of (5) with respect to t, then differentiate with respect to w_x and set $w_x = 0$, we obtain

$$(9) \qquad \begin{aligned} [\delta v_x/\delta w_x]_t &= -[\delta v_x/\delta v_{xo}]_t + 1, \\ [\delta v_y/\delta w_x]_t &= -[\delta v_y/\delta v_{xo}]_t. \end{aligned}$$

From this and (3.12) we obtain

$$(10) \qquad \begin{aligned} dv_x(w_x|y) &= \{-[\delta v_x/\delta v_{xo}]_y + 1\}w_x, \\ dv_y(w_x|y) &= -[\delta v_y/\delta v_{xo}]_y w_x. \end{aligned}$$

If the coordinates are regarded as functions of initial velocity, angle of departure, time, etc., instead of functions of v_{xo}, v_{yo}, time, etc., then in place of (1) we would have

$$x = x(t, w_x, v_o, \theta_o), \qquad y = y(t, w_x, v_o, \theta_o),$$

and in place of (5) we would have

$$(11) \qquad \begin{aligned} &x(t, w_x, v_o, \theta_o) \\ &\quad = w_x(t - t_o) + x(t, 0, \sqrt{(v_{xo} - w_x)^2 + v_{yo}^2}, \\ &\qquad\qquad \text{arc cot}\,\{(v_{xo} - w_x)/v_{yo}\}), \\ &y(t, w_x, v_o, \theta_o) \\ &\quad = y(t, 0, \sqrt{(v_{xo} - w_x)^2 + v_{yo}^2}, \\ &\qquad\qquad \text{arc cot}\,\{(v_{xo} - w_x)/v_{yo}\}). \end{aligned}$$

By differentiating with respect to w_x and setting $w_x = 0$ we obtain

$$(12) \qquad \begin{aligned} [\delta x/\delta w_x]_t &= t - t_o - [\delta x/\delta v_o]_t \cos\theta_o \\ &\qquad + [\delta x/\delta\theta_o]_t \sin\theta_o/v_o, \\ [\delta y/\delta w_x]_t &= -[\delta y/\delta v_o]_t \cos\theta_o \\ &\qquad + [\delta y/\delta\theta_o]_t \sin\theta_o/v_o. \end{aligned}$$

With the help of (3.12) these can be transformed into

$$(13)\quad \begin{aligned} [\delta x/\delta w_x]_y &= t - t_0 - [\delta x/\delta v_0]_y \cos\theta_0 \\ &\quad + (1/v_0)[\delta x/\delta\theta_0]_y \sin\theta_0, \\ [\delta t/\delta w_x]_y &= -[\delta t/\delta v_0]_y \cos\theta_0 \\ &\quad + (1/v_0)[\delta t/\delta\theta_0]_y \sin\theta_0. \end{aligned}$$

Next we suppose that there is a constant cross wind and no range wind, so that the components of wind are $(0, 0, w_z)$. We introduce a new set of axes coinciding with the original set at $t = t_0$ but fixed with respect to the air mass. If a particle has coordinates (x, y, z) at time t in the original coordinate system, its coordinates in the new system are

$$(14)\quad \begin{aligned} x^* &= x, \\ y^* &= y, \\ z^* &= z - w_z(t - t_0). \end{aligned}$$

Thus at $t = t_0$ the components of velocity of the projectile are v_{x0}, v_{y0}, $-w_z$. There is no wind with respect to the new axes, but conditions are still not standard because of the non-zero component of velocity along the z-axis. Therefore we rotate the axes about the y-axis through an angle ψ, where

$$(15)\quad \psi = \text{arc tan } w_z/v_{x0}.$$

There is no loss of generality in assuming that x_0 and z_0 are both 0. If we denote the new coordinates by X, Y, Z, the transformation is given by

$$(16)\quad \begin{aligned} x^* &= X\cos\psi + Z\sin\psi, \\ y^* &= Y, \\ z^* &= -X\sin\psi + Z\cos\psi. \end{aligned}$$

If we define

$$(17)\quad u_h = \sqrt{v_{x0}^2 + w_z^2},$$

we find from (16) that at $t = t_o$

$$(18) \qquad dX/dt = u_h, \; dY/dt = v_{yo}, \; dZ/dt = 0.$$

Thus there is no wind and the Z-component of velocity is zero in this newest coordinate system, and so by (1) the solution of the equations of motion is

$$(19) \qquad \begin{aligned} X &= x(t, 0, u_h, v_{yo}), \\ Y &= y(t, 0, u_h, v_{yo}), \\ Z &= 0. \end{aligned}$$

We now transform back to the original system through equations (16) and (15) obtaining

$$(20) \qquad \begin{aligned} x &= (v_{xo}/u_h)\, x(t, 0, u_h, v_{yo}), \\ y &= y(t, 0, u_h, v_{yo}), \\ z &= (t - t_o)w_z - (w_z/u_h)\, x(t, 0, u_h, v_{yo}). \end{aligned}$$

At $w_z = 0$ we see by (17) that

$$(21) \qquad u_h = v_{xo}, \; \partial u_h/\partial w_z = 0.$$

So by differentiation in (20) with respect to w_z and setting $w_z = 0$ we obtain

$$(22) \qquad \begin{aligned} [\delta x/\delta w_z]_t &= 0, \\ [\delta y/\delta w_z]_t &= 0, \\ [\delta z/\delta w_z]_t &= t - t_o - x/v_{xo}. \end{aligned}$$

We can easily eliminate the assumption that x_o is 0 by simply making a translation of axes. Then, with the help of (3.12) we find from the last of these equations that

$$(23) \qquad \begin{aligned} [\delta z/\delta w_z]_x = [\delta z/\delta w_z]_y &= [\delta z/\delta w_z]_t \\ &= t - t_o - (x - x_o)/v_{xo}, \end{aligned}$$

the differential effects of w_z on range and time of flight being zero.

5. Differential corrections for variable density.

There is one more non-standard (under the assumptions of the Siacci method) condition for which a differential correction can be computed. The method we shall use differs in execution from that used earlier in the chapter, and in fact offers a rather simple illustration of the method we must use in the following chapters.

We first recall the normal equations (V.1.8), in which x is the independent variable.

$$
(1) \qquad
\begin{aligned}
dt/dx &= 1/v_x, \\
dy/dx &= m, \\
dm/dx &= -\, g/v_x^2, \\
dv_x/dx &= -\, e^{-hy} G(v_x \sec \theta)/C_s.
\end{aligned}
$$

Under the assumptions of the Siacci method, $v_x \sec \theta$ can be replaced by $v_x \sec \theta_0$ and h is set equal to zero. We now make the first assumption and consider that for the normal trajectory h is zero, but that for the disturbed trajectory h is not zero. Thus the disturbed trajectory satisfies the equations

$$
(2) \qquad
\begin{aligned}
dt/dx &= \sec \theta_0/p, \\
dy/dx &= m, \\
dm/dx &= -\, g \sec^2\theta_0/p^2, \\
dp/dx &= -\, e^{-hy} \sec \theta_0\, G(p)/C_s,
\end{aligned}
$$

wherein we have replaced $v_x \sec \theta_0$ by the symbol p. The original trajectory satisfies these equations with h replaced by 0. We now wish to find the differential effect of the change in h when points on the two trajectories are matched by equal values of x. That is, we wish to find $[\delta t/\delta h]_x$, etc. But as remarked in Section 3, if the solutions of (2) are denoted by $t(x, h)$, $y(x, h)$, etc., these differential effects are the same as $\partial t/\partial h$, etc. We shall assume that the initial conditions are the same on the two trajectories since we wish to find only the effect of the change of density, and this does not alter the initial conditions. Hence for the initial conditions on the differential effects we have

$$\text{(3)} \quad \begin{aligned} &\text{at } x = 0, \\ &[\delta t/\delta h]_x = [\delta y/\delta h]_x \\ &\qquad = [\delta m/\delta h]_x = [\delta p/\delta h]_x = 0. \end{aligned}$$

The differential equations satisfied by the differential effects are obtained by differentiating (2) with respect to h, setting h equal to 0, and then replacing $\partial t/\partial h$ by $[\delta t/\delta h]_x$, and analogously for the other effects. The result is

$$\text{(4)} \quad \begin{aligned} &d[\delta t/\delta h]_x/dx = -(\sec\theta_o/p^2)[\delta p/\delta h]_x, \\ &d[\delta y/\delta h]_x/dx = [\delta m/\delta h]_x, \\ &d[\delta m/\delta h]_x/dx = 2g(\sec^2\theta_o/p^3)[\delta p/\delta h]_x, \\ &d[\delta p/\delta h]_x/dx = (y \sec\theta_o\, G(p)/C_s) \\ &\qquad\qquad - (dG(p)/dp)(\sec\theta_o/C_s)[\delta p/\delta h]_x. \end{aligned}$$

Herein, of course, p and y and G(p) can be regarded as known functions of x, having been determined in the process of computing the original trajectory. We have thus reduced the problem to that of solving the four equations (4) for the four differential effects, subject to the initial conditions (3). This solution can be accomplished by quadratures. For the last of equations (4) is a linear first-order equation in $[\delta p/\delta h]_x$, and can be solved by quadratures, and the solutions of the other three equations can be found at once from this, by quadratures. However, it is somewhat more convenient to change to p instead of x as independent variable. With the help of the last of equations (2), wherein we set h = 0 because we are computing disturbances about the original trajectory, we find

(5)
$$d[\delta t/\delta h]_x/dp = (C_s/p^2 G(p))[\delta p/\delta h]_x,$$
$$d[\delta y/\delta h]_x/dp = -(C_s \cos\theta_0/G(p))[\delta m/\delta h]_x,$$
$$d[\delta m/\delta h]_x/dp = -(2gC_s \sec\theta_0/p^3 G(p))[\delta p/\delta h]_x,$$
$$d[\delta p/\delta h]_x/dp = -y + (1/G(p))(dG(p)/dp)[\delta p/\delta h]_x.$$

These equations are easily solved—an integrating factor for the last one is $1/G(p)$. The solution, for the initial conditions (3), is:

(6)

$$[\delta p/\delta h]_x = -G(p)\int_{v_0}^{p}\{y(u)/G(u)\}\,du,$$

$$[\delta m/\delta h]_x = 2gC_s \sec\theta_0 \int_{v_0}^{p}\{1/r^3\} \cdot\left[\int_{v_0}^{r} y(u)/G(u)\,du\right]dr,$$

$$[\delta t/\delta h]_x = -C_s\int_{v_0}^{p}\{1/r^2\}\left[\int_{v_0}^{r}\{y(u)/G(u)\}\,du\right]dr,$$

$$[\delta y/\ h]_x = -2gC_s^2\int_{v_0}^{p}\{1/G(s)\}\left(\int_{v_0}^{s}\{1/r^3\} \cdot\left[\int_{v_0}^{r}\{y(u)/G(u)\}\,du\right]dr\right)ds.$$

The differential effects may now be obtained by the formulas:

$$dt(h|x) = h[\delta t/\delta h]_x, \tag{7}$$
$$dy(h|x) = h[\delta y/\delta h]_x.$$

Chapter VIII

DIFFERENTIAL EFFECTS AND WEIGHTING FACTORS

1. Functionals and their differentials.

If a projectile is launched at a time t_0, its position at a later time T will depend on the initial position, initial velocity, ballistic coefficient, gravity constant, and also on the relative air density and relative sound velocity functions. H(y) and a(y). But the dependence of, say, x(T) on these last is different from its dependence on, say, C; along with the other data, in order to determine x(T) we need to know the entire aggregate of functional values of H(y) and a(y) along the interval of values of y traversed by the projectile. So x(T) is a function of the functions H(y) and a(y), not of any individual values of these functions. A number whose value is determined by a function in its entirety, not necessarily by the value of that function at some specific spot, is called a functional of that function. We have already observed this situation in Section 3 of Chapter VII, and have introduced a notation in (VII.3.3). In accordance with that notation scheme, a number, say y, determined by the entire aggregate of the values of some function g(t) will be denoted by a symbol such as $y = f(g(\ ))$. This contrasts with a symbol such as f(g(t)), which would mean the value of a function f(x) of a real variable in which the real variable x is replaced by the numerical value g(t) of a function g at a specific spot t. The empty parentheses are meant to indicate that f is dependent on the whole function g, not on its value at a particular place t.

Many of the concepts associated with functions of several real variables can be extended to apply to functionals also. In order to extend the important concept of continuity, we recall that in the case of functions of several real variables, this definition involved the distance between two points in the space, or what amounts to the same thing, the length of a vector in the space. So to define continuity for a functional, we assume that to each function g which is an argument for the functional f(g( )) there is a "length," or "norm," N[g( )]. This will be assumed to have the following properties. If g is identically zero, then N[g( )] = 0; otherwise, N[g( )] is positive. If k is a real number, the norm of the function kg( ) which is everywhere equal to k times the function g satisfies

$$N[kg(\ )] = |k| N[g(\ )].$$

And finally, for any two functions g and h the inequality

$$N[g(\ ) + h(\ )] \leq N[g(\ )] + N[h(\ )]$$

is satisfied. This last is called the "triangle inequality," being the generalization of the statement that the sum of two sides of a triangle is at least equal to the third side. Two particular examples of a "norm" satisfying these requirements, and in fact the only two that we have any need of, are:

(1) N [g( )] = maximum value of |g(t)| for all t;

(2) N [g( )] = the greater of the two numbers {maximum value of |g(t)| for all t, maximum value of |g'(t)| for all t}.

Having such a definition of "norm," the generalization of the definition of continuity is immediate.

A functional $f(g(\ ))$ will be said to be continuous at a function $g_o(\ )$ if to each positive ε there corresponds a positive δ such that

$$| f(g(\ )) - f(g_o(\ )) | \leq \varepsilon$$

whenever

$$N[g(\ ) - g_o(\ )] < \delta.$$

Of course the meaning of this definition will change if we change the meaning of the "norm" involved in it.

A functional $f(g(\ ))$ is <u>linear</u> if it satisfies the following two requirements.

(3) If $g(\ )$ and $h(\ )$ are functions in the domain of arguments of the functional f, and a and b are real numbers, then $ag(\ ) + bh(\ )$ is a function in the domain of arguments of f, and

$$f(ag(\ ) + bh(\ )) = a\, f(g(\ )) + b\, f(h(\ )).$$

(4) There is a constant K such that for every function $g(\ )$ in the domain of arguments of f,

$$| f(g(\ )) | \leq K\, N[g(\ )].$$

The second of these requirements is a simple consequence of the first in the case of functions of several real variables with the usual definition of distance of two points (or length of a vector); but for functionals in general, it does not follow from (3), and must be stated as a separate hypothesis.

The standard definition of a differential, which we have repeated in Section 1 of Chapter VII (see the sentences containing (VII.1.3, 4)) can be extended almost verbatim to functionals. A functional $f(g(\ ))$ has a differential at the function $g_o(t)$ if there is a linear functional $L(g(\ ))$ which approximates $f(g(\ )) - f(g_o(\ ))$ to within an error which vanishes more rapidly than first order in $N[g(\ )]$.

That is, the ratio

$$(5) \qquad \frac{f(g(\ )) - f(g_0(\ )) - L(g(\ ))}{N[g(\ )]}$$

approaches zero when $N[g(\ )]$ approaches zero. Whenever this differential exists, we shall denote it by a symbol similar to that introduced in Section 1 of Chapter VII, namely $df(g(\ ) - g_0(\ ))$, or $df(\Delta g(\ ))$. Usually it would be necessary to amplify such a symbol to indicate the particular g_0 at which the differential is being taken, but in ballistics we shall always start with a specified normal trajectory in which the disturbances are zero, and the quantities investigated will be regarded as functionals of the disturbances; the only place at which we care to find the differential is at the particular argument "all disturbances = 0." So we may safely omit any indication of the place at which the differential is to be taken; the reader will remember that this is the zero function, "all disturbances = 0."

In the simpler case of disturbances depending on a single number, we found the use of differential effects convenient; the closeness of the approximation between differential and difference permitted the substitution of the differential for the difference with little error, provided the disturbance remained within small enough bounds, and the linearity of the differential effects made it easier to work with them than with the actual differences. All of these remarks apply to this more complicated case of functionals, and in fact in an intensified form. It is still permissible to replace the difference by the differential effect with only a small error, if the disturbance (or its norm) remains within small en ugh bounds; and the gain in simplicity is even more marked, because the differences are essentially more complicated, being functionals, than they were in the simpler case considered in the preceding chapter. Therefore it is important, first, to show that the differentials exist; second, to investigate methods by which they can be found and made available to the using services. The existence of the differentials will be established in the next section. The methods of the proof are due in large part to Professor G. A. Bliss, who seems to have been the first to realize the need of showing that the differentials exist. The proof of the existence will automatically furnish us with one feasible method of computing the differential effects; more practical methods will be exhibited in the next chapter. However the differential effects may be computed, it is clearly necessary that the bulk of the computational work be done in some establishment behind the lines, and that the results of the computations be furnished the services in such a form that they can be applied to the conditions prevailing at time of firing with little additional work. The means of furnishing the results of the computations to the using services in convenient form is considered in the latter part of this chapter.

2. Proof of the existence of differential effects.

Since the practical importance of being able to compute differential effects has been established beyond dispute, it is logically important to prove that differential effects exist.

Let us choose the axes as usual; to be specific, we shall choose the y-axis positive upward. Assume that there is a horizontal wind with components w_x, w_z which are functions of the altitude y. The ratio of air density to standard sea-level air density will be denoted as usual by H(y), and the ratio of sound velocity to standard by a(y). It will be assumed that there are small forces acting on the projectile, other than standard gravity and drag, which produce an acceleration with components a_x, a_y, a_z; these may be functions of the coordinates and the components of velocity. The components of velocity with respect to the axes are v_x, v_y, v_z, and v is the length of this vector. The components of velocity with respect to the air are $u_x = v_x - w_x$, $u_y = v_y - w_y$, $u_z = v_z - w_z$, and the length of this vector is the air speed u. The equations of motion can be written in the form

$$\begin{aligned} dx/dt &= v_x, \\ dy/dt &= v_y, \\ dz/dt &= v_z, \\ dv_x/dt &= -Eu_x + a_x, \\ dv_y/dt &= -Eu_y - g + a_y, \\ dv_z/dt &= -Eu_z + a_z, \end{aligned} \tag{1}$$

where

$$\begin{aligned} E &= \gamma a(y)H(y)G(u/a(y)) \\ &= \gamma H(y)uB(u/a(y)); \end{aligned} \tag{2}$$

or they may be written in any of a number of other ways which have been exhibited in Section 1 of Chapter V. The normal equations resemble these, but have the standard functions for H and a, and w_x, w_z, a_x, a_y, and a_z are all zero. Our task is to compare the solutions of (1) that have certain initial values with the solutions of the normal equations that have the same or slightly different initial values. Partly for notational simplicity, and partly to have a formulation that will also cover the other possible ways of writing the equations of motion, we shall now change the notation. The variables x, y, z, v_x, v_y, v_z will be given the new names y_1, y_2, y_3, y_4, y_5, y_6 respectively; and whenever we wish, we shall use the single letter y to denote the vector, or point of six-dimensional space, $(y_1, \ldots, y_6)$. The right members of the six equations (1) are functions of the six variables y_i, and will be denoted by the symbols $F_1(t, y), \ldots, F_6(t, y)$ in order, from the top down.

Thus all six equations (1) can be condensed into the compact form

$$(3) \qquad dy_i/dt = F_i(t, y) \qquad (i = 1, \ldots, n),$$

where n happens to be 6. The normal equations are similarly condensed into an analogous form, but the right members are, of course, not the same functions of the independent variables as the right members of (3), which correspond to the perturbed equations. We denote the right members of the normal equations by $f_i(t, y)$, $i = 1, \ldots, 6$, so that the normal equations take the form

$$(4) \qquad dy_i/dt = f_i(t, y) \qquad (i = 1, \ldots, n).$$

Had we started with any of the other forms of the equations of motion, say with slope as independent variable, we could have introduced a change of notation that would have brought the disturbed equations and the normal equations into the respective forms (3) and (4); however, in this case the independent variable t would not have the physical interpretation of time, but would be whatever quantity we had selected as independent variable. This, in fact, is the reason that we have left in t as an argument of F and f; in the equations (1), the right members do not depend on the independent variable, but if slope or y were independent variable this would no longer be true.

We can easily agree on bounds for the initial altitude and velocity and an upper bound for wind speed, from which simple physical considerations show that each coordinate and each component of velocity will remain between certain bounds. Within this region, we assume that the standard functions $f(t, y)$ have continuous partial derivatives of first and second order, and the perturbed functions $F(t, y)$ have continuous partials of first order. The functions to be compared are on the one hand a solution $y_i = y_i(t)$

of the normal equations (4) with certain initial values

$$y_1(0), \ldots, y_n(0), \tag{5}$$

and on the other hand a solution $y_i = Y_i(t)$ of the disturbed equations (3) with initial values

$$Y_1(0), \ldots, Y_n(0). \tag{6}$$

Let us define

$$\Delta y_i(t) = Y_i(t) - y_i(t), \tag{7}$$

$$\Delta f_i(t, y) = F_i(t, y) - f_i(t, y). \tag{8}$$

The amount by which the equations have been changed in replacing f_i by F_i can reasonably be measured by the greatest value of the length of the vector from $(f_1(t, y), \ldots, f_n(t, y))$ to $(F_1(t, y), \ldots, F_n(t, y))$ as t and the y_i each vary over the interval of values between its least and its greatest value. The amount by which the initial conditions have been changed can be specified by stating the distance between the points (5) and (6). The norm of the disturbance can be specified as the greater of these two numbers, namely the amount by which the equations were changed and the amount by which the initial conditions were changed. Therefore we define

$$N_1 = \text{greater of } \sqrt{\sum_1^n [\Delta y_i(0)]^2} \text{ and } \max \sqrt{\sum_1^n [\Delta f_i(t, y)]^2}. \tag{9}$$

The proof of the possibility of good linear approximations to effects of small disturbances depends on the following lemma.

(10) Lemma. *Given any fixed number T, there is a constant A such that the inequality*

$$\sqrt{\sum_{1}^{n} [\Delta y_i(t)]^2} \leq AN_1$$

is satisfied for all t in the interval $0 \leq t \leq T$.

For simplicity of notation we shall denote the left member of the inequality in (10) by $r(t)$. The functions y_i and Y_i satisfy equations (4) and (3) respectively, so that

$$\begin{aligned} d\Delta y_i/dt &= F_i(t, Y) - f_i(t, y) \\ &= \Delta f_i(t, Y) + f_i(t, Y) - f_i(t, y) \qquad (11) \\ &= \Delta f_i(t, Y) + \sum_{j=1}^{n} [\partial f_i(t, \bar{y})/\partial y_j][Y_j - y_j], \end{aligned}$$

where $\bar{y}$ is some point on the line segment from y to Y. Let us multiply both members of this equation by Δy_i and sum for $i = 1, \ldots, n$. The left member is then half the derivative of r^2 with respect to t, so that

$$\begin{aligned} r\,dr/dt &= \sum_{i=1}^{n} \Delta y_i(t) \Delta f_i(t, Y) \qquad (12) \\ &+ \sum_{i,j=1}^{n} [\partial f_i/\partial y_j] \Delta y_i \Delta y_j. \end{aligned}$$

The first term on the right is the inner product of two vectors, the first of which has length r and the second of which has length at most N_1, by definition of N_1. So the first term cannot exceed $N_1 r$. The quadratic form

$$\sum_{i,j=1}^{n} [\partial f_i(t,y)/\partial y_j] k_i k_j$$

has a finite upper bound, which we call M, as (t, y) ranges over all values within the bounds we have permitted and $k_1, \ldots, k_n$ vary over all values the sum of whose squares is 1. So the last term in the right member of (12) cannot exceed Mr^2. This furnishes the estimate

$$dr/dt \leq N_1 + Mr, \tag{13}$$

valid whenever r is not 0. Now let t* be any number between 0 and T. If r vanishes for some t between 0 and t*, let t_1 be the largest such t; otherwise let t_1 be 0. In either case (13) holds for t between t_1 and t*. If we write (13) in the form

$$(dr)/(r + N_1/M) \leq M\, dt$$

and integrate from t_1 to t^*, we find

$$r(t^*) + N_1/M \leq e^{M(t^* - t_1)}[r(t_1) + N_1/M]. \tag{14}$$

Here $t^* - t_1$ cannot exceed T. Also, either t_1 is 0, in which case $r(t_1)$ is at most N_1 by definition of r and N_1, or else t_1 is a place at which r vanishes, in which case it is still true that $r(t_1)$ cannot exceed N_1. Hence (14) implies

$$r(t^*) \leq e^{TM}[1 + 1/M]N_1. \tag{15}$$

If we let A stand for the coefficient of N_1 in the right member, this is inequality (10), and the lemma is established.

This lemma shows that if we have a solution of a differential equation, and then change the functions in the equation or the initial conditions by small amounts, the solution will also be changed by a small amount — not more than a preassigned multiple of the amount by which functions or initial conditions were changed. In particular, the value of the solution varies in a continuous way as functions and initial values are altered. To show that it varies in a differentiable way, we must also take the derivatives of the functions Δf_i into consideration. This we do by defining a new kind of estimate of the amount by which the equations are disturbed; we define

N_2 = greatest of the numbers

$$(16) \qquad N_1, \ \max \sqrt{\sum_{j=1}^{n} [\partial f_1(t, y)/\partial y_j]^2}, \ \ldots,$$

$$\max \sqrt{\sum_{j=1}^{n} [\partial f_n(t, y)/\partial y_j]^2},$$

the maxima being taken over the entire range of permissible values of $t, y_1, \ldots, y_n$.

We shall now show that if the functions $y_i(t)$ satisfy (4) with initial values (5) and the functions $Y_i(t) = y_i(t) + \Delta y_i(t)$ satisfy (3) with initial values (6), then the differences $\Delta y_i(t)$ are approximated by the solutions $\eta_i(t)$ of the equations

$$(17) \quad d\eta_i/dt = \sum_{j=1}^{n} [\partial f_i/\partial y_j]\eta_j(t) + \Delta f_i(t, y(t)),$$

with the initial values

$$(18) \qquad \eta_i(0) = \Delta y_i(0),$$

the partial derivatives in (17) being evaluated for arguments (t, y(t)). Precisely, <u>for each fixed value</u> T, <u>there is a constant</u> B <u>such that</u>

$$(19) \quad \sqrt{\sum_{i=1}^{n} [\Delta y_i(t) - \eta_i(t)]^2} \leq BN_2^{\,2} \quad (0 \leq t \leq T).$$

Thus the functions η_i approximate the changes Δy_i accurately to the first order in N_2, the error being of the second order in N_2.

The functions η_i and Δy_i satisfy equations (17) and (11) respectively, and they have the same initial values. We can apply (10) to estimate the difference between the η_i and the Δy_i. But this will be easier to do if we first make some changes in notation. First, the right member $F_i(t, Y) - f_i(t, y)$ of equation (11) will be expanded by the theorem of mean value. Equation (11) then takes the form

$$
\begin{aligned}
d\Delta y_i/dt = \Delta f_i(t, y(t)) &+ \sum_{j=1}^{n} c_{ij}(t)\Delta y_j \\
(20) \qquad &+ \sum_{j=1}^{n} [\partial f_i/\partial y_j]\Delta y_j \\
&+\tfrac{1}{2} \sum_{j,k=1}^{n} b_{ijk}(t)\Delta y_j \Delta y_k,
\end{aligned}
$$

where the partial derivative in the second sum on the right is evaluated at $(t, y(t))$, the coefficient $c_{ij}(t)$ is the value of $\partial \Delta f_i/\partial y_j$ at some point (t, y^*) on the line segment joining $(t, y(t))$ to $(t, Y(t))$, and the coefficient b_{ijk} is the value of the partial derivative $\partial^2 f_i/\partial y_j \partial y_i$ at some point (t, y^{**}) on the same line segment. Let us define

$$
\begin{aligned}
&\Phi_i(t, v_1, \ldots, v_n) \\
(21) \qquad &= \sum_{j=1}^{n} c_{ij}(t)v_j + \sum_{j=1}^{n} [\partial f_i/\partial y_j]\, v_j \\
&\quad +\tfrac{1}{2} \sum_{j,k=1}^{n} b_{ijk}(t)\, v_j v_k + \Delta f_i(t, y(t)),
\end{aligned}
$$

$$
\begin{aligned}
&\phi_i(t, v_1, \ldots, v_n) \\
(22) \qquad &= \sum_{j=1}^{n} [\partial f_i/\partial y_j]\, v_j + \Delta f_i(t, y(t)).
\end{aligned}
$$

Then Δy and η are respectively the solutions of the equations

(23) $$dv_i/dt = \Phi_i(t, v),$$

(24) $$dv_i/dt = \phi_i(t, v)$$

with the initial values $\Delta y_1(0), \ldots, \Delta y_n(0)$. The difference $\Delta f_i(t, y) = F_i(t, y) - f_i(t, y)$ is replaced by the difference

(25) $$\Phi_i(t, v) - \phi_i(t, v) = \sum_{j=1}^{n} c_{ij}(t)\, v_j + \tfrac{1}{2} \sum_{j,k=1}^{n} b_{ijk}(t)\, v_j\, v_k.$$

Since the initial values of Δy_i are the same as those of η_i, the formula (9) for the norm of the disturbance yields the number

(26) $$\nu_1 = \max \sqrt{\sum_{i=1}^{n} [\Phi_i(t, v) - \phi_i(t, v)]^2},$$

where the maximum is to be taken over all the permissible values of t and v. The permissible values of t are those in the interval $0 \leq t \leq T$, while by (10) the permissible values of the v_i are those which form vectors of length at most AN_1. Let N be the upper bound of the last term in (25) as the vector v varies over all unit vectors and t varies from 0 to T; this upper bound is finite, since we have assumed that the second partial derivatives of the function f with respect to the y_i are continuous. Then the last term in (25) cannot exceed N times the square of the length of v, and therefore is at most $N(AN_1)^2$. For each i, the numbers $(c_{i1}, \ldots, c_{in})$ form a vector of length at most N_2, by (16). Hence the first term in the right member of (25) cannot exceed AN_2N_1. Since N_2 is at least as great as N_1, this shows that the left member of (25) cannot exceed a constant multiple of N_1N_2.

If we square these quantities, add and take the square root, we obtain a number which cannot exceed a constant multiple of N_1N_2. Hence by (26),

$$\nu_1 \leq CN_1N_2, \tag{27}$$

where C is a constant. Now by (10) we see that the left member of (19) cannot exceed a constant multiple of ν_1, therefore by (27) cannot exceed a constant multiple of N_1N_2. This is in fact a stronger statement than the inequality (19) which we set out to prove, since N_1 cannot be greater than N_2.

It remains to show that for each fixed t, the solutions $(\eta_1(t), \ldots, \eta_n(t))$ of equations (17) are the differentials of $y_1(t), \ldots, y_n(t)$ respectively. First it must be shown that each $\eta_i(t)$ is a linear functional of the disturbances. If the disturbances were all zero, so that $\Delta y_i(0) = 0$ and the Δf_i vanish identically, the solution of (17) would be $\eta_i(t) = 0$. These values we compare with the solutions of equations (17) as written, with initial values (18). Since the first set of η_i is 0, the difference $\Delta\eta_i$ is the same as η_i. By (10), this cannot exceed a constant multiple of N_1, which in turn cannot exceed the same constant multiple of N_2. So (1.4) is satisfied. Next, suppose that a first set of disturbances consists of changes $\Delta y_{i1}(0)$ in the initial conditions and changes $\Delta f_{i1}(t, y)$ in the equations, and that the corresponding solutions of (17) are $\eta_{i1}(t)$; and that a second set of disturbances consists of changes $\Delta y_{i2}(0)$ in the initial conditions and changes $\Delta f_{i2}(t, y)$ in the equations, the corresponding solutions of (17) being $\eta_{i2}(t)$. If a and b are any real numbers, it is easily verified by substitution that corresponding to the disturbances consisting of changes $a\Delta y_{i1}(0) + b\Delta y_{i2}(0)$ in the initial conditions and changes $a\Delta f_{i1}(t, y) + b\Delta f_{i2}(t, y)$ in the equations, the solutions of equations (17) are

$$a\eta_{i1}(t) + b\eta_{i2}(t).$$

Hence the $\eta_i(t)$ have property (1.3), which completes the proof that they are linear functionals of the disturbances. The statement that the ratio (1.5) approaches 0 with N_2 is a consequence of (19), and so the $\eta_i(t)$ are the differentials of the $y_i(t)$, as was to be proved.

Equations (17) need one minor change to make them more readily applicable to the situation in ballistic applications. In studying the effects of non-standard conditions on the motion of the projectile, we have to consider equations such as (1). Here each right member is formed from a function which involves the variables t and y_i and also certain other variables; these latter are themselves replaced by certain functions of t and the y_i, and the result is the function $F_i(t, y)$ of the foregoing pages. For example, if conditions are standard the right members of (1) are certain functions of t, v_x, ..., z. If we denote the departures of sound velocity and density from standard by Δa and ΔH and the wind components by w_x and w_z, the right member of each of equations (1) appears as a function of the fourteen variables t, x, y, z, v_x, v_y, v_z, Δa, ΔH, w_x, w_z, a_x, a_y, a_z. When the last seven are set equal to zero, equations (1) are the normal equations. When they are replaced by functions of the first seven variables, we have the equations of motion under non-standard conditions, and the right members of (1) are the functions F_i of equations (3). To be specific, the right member of the last of equations (1) is a function $f_6(t, \ldots, a_z)$ of the fourteen variables already listed; and the difference $\Delta f_6(t, y)$ is the same as

$$f_6(t, x, \ldots, v_z, \Delta a, \Delta H, \ldots, a_z) - f_6(t, x, \ldots, v_z, 0, \ldots, 0), \tag{28}$$

wherein Δa, etc., are to be regarded as specified as functions of t, x, ..., v_z. The difference (28) and its five analogues with subscripts 1, ..., 5 can be approximated by means of a Taylor expansion to linear terms. We thus find that to linear terms

$$\Delta f_6 = [\partial f_6/\partial a]\Delta a + [\partial f_6/\partial H]\Delta H + \dots + [\partial f_6/\partial a_z]a_z, \tag{29}$$

where the partial derivatives are to be evaluated for the arguments $(t, x, \dots, v_z)$ belonging to the normal trajectory. The magnitude of Δf_6 is not more than a certain constant multiple of the greatest of the numbers $\max|\Delta H|, \dots, \max|a_z|$, and the first-order partial derivatives of Δf_6 do not exceed some multiple of the greatest of these numbers and the numbers $\max|\partial \Delta H/\partial y|$, etc. Hence the latter of these may be used to replace N_2 in (19). The error in the expansion (29) does not exceed some multiple of the square of the greatest of $|\Delta H|$, etc., hence does not exceed some multiple of N_2^2. So, by (10), if we replace Δf_6 by the right member of (29) in equation (17), and treat Δf_1, etc., similarly, the resulting error does not exceed some multiple of N_2^2. Thus the use of the approximation (29) does not injure the order of accuracy with which the solutions of (17) approximate the changes Δy_1, etc., in the solutions. But now the right members of (17) are linear in the disturbing functions Δa, ΔH, etc. The solutions η_i of (17) have already been seen to be linear as functions of the Δf_i, so now the $\eta_i(t)$ have been made to depend linearly on the disturbances Δa, etc.

The changes $\Delta y_i(t)$ are functionals of the disturbances $w_x(y)$, etc., and the $\eta_i(t)$ are their differentials. According to the symbolism mentioned in the preceding section, if the disturbance consists, say, of a range wind w_x alone, all other conditions being standard, the resulting $\eta_i(t)$ should be designated by the symbols $dy_i(w_x)$; if the disturbance were a departure ΔH from standard density, the $\eta_i(t)$ would be designated by $dy_i(\Delta H)$; and so on. But once again we observe that these differential effects would be different if some other variable had been selected as independent variable in the computations. The differential effects are those found by matching points of equal values of t on the disturbed and undisturbed trajectories. As before, we introduce a symbol to indicate which variable is used in the matching of the points of the two trajectories. Thus if the density differs from normal by ΔH, the corresponding differential effect will be denoted by $dy_1(\Delta H|t)$. The symbol after the vertical bar indicates the variable used in establishing the correspondence between points of the disturbed and the undisturbed trajectories.

An important consequence of the linearity of the differential corrections is the "superposability of differential corrections." Let p be one departure from normal conditions (for example, a range wind, variable with position) and q another such departure. Then because of linearity

$$dx(p + q|t) = dx(p|t) + dx(q|t).$$

The differential effect of the two disturbances acting simultaneously is the sum of the differential effects of the two disturbances acting separately. This permits us to consider different types of disturbance one by one, and finally to find the differential effect of the aggregate by simply adding the results.

The equations with which we finished are the same as we would have obtained had we simply expanded everything involved in the equations by Taylor's theorem, stopping with linear terms. But this process would have been logically inadequate. It would have left us without information as to the magnitude of the errors that might result. Such a difficulty can occur even in the study of functions of two real variables, which are much simpler than the functions of functions which we have been investigating. For example, it might be thought that if a function has a directional derivative at the origin which is zero whatever the direction, it must necessarily be closely approximated by a constant on some region about the origin. But this is false; functions with such directional derivatives can nevertheless be discontinuous at the origin. Expansions of our equations to terms linear in, say ΔH, are analogous to directional derivatives, and are incapable of furnishing adequate information about the behavior of the changes Δy_i.

A difficulty of a different kind is raised by the presence of the partial derivatives in the definition (16) of the norm N_2. If, for example, we use t as independent variable and wish to find the differential effects of winds, the differential involves the norm N_2, which in turn involves the rate of change of wind with respect to y. This is undesirable both mathematically and physically; the latter, because the measurement of wind is performed by measuring the travel of a balloon in various time intervals, and would not reveal the presence of an extremely thin zone in which the rate of change of wind with altitude is very large. If it is possible to use y as independent variable this difficulty disappears, for in (16) the partial derivatives with respect to the independent variable do not occur, so by using y as independent variable we avoid having to consider dw_x/dy, etc., in the norm N_2. If the trajectory has two branches this simple device cannot be used. However, by means of a more intricate analysis it can nevertheless be shown that the partial derivatives with respect to y can be disregarded in defining the norm N_2 without destroying the conclusion of the existence of the differential. We shall not attempt to reproduce this rather difficult proof here; it can be found in Vol. 17 of the Duke Mathematical Journal (1950), pp. 115-134 (E. J. McShane: "The Differentials of Certain Functions in Exterior Ballistics").

3. The equations of variations based on time.

In order that we can work with the equations (2.17) which are the "equations of variation," we need to know the specific expressions for the various partial derivatives occurring in the right member. Since the functions $f_1, \ldots, f_6$ are the right members of equations (2.1) when time is used as independent variable, it is clear that

$$(1) \qquad \partial f_1/\partial v_x = \partial f_2/\partial v_y = \partial f_3/\partial v_z = 1,$$

all the other partials of f_1, f_2 and f_3 being zero. The right members of the first three of equations (2.1) will not be affected by any change in wind, density, etc., so

$$\Delta f_1 = \Delta f_2 = \Delta f_3 = 0.$$

Thus the first three of equations (2.17) reduce to

$$(2) \quad d\eta_1/dt = \eta_4, \quad d\eta_2/dt = \eta_5, \quad d\eta_3/dt = \eta_6.$$

We shall introduce the new notation

$$(3) \qquad \xi = \eta_1, \quad \eta = \eta_2, \quad \zeta = \eta_3.$$

Then by (2) we have

$$(4) \qquad \dot{\xi} = \eta_4, \quad \dot{\eta} = \eta_5, \quad \dot{\zeta} = \eta_6.$$

It will be assumed that the standard density law is

$$(5) \qquad H(y) = e^{-hy},$$

and that the standard temperature (or sound velocity) law is

$$(6) \qquad a(y) = e^{-a_1 y}.$$

Along the normal trajectory we have $\dot{x} = v_x$, etc., and by (VI.9.8) we find that

(7) $$\partial E/\partial y = [-h + (n - 2)a_1]E,$$

wherein E is to be evaluated along the normal trajectory and n is the Mayevski n (see (VI.9.6)) evaluated for the argument v/a(y). From the identity

$$v^2 = v_x^2 + v_y^2 + v_z^2$$

we obtain

$$\partial v/\partial v_x = v_x/v,$$

and from this and (VI.9.9)

(8) $$\partial E/\partial v_x = (n - 1)(v_x/v^2)E,$$

with two analogous equations for the partial derivatives of E with respect to v_y and v_z (the latter being identically zero along a normal trajectory). We can now compute the first-order partial derivatives of the three functions f_4, f_5, f_6 with respect to the seven variables t, x, ..., v_z. The only ones which are not identically zero are

(9) $$\begin{aligned} \partial f_4/\partial y &= [h - (n - 2)a_1]E\dot{x}, \\ \partial f_5/\partial y &= [h - (n - 2)a_1]E\dot{y}, \\ \partial f_4/\partial v_x &= -[1 + (n - 1)(\dot{x}^2/v^2)]E, \\ \partial f_4/\partial v_y = \partial f_5/\partial v_x &= -(n - 1)(\dot{x}\dot{y}/v^2)E, \\ \partial f_5/\partial v_y &= -[1 + (n - 1)(\dot{y}^2/v^2)]E, \\ \partial f_6/\partial v_z &= -E. \end{aligned}$$

With the help of equations (9), the first term in the right member of (2.17) can be written explicitly. As mentioned at the end of Section 1, instead of the Δf_i themselves we shall use the linear approximations defined as in (2.29). These we shall denote by e_1, ..., e_6 respectively. Since the first three equations (2.1) are not affected by any of the departures from standard conditions, we have at once

(10) $$e_1 = e_2 = e_3 = 0.$$

The equations of variation are now

$$d\xi/dt = \dot{\xi},$$

$$d\eta/dt = \dot{\eta},$$

$$d\zeta/dt = \dot{\zeta},$$

$$\begin{aligned} d\dot{\xi}/dt = {} & [h - (n - 2)\, a_1]E\dot{x}\,\eta \\ & - [1 + (n - 1)(\dot{x}^2/v^2)]E\dot{\xi} \\ & - (n - 1)(\dot{x}\dot{y}/v^2)E\dot{\eta} + e_4, \end{aligned} \qquad (11)$$

$$\begin{aligned} d\dot{\eta}/dt = {} & [h - (n - 2)\, a_1]E\dot{y}\,\eta \\ & - (n - 1)(\dot{x}\dot{y}/v^2)E\dot{\xi} \\ & - [1 + (n - 1)(\dot{y}^2/v^2)]E\dot{\eta} + e_5, \end{aligned}$$

$$d\dot{\zeta}/dt = - E\dot{\zeta} + e_6.$$

The forms of the terms e_4, e_5, e_6 will now be computed for a disturbance which includes all the types of disturbance already mentioned and also allows for a change in the drag function G and a change in the reciprocal ballistic coefficient γ. The last mentioned is of rather secondary importance. But the change in drag function could be of great utility if, for example, a large ballistic table had been completed, and some time after its completion it was found that new projectiles had to be considered whose drag function differed more than trivially from that used in the preparation of the tables. The method of differential corrections would then permit us to prepare a small table of amendments to the large table, at much less cost in time and labor than would be required to prepare a complete new set of ballistic tables with the new drag function.

To fix the notation, we assume the following departures from standard conditions.

(12) The reciprocal ballistic coefficient for the disturbed trajectory exceeds by an amount $\Delta\gamma$ the reciprocal ballistic coefficient γ along the normal trajectory.

(13) At each value of v, the drag function along the disturbed trajectory exceeds by an amount $\Delta_0 G(v)$ the drag function $G(v)$ used in computing the normal trajectory.

(14) At altitude y, the density is $[1 + \kappa(y)]$ times the standard density at altitude y.

(15) At altitude y, the absolute temperature exceeds by an amount $\Delta\Theta$ the standard absolute temperature Θ by altitude y.

(16) At altitude y, there is a horizontal wind with components $w_x(y)$ and $w_z(y)$ along the x- and z-axes respectively.

(17) When the projectile has position (x, y, z) and velocity components (v_x, v_y, v_z), it is acted upon by a force, other than drag and standard gravity, which produces an acceleration (a_x, a_y, a_z).

From (15), it follows that on the disturbed trajectory, at altitude y the velocity of sound is

$$\sqrt{1 + (\Delta\Theta/\Theta)}$$

times the standard velocity, since the velocity of sound in air is proportional to the square root of the absolute temperature. To linear terms, therefore, the new relative velocity of sound is $[1 + (\Delta\Theta/2\Theta)]$ times the standard value, a(y).

The quantities e_4, e_5, e_6 have been defined to be the linear approximations to the Δf_i defined by (2.29).

These are obtained by computing the derivatives of the right members of the fourth, fifth and sixth of equations (2.1) with respect to γ, G, H, a, w_x, w_z, a_x, a_y, a_z, multiplying these partial derivatives by the respective changes $\Delta\gamma$, $\Delta_0 G$, $H\kappa(y)$, $\frac{1}{2}a\Delta\Theta/\Theta$, w_x, w_z, a_x, a_y, a_z and summing. The results are

$$
\begin{aligned}
(18)\quad e_4 &= -E\dot{x}[\Delta\gamma/\gamma + \Delta_0 G/G + \kappa(y) + n\Delta\Theta/2\Theta] \\
&\quad + E[1 + (n-1)(\dot{x}^2/v^2)]w_x + a_x, \\
e_5 &= -E\dot{y}[\Delta\gamma/\gamma + \Delta_0 G/G + \kappa(y) + n\Delta\Theta/2\Theta] \\
&\quad + E[(n-1)\dot{x}\dot{y}/v^2]w_x + a_y, \\
e_6 &= Ew_z + a_z.
\end{aligned}
$$

Now we are in a position to compute the differential effect of any of the departures from standard conditions listed in (12) to (17). For when the quantities (18) are substituted in (11), the latter become a sixth-order system with all terms known, and the solutions with the initial values

$$
(19)\quad
\begin{aligned}
\dot{\xi}(0) &= \Delta v_x(0), \quad \dot{\eta}(0) = \Delta v_y(0), \quad \dot{\zeta}(0) = \Delta v_z(0), \\
\xi(0) &= \Delta x(0), \quad \eta(0) = \Delta y(0), \quad \zeta(0) = \Delta z(0)
\end{aligned}
$$

constitute the differential effects sought. For example, if a range wind w_x is the departure from standard conditions whose effect is being computed, then

$$
(20)\quad \dot{\xi}(T) = dv_x(w_x \rfloor T), \; \ldots, \; \zeta(T) = dz(w_x \rfloor T).
$$

The process of solving the equations (11) is not a totally impracticable undertaking, although it is not to be recommended in view of better techniques for handling equations (11) which will be exhibited in the next chapter. It is true that (11) is a sixth-order system, but it need not be handled as such. To begin with, the sixth is directly solvable by quadrature and the third may next be solved, also by quadrature. Next, the second, fourth and fifth equations form a third-order system. After this is solved, the solution of the first equation requires merely a quadrature.

Ordinarily we are much less interested in the differential effects of disturbances computed at points of equal time than we are in the differential effects at points of equal ordinate y. But this conversion is simple. It can be made by means of equation (VII.1.8), which remains valid for the differential effects of functionals. In fact, the proof of that equation was so arranged that it applies to the present situation with no change other than replacing the symbol N by N_2. We are interested in the special case of (VII.1.8) in which B is y and A is t. Then for any disturbance q and any variable C we have

(21) $$dC(q|y) = dC(q|t) - (dC/dy)\, dy(q|t).$$

If we let C be replaced successively by x, z, t, v_x, v_y, v_z and substitute in (21) the appropriate derivatives of these quantities with respect to y, (21) yields the six equations

(22)
$$\begin{aligned}
dx(q|y) &= dx(q|t) - \cot\theta\, dy(q|t),\\
dz(q|y) &= dz(q|t),\\
dt(q|y) &= -(1/v_y)\, dy(q|t),\\
dv_x(q|y) &= dv_x(q|t) + E\cot\theta\, dy(q|t),\\
dv_y(q|y) &= dv_y(q|t) + E(1 + g/v_y)\, dy(q|t),\\
dv_z(q|y) &= dv_z(q|t).
\end{aligned}$$

Since the only solutions of a set of homogeneous linear differential equations having initial values 0 is the system of functions all identically zero, we see readily that

(23) The differential effects on deflection z due to disturbances $\Delta x(0)$, $\Delta y(0)$, $\Delta v_x(0)$, $\Delta v_y(0)$, $\Delta\Theta$, ΔH, $\Delta_0 G$, w_x, a_x and a_y are all zero.

(24) The differential effects on range, time of flight and v_x and v_y at impact due to the disturbances $\Delta z(0)$, $\Delta v_z(0)$, w_z and a_z are all zero.

Thus our problem may be split into two parts capable of separate discussion. The simpler of these problems is the computation of effects on z and v_z produced by disturbances $\Delta z(0)$, $\Delta v_z(0)$, w_z and a_z. The more difficult problem is the evaluation of the differential effects on range, time of flight, and x- and y-components of striking velocity, caused by the disturbances $\Delta x(0)$, $\Delta y(0)$, $\Delta v_x(0)$, $\Delta v_y(0)$, $\Delta \Theta$, ΔH, $\Delta {}_0G$, w_x, a_x and a_y.

4. The equations of variation based on slope.

The traditional treatment of differential variations has long been that of the preceding section, in which points are matched which correspond to equal values of time. This is of course unnecessary; for example, in connection with his proposal to use x as independent variable in trajectory computations, Dr. L. S. Dederick worked out the form of the equations of variation when points on normal and disturbed trajectory were matched when they had equal values of the x-coordinate. In fact, along with each choice of independent variable in the computation there enters the natural choice of this same variable as matching variable. It will now be shown that there is considerable computational advantage in matching points of equal slope.

The equations for effects on z separate from the other equations, and we could hardly hope to find anything simpler than the third and sixth of equations (3.11). So henceforth we put aside the equations for effects on z, and restrict our attention to the equations involving the other coordinates. In terms of slope as independent variable, these are the first four of equations (V.1.9). It is notationally convenient to introduce a new symbol r for the x-component of velocity,

(1) $$r = v_x,$$

and it is advantageous to write the four equations in the reversed order:

$$
(2)\quad
\begin{aligned}
dr/dm &= (Er - Ew_x - a_x)r/(g - a_y + ma_x + Emw_x),\\
dy/dm &= -\,mr^2/(g - a_y + ma_x + Emw_x),\\
dx/dm &= -\,r^2/(g - a_y + ma_x + Emw_x),\\
dt/dm &= -\,r/(g - a_y + ma_x + Emw_x).
\end{aligned}
$$

Here

$$
(3)\quad
\begin{aligned}
E &= \gamma HaG(u/a)\\
&= \gamma HaG(\sqrt{(r - w_x)^2 + (mr)^2}/a).
\end{aligned}
$$

The independent variable m takes the place of the independent variable t of Section 2, and the right members of equations (2) constitute the four functions $f_1, \ldots, f_4$ of the variables m, r, y, x, t when the disturbances are all set equal to zero. In the right members of equations (2.17) there occur the partial derivatives of the four functions f_i with respect to the four variables y_i, which in the present notation are r, y, x and t. Of these sixteen partial derivatives, eleven are identically zero; the others are

$$
(4)\quad
\begin{aligned}
\partial f_1/\partial r &= r(n + 1)E/g,\\
\partial f_1/\partial y &= [-\,h + (n - 2)a_1]Er^2/g,\\
\partial f_2/\partial r &= -\,2mr/g,\\
\partial f_3/\partial r &= -\,2r/g,\\
\partial f_4/\partial r &= -\,1/g.
\end{aligned}
$$

These equations permit us to write out the first term in the right member of (2.17) explicitly. As before, instead of the Δf_i the second term in the right member will be replaced by the linear parts of the expansions of the Δf_i, which will be denoted by ϵ_i, $i = 1, \ldots, 4$.

Thus equations (2.17) take the form

$$
\begin{aligned}
(5)\quad & d\xi_1/dm = [r(n+1)E/g]\xi_1 \\
& \qquad + [-h + (n-2)a_1 E r^2/g]\xi_2 + \varepsilon_1, \\
& d\xi_2/dm = -(2mr/g)\xi_1 + \varepsilon_2, \\
& d\xi_3/dm = -(2r/g)\xi_1 + \varepsilon_3, \\
& d\xi_4/dm = -(1/g)\xi_1 + \varepsilon_4.
\end{aligned}
$$

If we consider the same set of disturbances as were already listed in (3.12 to 17) and expand the right members of equations (2) to linear terms in these disturbances, for fixed values of m, r and y, we obtain

$$
\begin{aligned}
(6)\quad & \varepsilon_1 = (Er/g)[r\Delta\Upsilon/\Upsilon + r\kappa \\
& \qquad - r\Delta\Theta(n-2)/2\Theta + r\Delta_0 G/G] \\
& \qquad - (Er/g)[(n-1)r^2/v^2 + 1 + Erm/g]\,w_x \\
& \qquad - (r/g)(1 + Erm/g)a_x + (Er^2/g^2)a_y, \\
& \varepsilon_2 = (E\,m^2r^2/g^2)w_x + (m^2r^2/g^2)a_x - (mr^2/g^2)a_y, \\
& \varepsilon_3 = (Emr^2/g^2)w_x + (mr^2/g^2)a_x - (r^2/g^2)a_y, \\
& \varepsilon_4 = (Emr/g^2)w_x + (mr/g^2)a_x - (r/g^2)a_y.
\end{aligned}
$$

If we let q stand for any disturbance, consisting either of one of those listed in (3.12 to 17) or else of a change in initial conditions, and find the corresponding ε_i by (6) and then solve equations (5), the solutions will represent the differential effects of the disturbance q on the variables $r = v_x$, y, x, t, respectively, where corresponding points of the disturbed and undisturbed trajectories are understood to be points of equal slope. Thus

$$
\begin{aligned}
(7)\quad & \xi_1(m) = dv_x(q|m), \; \xi_2(m) = dy(q|m), \\
& \xi_3(m) = dx(q|m), \; \xi_4(m) = dt(q|m).
\end{aligned}
$$

However, the process of solving equations (5) is seriously hampered by the fact that in them the slope is the independent variable, while the trajectories are ordinarily computed with time as independent variable. To extricate ourselves from this difficulty we need only introduce t as independent variable in the $\xi_i(m)$, thus defining four new functions:

$$(8)\qquad \begin{aligned} \rho(t) &= \xi_1(m(t)), \quad \eta(t) = \xi_2(m(t)), \\ \xi(t) &= \xi_3(m(t)), \quad \tau(t) = \xi_4(m(t)). \end{aligned}$$

(The letters ξ and η have been used in the preceding section, but with entirely different meanings.) It should be emphasized that this is a completely different procedure from matching points at which times are equal. By (7) and (8),

$$(9)\qquad \begin{aligned} \rho(t) &= dv_x(q|m = m(t)), \quad \eta(t) = dy(q|m = m(t)), \\ \xi(t) &= dx(q|m = m(t)), \quad \tau(t) = dt(q|m = m(t)). \end{aligned}$$

In the coefficients in the right members of (5), the functions are computed along the normal trajectory. Hence, in particular, $dt/dm = -r/g$. It follows that

$$(10)\quad (d\rho/dt) = (d\xi_1/dm)(dm/dt) = -(g/r)(d\xi_1/dm),$$

with like equations for the derivatives of the other functions defined in (8). Accordingly, from (5) and (6) we deduce

$$(11)\qquad \begin{aligned} d\rho/dt &= -(n+1)E\rho \\ &\qquad + [h - (n-2)a_1] E\dot{x}\eta + e_1, \\ d\eta/dt &= 2\dot{y}\rho/\dot{x} + e_2, \\ d\xi/dt &= 2\rho + e_3, \\ d\tau/dt &= (1/\dot{x})\rho + e_4, \end{aligned}$$

where the e_i (different from those of the preceding section) are defined by the equations

$$e_1 = -E\dot{x}\Delta\gamma/\gamma - E\dot{x}\kappa + E\dot{x}(n-2)\Delta\Theta/2\Theta$$

$$- E\dot{x}\Delta_0 G/G$$

$$+ E[(n-1)\dot{x}^2/v^2 + 1 + E\dot{y}/g]w_x$$

(12) $$+ (1 + E\dot{y}/g)a_x - E\dot{x}/g)a_y,$$

$$e_2 = -(E\dot{y}^2/g\dot{x})w_x - (\dot{y}^2/g\dot{x})a_x + (\dot{y}/g)a_y,$$

$$e_3 = -(E\dot{y}/g)w_x - (\dot{y}/g)a_x + (\dot{x}/g)a_y,$$

$$e_4 = -(E\dot{y}/g\dot{x})w_x - (\dot{y}/g\dot{x})a_x + (1/g)a_y.$$

There are two common ways of stating the initial conditions for a trajectory. One is in terms of position and velocity components at time $t = t_o$; the other is in terms of position, initial velocity and angle of departure. Corresponding to each of these systems we shall derive the expression for the initial values of the functions in (8).

(13) _If at time $t = t_o$ the position and velocity components of the projectile are x_o, y_o, v_{xo}, v_{yo} on the normal trajectory and $x_o + \Delta x_o$, $y_o + \Delta y_o$, $v_{xo} + \Delta v_{xo}$, $v_{yo} + \Delta v_{yo}$ on the disturbed trajectory, the corresponding initial values of the differential effects (8) are_

$$\rho(t_o) = \Delta v_{xo} - (E/g)(v_{xo}\Delta v_{yo} - v_{yo}\Delta v_{xo}),$$

$$\eta(t_o) = \Delta y_o + (v_{yo}/gv_{xo})(v_{xo}\Delta v_{yo} - v_{yo}\Delta v_{xo}),$$

$$\xi(t_o) = \Delta x_o + (1/g)(v_{xo}\Delta v_{yo} - v_{yo}\Delta v_{xo}),$$

$$\tau(t_o) = (1/gv_{xo})(v_{xo}\Delta v_{yo} - v_{yo}\Delta v_{xo}).$$

To prove this we observe that if q denotes the disturbance consisting of the changes

$$(\Delta x_o, \Delta y_o, \Delta v_{xo}, \Delta v_{yo}),$$

it is obvious that

$$dx(q|t = t_o) = \Delta x_o,$$
$$dy(q|t = t_o) = \Delta y_o,$$
$$dv_x(q|t = t_o) = \Delta v_{xo},$$

while from the equation

$$\Delta m = \frac{v_{yo} + \Delta v_{yo}}{v_{xo} + \Delta v_{xo}} - \frac{\Delta v_{yo}}{v_{xo}}$$

we deduce

$$dm(q|t = t_o) = (v_{xo}\Delta v_{yo} - v_{yo}\Delta v_{xo})/v_{xo}^2.$$

From this and (VII.1.8) we obtain

$$dC(q|m = m_o) = dC(q|t = t_o)$$
$$- (dC/dm)(v_{xo}\Delta v_{yo} - v_{yo}\Delta v_{xo})/v_{xo}^2.$$

In this we successively let C be r (= v_x), y, x and t; we thus find that the quantities

$$dr(q|m = m_o),\ dy(q|m = m_o),$$
$$dx(q|m = m_o),\ dt(q|m = m_o)$$

are the same as the right members of the equations in (13). But by (9) they are also the same as the left members of these equations, and (13) is proved.

The corresponding initial values, for the case in which the initial position (x_o, y_o), initial velocity v_o and angle of departure θ_o are given, can be deduced without trouble from (13). All that is needed is to make the substitutions of $\Delta v_o \cos\theta_o - \Delta\theta_o v_o \sin\theta_o$ for Δv_{xo} and $v_o^2\Delta\theta_o$ for $(v_{xo}\Delta v_{yo} - v_{yo}\Delta v_{xo})$ in (13).

Equations (9) do not make any explicit mention of v_y, but the differential correction to v_y is easily deduced from (9). Since $v_y = mv_x$, and the matching of points on the two trajectories is by equal values of m, the difference of values of v_y at matched points is m times the difference of values of v_x at the same points. Hence

(14) $$dv_y(q|m) = m\, dv_x(q|m) = m\rho(t).$$

The important application of the formulas is to the computation of differential effects at equal values of y. From (VII.1.8), if t is the time and m the slope at the point whose ordinate is y,

$$(15) \qquad dC(q|y) = dC(q|m) - (dC/dy)\ dy(q|m).$$

In this we replace C by x, t, v_x, v_y successively. With the help of (9) and (14), we obtain

$$(16) \qquad \begin{aligned} dx(q|y) &= \xi(t) - (\cot\theta)\ \eta(t), \\ dt(q|y) &= \tau(t) - (1/\dot{y})\ \eta(t), \\ dv_x(q|y) &= \rho(t) + (E\cot\theta)\ \eta(t), \\ dv_y(q|y) &= (\tan\theta)\ \rho(t) + (E + g/\dot{y})\ \eta(t). \end{aligned}$$

In the preceding section, in which time was used as the basis for matching points on the two trajectories, the fourth-order system consisting of the first, second, fourth and fifth of equations (3.11) was seen to subdivide into a third-order system, consisting of the second, fourth and fifth of the equations, followed by a quadrature to obtain the remaining function, $\xi(t)$. When slope is used to match points, as in this section, the splitting up of the problem is even more marked. The first and second of equations (11) can be solved together, as a second-order system, and afterwards the two remaining functions ξ and τ can be obtained by quadratures. Even the quadratures are easier than might be anticipated, since the right members of the last two of equations (11) are obtained without difficulty after the first pair have been solved. For in solving the first two we necessarily compute the right member of the second of equations (11), namely $2\dot{y}\rho/\dot{x} + e_2$. By (11) and (12), if we multiply this by $\dot{x}/\dot{y}$ we obtain the right member of the third of equations (11); while the right member of the last of (11) is the sum of $(1/2\dot{y})$ times the first term in the right member of the second equation and $(1/\dot{y})$ times the second term.

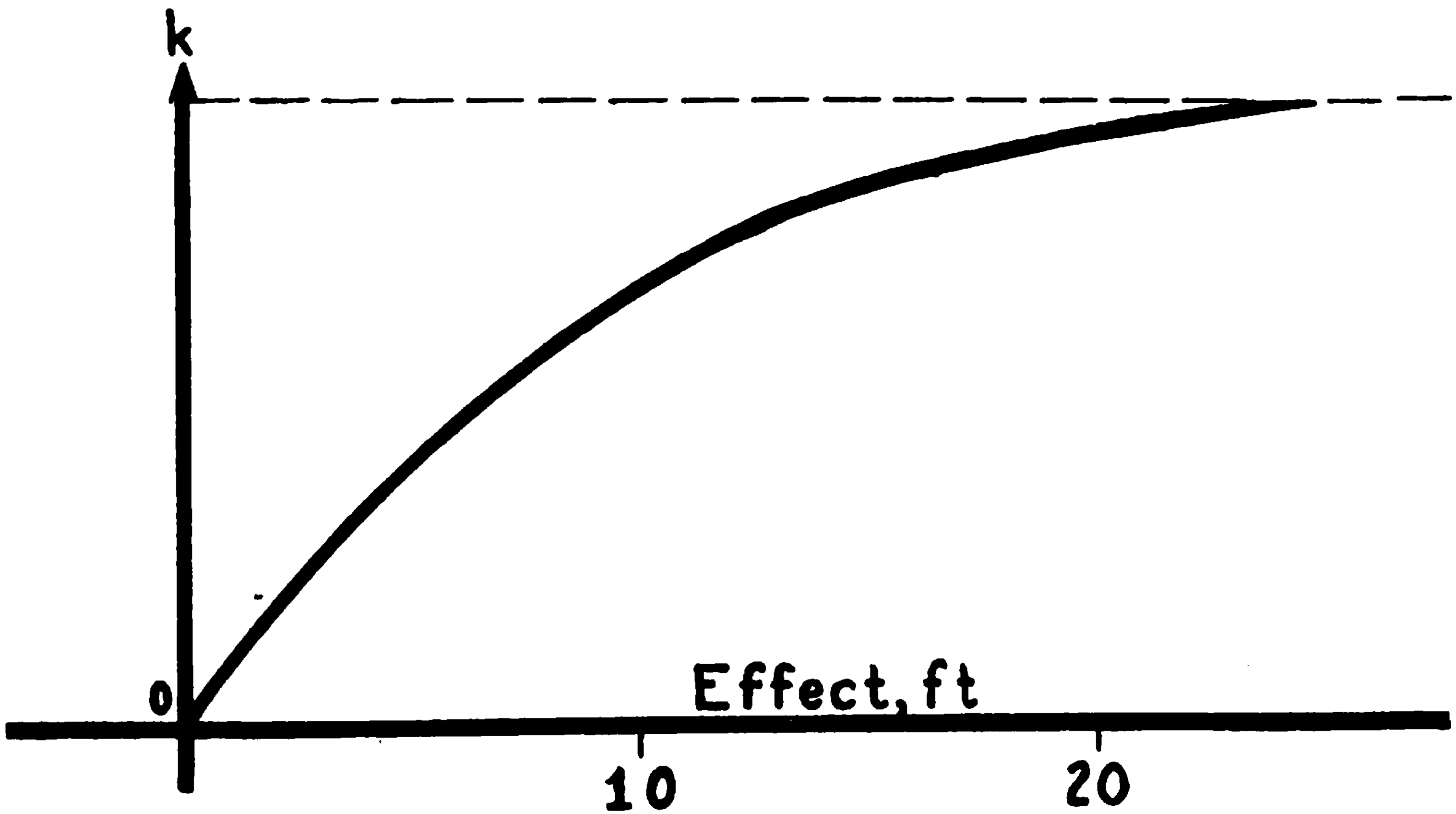


Figure VIII.5.1

5. Weighting factors.

It is evidently true that the two methods of treating differential effects already discussed are much too complex for use in the field, and that the problem is sufficiently difficult to leave us no hope that any method of solution of the equations can be found simple enough for use under the conditions of service. Therefore some method must be devised by which the bulk of the computations of differential effects can be carried out in some organization remote from the front, and the results of the computations recorded in such a way that the application to the conditions at any particular moment requires nothing but very elementary calculations.

To be specific, we shall suppose first that we are interested in the differential effects of range wind w_x on the range of a bomb having a certain reciprocal ballistic coefficient γ, launched horizontally with velocity v_0 from an altitude Y. For the time being we omit the subscript x from the w, leaving it for the reader to remember that we are considering a range wind. Let t^* be any time between time of release t_0 and time of impact T. We first imagine a wind which at all times before t^* (which is the same as saying at all levels above $y(t^*)$) is zero, and at all times after t^* (or at all levels below $y(t^*)$) is + 1. Either by one of the methods already explained or by one of the more convenient methods to be explained in the next chapter we compute the differential effect of this wind on the range; for the moment we denote it by the letter g. Now we plot the point whose ordinate is $k = y(t^*)/Y$ and whose abscissa is g. This process we repeat for a collection of different values of t^*. For each value of $k = y(t^*)/Y$ we obtain a value of g, and thus are able to draw the graph of the function g(k) for k ranging from 0 to 1. This gives us a curve such as is shown in Figure 1. We have followed the usual ballistic custom of plotting the values of the independent variable k as ordinates rather than as abscissas. This is meant as a reminder that k is a multiple of the altitude y. For the particular projectile we are considering, dropped from height Y at speed v_0, this graph contains all the information we need to compute the differential effect on range of any range wind w(y) which varies continuously with the altitude y. We shall show this in four steps. Suppose first that w(y) is 1 at all levels between $y_1 = k_1Y$ and $y_2 = k_2Y$ (we suppose the latter to be the greater) and that w(y) is zero at all other levels. We can think of w(y) as $w_2(y) - w_1(y)$, where $w_2(y)$ is 0 above y_2 and is + 1 below it, and $w_1(y)$ is 0 above y_1 and + 1 below it. Then the differential effect of w(y) is

$$dx(w|y) = dx(w_2|y) - dx(w_1|y) = g(k_2) - g(k_1),$$

by the definition of g(k). Suppose next that w(y) is equal to W between y_1 and y_2 and is zero elsewhere. This is W times the function first considered, so its differential effect is W times as great, namely, $W[g(k_2) - g(k_1)]$. Suppose next that the region of the atmosphere between levels 0 and Y can be split into a first zone, between levels 0 and $y_1 = k_1Y$, in which w(y) is constantly equal to w_1, a second zone between levels y_1 and $y_2 = k_2Y$ in which w(y) is constantly equal to w_2, and so on, up to an uppermost zone between levels $y_{n-1} = k_{n-1}Y$ and $y_n = Y$ in which w(y) is constantly equal to w_n. This function w(y) can be regarded as the sum of n functions each of which has one of the values w_1, etc., in the appropriate zone and is zero elsewhere. So the differential effect of this w(y) on the range is the sum of the effects of these individual functions, namely

$$dx(w|y) = w_1[g(k_1) - g(0)] + \dots + w_n[g(1) - g(k_{n-1})]. \tag{1}$$

Finally, by (2.10) we know that there is a constant A such that

$$|dx(w|y)| \leq A \max |w(y)|.$$

Now let $w(y)$ be any continuous function and ε any positive number. We can subdivide the atmosphere into zones between levels

$$0, y_1 = k_1 Y, \ldots, y_{n-1} = k_{n-1} Y, y_n = Y$$

so thin that in each zone the values of $w(y)$ vary by less than ε. Let w_i be the average of $w(y)$ at top and bottom of the i-th zone

$$w_i = [w(y_i) + w(y_{i-1})]/2.$$

The function $w^*(y)$ which in each zone is constant, having the value w_i in the i-th zone, cannot differ from $w(y)$ by as much as ε at any point. Therefore

(2) $$|dx(w|y) - dx(w^*|y)| = |dx(w - w^*|y)| < A\varepsilon.$$

That is, the differential effect $dx(w|y)$ is approximated by the sum (1) with an error less than $A\varepsilon$, which is as small as we choose. Our statement is now established; we can use the graph of the function $g(k)$ to find the differential effect on range produced by any range wind $w(y)$ which is a continuous function of the altitude y.

We have shown that if $w(y)$ is continuous, the differential effect $dx(w|y)$ is the limit of sums of the type of the right member of (1), as the intervals $k_{i-1} \leq k \leq k_i$ all approach zero in length. But the limit of such a sum is well known in analysis; it is the Stieltjes integral, which we have already met in (I.16.7, 8). In the notation of the Stieltjes integral, the results of the preceding paragraph can be summarized in the equation

(3) $$dx(w|y) = \int_0^1 w(kY)\, dg(k).$$

But if the reader prefers to avoid the idea of the Stieltjes integral, he need only remember that (3) is merely a condensed form of the statement that $dx(w|y)$ can be approximated as closely as desired by sums like the right member of (1).

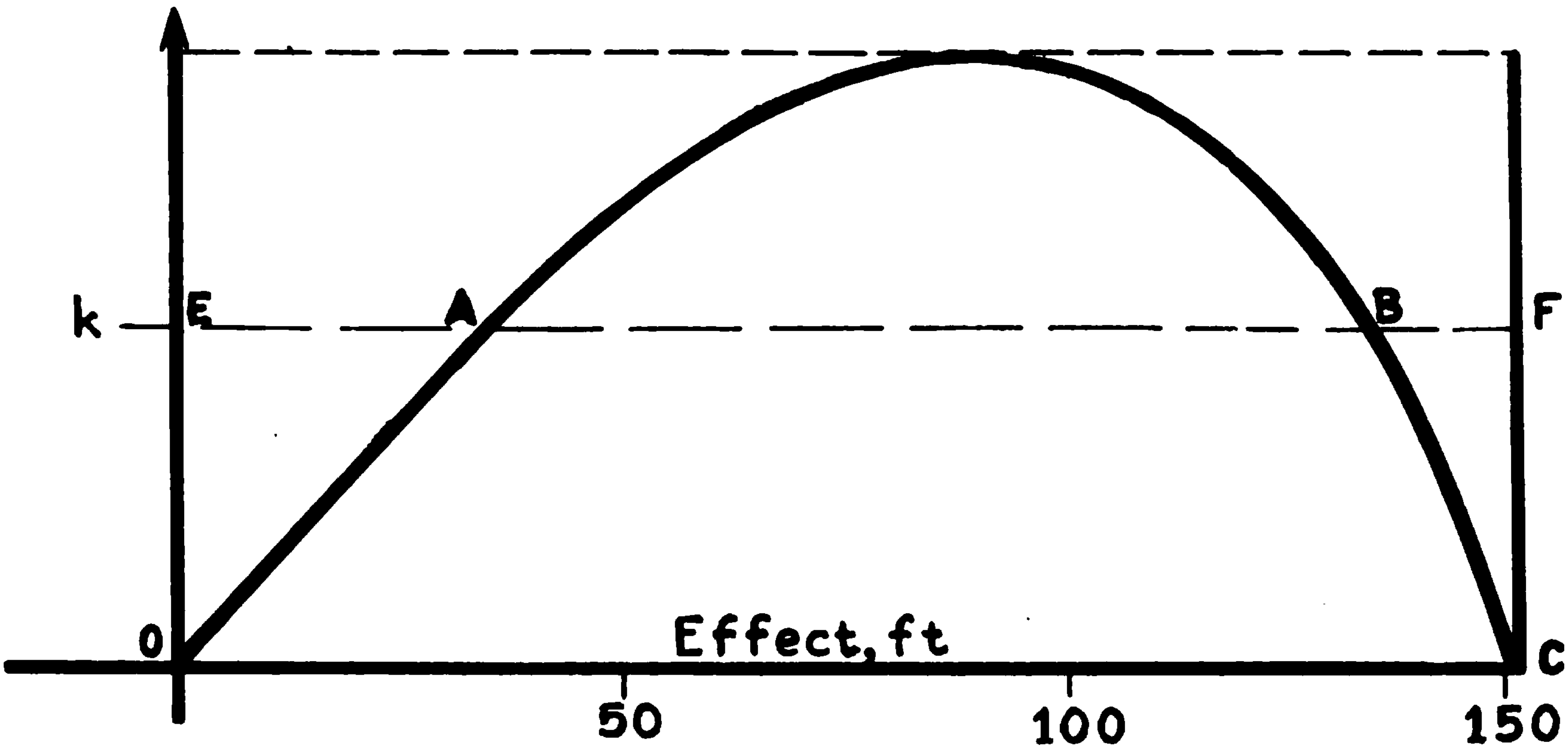


Figure VIII.5.2

Since we shall wish to refer to the graphs of $g(k)$ and its analogues for the differential effects of other disturbances, we shall give these curves the name of "effect curves." This is not a standard name. In fact, there is no standard nomenclature, because the functions $g(k)$ ordinarily appear only as intermediate stages in the computation of the "weighting factor curves" which we shall shortly define.

Computing the effect curves for artillery projectiles is complicated slightly by the fact that to each altitude y between that of the summit and that of muzzle and target there are two values of t, one on the ascending branch and one on the descending branch of the trajectory. We shall repeat the process described in the second paragraph of this section; for each t^* between t_o and T we compute the differential effect of a wind which is 0 up to time t^* and is + 1 thereafter. Temporarily we use the symbol $h(t^*)$ to denote this differential effect; and we use Y to stand for the altitude of the summit of the trajectory. Now we plot the point whose abscissa is $h(t^*)$ and whose ordinate is $y(t^*)/Y$. This process is repeated for a collection of values of t^* between t_o and T. The collection of plotted points enables us to draw the graph of a function, such as is shown in Figure 2. This is the "two-branched effect curve." Now suppose that we wish to find the differential effect $g(k)$ of a wind which is equal to 0 at all levels above kY and is + 1 at all levels below kY. The projectile will first reach height kY at a certain time t' and will again pass through this same height on the descending branch at a time t''. So w will be + 1 between times 0 and t' and between times t'' and T, and will be zero at other times. Thus w can be thought of as the superposition of three range winds, the first being + 1 between 0 and T, the second being - 1 between times t' and T and zero at other times, and the third being + 1 at times between t'' and T and zero at other times. It follows that its differential effect $g(k)$ is

$$h(0) - h(t') + h(t'').$$

This is the same as [abscissa of C - abscissa of B + abscissa of A] in Figure 2; it is also the same as OC - AB. The function g(k) being thus determined, its graph can be plotted. The result is a curve similar in shape to that in Figure 1. A similar process can be applied to the "two-branched curve" for the effects of any disturbance, yielding an effect curve from which in turn the differential effects of any continuous disturbance can be computed.

However, along with their advantages the effect curves have a disadvantage which amounts to a serious inconvenience to the ballistician and is even more serious from the point of view of the artillery officer. The disadvantage is that under change of altitude, initial velocity or ballistic coefficient, the effect curves change greatly. For example, for large bombs dropped at a given speed from a given altitude the effect of density on range will be roughly proportional to γ. From the ballistician's point of view, this makes it hard to interpolate between given curves in order to find the effect curve for some bomb whose ballistic coefficient or altitude of release or initial velocity happens to differ from those for which effect curves have been tabulated. Fortunately, however, it is true that over fairly wide intervals of values of γ, v_0 and Y the effect curves of any one type (such as curves for effect of density on range) differ chiefly in scale, so that for any two effect curves g(k) and $g_1(k)$ it will be true that a properly chosen constant multiple of g(k) will be a good approximation to $g_1(k)$. We thus have the following empirical statement, whose truth can not readily be seen by inspection of the equations of variation, but is apparent from an inspection of a collection of the effect curves once they have been computed: given a collection of effect curves all of the same type (such as effect of wind on range), corresponding to various values of γ, v_0 and height of summit Y, we can divide each g(k) by a properly chosen number, depending on γ, v_0 and Y, so that the quotient is a slowly varying function of γ, v_0 and Y.

The number referred to in the preceding sentence depends on γ, v_0 and θ_0, and also depends on the type of disturbance q and on the quantity (x, t, v_y or v_x) whose differential change is being computed. So it properly should be denoted by some symbol such as $N[q, x, \gamma, v_0, \theta_0]$, the first symbol in the brackets showing the nature of the disturbance and the second the quantity whose change is being computed. However, this may safely be abbreviated to N[q, x] or even to N, leaving it to the reader to remember the variables on which it depends.

Before we discuss the choice of the numbers N, we digress to show the usefulness of the empirical property just stated, first in the ballistic laboratory, and second in the field. The ballistician who needs to compute differential effects, say for the reduction of range firing experiments, will prepare tables of N[q, x] and N[q, t] for an assortment of values of γ, v_0 and Y or θ_0. Also, for several values of these same variables he will prepare the effect curves; but these are now merely intermediaries to be used in computing the functions

$$p(k) = g(k)/N. \tag{4}$$

These will also depend on γ, v_0 and θ_0 or Y, as well as on the nature of the disturbance and the quantity whose change is being found. But if the N[q, x] and N[q, t] have been well chosen, the p(k) will vary only slowly with change in γ, v_0 and θ_0 or Y. Hence if the projectile which is being range-fired has its γ, v_0 and θ_0 different from those tabulated, the corresponding p(k) can easily be found by interpolation. To be specific, let us suppose that we are investigating the differential effect of wind on range. The region of the atmosphere between 0 and Y is cut into zones in each of which the wind departs only slightly from its average value in the zone. Let the zone boundaries be 0, k_1Y, k_2Y, ..., $k_{n-1}Y$, Y, the average wind in the i-th zone being w_i.

We form the sum

$$(5) \qquad w_1[p(k_1) - p(0)] + w_2[p(k_2) - p(k_1)] + \ldots + w_n[p(1) - p(k_{n-1})].$$

By (4), this is 1/N times the right member of (1), and therefore is a close approximation to $dx(w|y)/N$. Therefore all that remains is to look in the tables for the N that corresponds to the γ, v_0 and θ_0 (or Y) of the projectile and multiply this N by the quantity (5); the product will be the differential effect $dx(w|y)$ sought.

In the field the importance of the empirical property is much more marked. Within a single division there will be a great assortment of artillery pieces, firing at targets at various ranges. It would be too much of an undertaking to attempt to compute (5) for each piece separately. Obviously the sum (5) must be computed for several altitudes Y, since the wind varies greatly with altitude. It is no loss to use with each Y a curve p(k) proper for that Y. But already this means several computations. If in addition it were necessary to use several different p(k) for each altitude the time of computation might be prohibitive. The ideal situation, from the point of view of the service whose duty is to compute the sums (5), (in the U. S. Army, this is the Signal Corps) would be to have a single p(k) for each altitude Y, independent of γ and v_0, independent too of the fact that the different projectiles have different drag functions G(v). This is a bit too much to ask. Nevertheless, it has been found practicable to use a total of three curves p(k) for each altitude. The artillery pieces are classified into three large classes; the artillery officer in charge of a piece notes only the meteorological message giving the sum (5) for the class of pieces to which his belongs. This message will give the sum (5) at a series of different summital altitudes Y. Now if the piece is to fire at a given target, from the firing table the artillery officer finds the summital altitude Y, and from the meteorological message he reads the particular sum (5) for that Y. Next he has to find N. For the ballistician, this N is a function of Υ, v_0 and θ_0. But for the artillery officer two of these variables are fixed; the piece will have a certain specified muzzle velocity (or a specified set for different zones of fire if it is a mortar) and the Υ is determined by the projectile being fired. So N is a function of θ_0 alone, which has been tabulated and printed in the firing table. The artillery officer multiplies this N from his firing table by the sum (5) from the meteorological message, and the product is the differential effect of the range wind. (This is an intentional oversimplification; the Signal Corps has to treat the w_i in (5) as vectors and report the direction as well as the magnitude of the sum, and the artillery officer has to find the component of this vector sum along his direction of fire, and also across the line of fire in order to make the correction for cross wind. But this does not alter the essentials of the situation.) Thus the fact that for a given Y the functions p(k) are not sensitive to changes in v_0 and Υ has the operationally important consequence that three meteorological messages convey the essential information about winds at all altitudes, for all pieces, instead of there having to be a separate message for each type of piece.

To return now to the method of choosing the multipliers N: it is essential that they be chosen so that the resulting functions p(k) lie close together; it is desirable that they should have some rational type of definition in terms of the "effect curves" g(k). Each type of disturbance is expressed in terms of a unit consistent with the units of length, etc.; for example, the natural unit for w_x is one unit of length per unit of time, the natural unit for departure from standard density $\Delta H/H$ or κ is the pure number 1, and likewise for departure from standard absolute temperature $\Delta \Theta / \Theta$. These may not be the most convenient in practice, and we may wish to change them later, but they are appropriate for the present discussion. We suppose that these natural unit disturbances were used in computing the effect curves, as in fact was specifically mentioned in the example of the range wind. Then, by the definition of the effect curves, if the amount of a disturbance is one unit at all levels, the differential effect of the disturbance is g(1). This is given the self-suggesting name of "unit effect." It is customary in ballistics to choose this unit effect as the number N of our previous discussion, so that the function p(k) of equation (4) is

$$p(k) = g(k)/g(1). \tag{6}$$

The graph of this function p(k) is called the "weighting factor curve" for the differential effect of the disturbance on the quantity x, t, v_x or v_y as the case may be. It is obvious that

$$p(0) = 0, \; p(1) = 1.. \tag{7}$$

These definitions clearly have the quality spoken of as desirable; they are rationally and simply defined in terms of the effect curves. It cannot be seen without examination of a collection of weighting factor curves, but is nevertheless true, that for the important instances of differential effects of range wind and density on range and on time of flight, and of cross wind on deflection, they also satisfy the essential requirement that the weighting factor curves lie close together. By (7) we see that each weighting factor curve reaches from (0, 0) to (1, 1), so they all have two points in common. Moreover, the specific instances mentioned all have monotonic increasing weighting functions p(k), increasing steadily from the value 0 at k = 0 to the value 1 at k = 1, except for the differential effect of density on time of flight; for bombs of low ballistic coefficient this has a perceptible reversal near k = 1, p(k) being a little greater than 1 for k slightly below 1. This would lead to the expectation, verified by investigation, that the weighting factor curves for a given type of disturbance at a given Y resemble each other closely.

Suppose again that we are interested in the differential effect of a range wind on the range. Let $w(y)$ be the range wind at altitude y. By definition, the "ballistic wind" (which more properly should be called the ballistic mean wind, or range-mean range wind, but is not so named) is that constant wind $\bar{\bar{w}}$ which has the same differential effect on range as the actual wind $w(y)$ has. By (3) and (4), the differential effect of $w(y)$ is

$$(8) \qquad dx(w|y) = N \int_0^1 w(kY)\, dp(k),$$

while the differential effect of $\bar{\bar{w}}$ is

$$(9) \qquad dx(\bar{\bar{w}}|y) = N \int_0^1 \bar{\bar{w}}\, dp(k).$$

This last integral is computed as the limit of sums such as (5) with all w_i equal to $\bar{\bar{w}}$. But by (7) these sums are always equal to $\bar{\bar{w}}$, however we choose the points k_i, so

$$(10) \qquad dx(\bar{\bar{w}}|y) = N\bar{\bar{w}}.$$

By definition of $\bar{\bar{w}}$ the left members of (8) and (10) are equal, so

$$(11) \qquad \bar{\bar{w}} = \int_0^1 w(kY)\, dp(k).$$

In other words, the quantity whose approximate value (5) is computed by the Signal Corps for artillery use is the same as what we have just named the ballistic wind.

In a similar manner we define a quantity which we shall call the ballistic density-excess, but which properly should have some more descriptive name, such as range-mean excess density ratio. This is the constant value of the ratio $\Delta H/H$ which would have the same differential effect on range as the actual ratio $\Delta H/H$, which depends on y. A discussion like the preceding applies in this case too. The ballistic density-excess is given by the formula

$$(12) \qquad \overline{\Delta H/H} = \int_0^1 [\Delta H(kY)/H(kY)]\, dp(k),$$

where now the p(k) is the weighting function for the differential effect of density on range. The integral in the right member of this equation is in practice replaced by its approximate value computed by a formula like (5), with $\Delta H/H$ in place of w and the p(k) being the weighting function for effect of density on range.

However, when we try to apply a similar procedure to the differential effects of departure from standard temperature, and also to several other differential effects of less importance, we encounter a serious difficulty. The effect curves for differential effect of temperature on range, to select the most important example, are not the graphs of monotonic functions. As k rises from 0 to 1, g(k) may increase up to a certain value of k and then decrease; or it may increase a while, then decrease for a while, and finally increase again. This is so marked a phenomenon that even the sign of g(1) may be different for different values of Υ or Y or v_0. In particular, when g(1) happens to be near 0 the function p(k) defined by (6) may have exceedingly large values for some values of k. So choosing N = g(1) fails in the primary purpose; the resulting curves p(k) do not lie close together, but instead are spread farther apart. The corrections for differential effects of temperature have consequently been decidedly more troublesome than those for effects of wind and departure from standard density.

To remedy this difficulty, we suggest a somewhat different approach which succeeds in bringing together the curves p(k) for each type of disturbance and at the same time in leaving the curves p(k) unchanged for the important instances of effects of range wind on range, effect of density on range and effect of cross wind on deflection. Recall that the unit effect for any type of disturbance was defined to be the differential effect of a disturbance whose value at all levels was equal to 1. We shall replace this by a quantity which we shall name the <u>norm effect</u>. This we define to be the maximum differential effect of a disturbance which at all levels is between - 1 and + 1 inclusive. It is this quantity which we shall use as the N of equation (4).

First we must show how this norm effect can be found from the effect curves. Let $q(y)$ be any kind of disturbance, and let $g(k)$ be the corresponding effect curve, say for the effect of q on range. We wish to find the maximum value N of the integral

$$\int_0^1 q(kY)\, dg(k), \tag{13}$$

subject to the restriction

$$-1 \leq q(y) \leq +1 \tag{14}$$

for all y. Let k' and k" (> k') be two numbers between 0 and 1. The part of the integral (13) between limits k' and k" has the approximate value

$$\begin{aligned} \int_{k'}^{k''} q(kY)\, dg(k) \doteq\ & q_1[g(k_1) - g(k')] \\ & + q_2[g(k_2) - g(k_1)] \\ & + \dots + q_n[g(k'') - g(k_{n-1})], \end{aligned} \tag{15}$$

where

$$k' < k_1 < k_2 \quad \dots < k_{n-1} < k''$$

and the q_i are average values of the function $q(y)$ on the interval

$$k_{i-1}Y < y < k_iY.$$

If $g(k)$ is an increasing function of k on this interval, the number in each square bracket is positive, and subject to the restriction (14) the maximum value of the sum in (15) is obtained by putting all the q_i equal to $+1$. The sum on the right is then $g(k'') - g(k')$. If the function $g(k)$ is a decreasing function, the number in each square bracket is negative, and the greatest value the sum can have subject to (14) is reached by putting all the q_i equal to -1. The sum on the right is then $g(k') - g(k'')$. Both cases are covered by the statement that if $g(k)$ is monotonic (increasing or decreasing) between k' and k'', the maximum value of the integral (15) is

$$(16) \qquad |g(k'') - g(k')|.$$

Suppose now that $g(k)$ is not monotonic, but that the interval from 0 to 1 can be split into several parts on each one of which $g(k)$ is monotonic. To be specific, we suppose that $g(k)$ increases from 0 to k', decreases from k' to k'', and increases again from k'' to 1. Then

$$(17) \quad \int_0^1 q(kY)\, dg(k) = \int_0^{k'} + \int_{k'}^{k''} + \int_{k''}^1 q(kY)\, dg(k).$$

The integral in the left member reaches its greatest value subject to (14) when each of the three integrals in the right member reaches its greatest value. But as we have just shown, this means that the greatest value of the integral is

$$(18) \quad \begin{aligned} N = {} & |g(k') - g(0)| \\ & + |g(k'') - g(k')| + |g(1) - g(k'')|. \end{aligned}$$

This value is attained if we set $q = +1$ on each stretch on which $g(k)$ is increasing and $q = -1$ on each stretch on which $g(k)$ is decreasing. (Mathematicians will recognize that we have in fact defined N as the total variation of the function $g(k)$.)

Having the numbers N, the "norm effects" for several different values of γ, v_0 and Y, we can define p(k) by (4). However, if the effect curves mostly lie in the region of negative abscissas, it will be slightly more convenient to define p(k) by the equation

$$p(k) = - g(k)/N. \tag{19}$$

For each type of disturbance we select either (4) or (19) to define the functions p(k), and we graph these functions. The resulting graphs will be referred to as the "normalized effect curves."

In the important cases in which g(k) is monotonic, the integral (13) reaches its greatest value subject to (14) if we set q = + 1 for all y or q = - 1 for all y, according as g is increasing or decreasing. This greatest value is $|g(1) - g(0)| = |g(1)|$. If g(1) happens to be positive, we define p(k) by (4); if negative, we define p(k) by (19). In either case, we have definition (6) back again. So for the cases in which g(k) is a monotonic function our new definition of N leaves everything unchanged. This is desirable, because the traditional treatment of these disturbances has been satisfactory.

In the other cases, in which g(k) is not monotonic, our norm effect N will be greater than the unit effect g(1) of the usual theory. It will no longer be true that p(1) = 1. Instead, the graphs of the functions p(k) have the following property. Imagine that a point P traverses the curve from (0, 0) to its end (1, p(1)). Let P' be the perpendicular projection of P on the axis of abscissas. Then as P traverses the normalized effect curve, P' will travel a total distance of one unit of length. This can be seen from (18). As a point Q traverses the effect curve, its projection Q' will travel on the axis of abscissas from 0 to g(k'), from g(k') to g(k"), and from g(k") to g(1), a total travel equal to N, as (18) shows.

Since p(k) is obtained from g(k) by division by N, the distance travelled by P' is 1, as stated. As a corollary, we see that the functions p(k) must lie between - 1 and + 1.

The normalized effect curves are used in the same way as the weighting factor curves. If q is a departure from normality, say a temperature excess $\Delta\Theta/\Theta$, the integral (13) is computed, or to be more precise, is approximated by a sum similar to that in (15). This we shall name the "mean effective" disturbance; thus if the disturbance is a departure from standard temperature, the integral (13) is the "mean effective excess temperature ratio."

When the mean effective disturbance is multiplied by the norm effect, the product is either the differential effect of the disturbance (if (4) has been used to define p) or is the negative of the differential effect (if (19) has been used to define p). Consequently the table of values of norm effect should bear a heading "To be added if mean effective ... is positive," or a heading "To be subtracted if mean effective ... is positive," according as (4) or (19) was used in computing the functions p(k). This accords with present practice in tabulating unit effects.

In computing the ballistic wind it is computationally more convenient to express the magnitude of the wind in tens of miles per hour, rather than in feet per second. To compensate for this, the unit effect is multiplied by 14.6666... and entered as "unit effect of ten mile per hour range wind on range." Likewise, it is more convenient to express the excess of density over standard in terms of per cent. To compensate, the unit effect is divided by 100 and entered as "unit effect of one per cent excess density ratio." A similar remark would apply to the temperature also.

A number of the weighting factor curves have horizontal tangents at the top. Thus, unless the topmost zone is excessively deep, the graph of the weighting factor curve will be approximately that of the function $1 - k = c(1 - p)^2$, where c is some constant. If the disturbance q is roughly linear from top to bottom of the zone, we find by substitution of the above approximation for p that

$$(20) \quad \int_{k'}^{1} q(kY)\, dp(k) \doteq \left[(2/3)w(Y) + (1/3)w(k'Y) \right] \left[p(1) - p(k') \right].$$

Thus in computing ballistic wind and density it is possible to gain accuracy by replacing the arithmetic means of the top and bottom values, used in the other zones, by the weighted mean shown in (20), for the top zone.

If this is carried to its very extreme, and the ballistic wind is estimated by (20) with $k' = 0$, with no intermediate points of subdivision at all, we can hardly hope for much accuracy. The result, recalling (7) and the assumption that the wind is a linear function of the altitude, turns out to be that the ballistic wind is roughly the wind at two-thirds the summital altitude. It is an interesting fact that this same rule, obtained of course by entirely different reasoning, was in wide-spread use in the early part of the First World War.

Chapter IX

METHODS OF COMPUTING DIFFERENTIAL CORRECTIONS

1. Some remarks on linear differential equations.

It is true that the differential effect of any disturbance of the types considered can be found by solving equations (VIII.3.11, 18, 22) or equations (VIII.4.11, 12, 16). But solving differential equations is too difficult a task to repeat frequently unless it is unavoidable, and it is therefore desirable to devise methods of treating these equations which will avoid numerous integrations of the differential equations.

Trajectories may be classified into two different types, the first type having a definite end point and the second type not having a definite end point. The first type includes the trajectories of projectiles fired from the ground at targets on the ground (or water). These trajectories terminate at the second point at which $y = 0$, slight differences in elevation of gun and target being handled by small correction terms. The second type includes anti-aircraft and bomb trajectories. Anti-aircraft trajectories normally start from sea-level (departures being handled by small corrections), but have no end point as far as the tables are concerned. The target may be at any distance up to the maximum range of the weapon. As for bombs, it was pointed out in Section 2 of Chapter IV that with the assumption $a = 1$, it was feasible to compute trajectories with initial values

$x = y = 0$ and various values of v_0 and γ_s, and then for each Y to obtain the trajectory with initial speed v_0, reciprocal ballistic coefficient γ and initial altitude Y by simply replacing γ by γ_s through (IV.2.6) and taking the first Y feet of drop as already computed (or interpolated) for the projectile with initial speed v_0 and summital reciprocal ballistic coefficient γ_s. In this sense the trajectories of bombs have no definite end point; in each instance of a bomb dropping there is an end point, but in the numerical computation of bomb trajectories any point of the computed trajectory might serve as the end point of the trajectory for some particular dropping.

To each of these two types of trajectory there is an appropriate method of handling the differential effects of disturbances. In Sections 2 to 6 we shall present the original method of the adjoint system, due to Bliss, and two modifications of this method. This type of computation method is well adapted to the first type of trajectory. If only differential effects on range are desired, a single numerical integration of a system of differential equations is all that is needed; after that is done, differential effects can be found by means of quadratures. However, the adjoint system is not nearly so well adapted to the second type of trajectory. More convenient methods for finding differential effects on trajectories of the second type will be discussed in Sections 7 and 8. It should be emphasized that the distinction between the two types of method is a matter of convenience only. Any one of the methods to be presented in this chapter can be applied to any trajectory, but the difference in the amount of computation can be rather considerable.

In this section we shall collect some remarks on linear differential equations that will be convenient for use in the rest of the chapter.

Equations (VIII.3.11) and (VIII.4.11) are both of the type

$$(1)\ d\xi_i/dt = \sum_{j=1}^{n} A_{ij}(t)\ \xi_j + e_i(t) \quad (i = 1, \ldots, n).$$

If only the initial conditions are altered, the e_i are identically zero both in (VIII.3.11) and in (VIII.4.11), and these equations are then of the homogeneous type

$$(2)\ d\xi_i/dt = \sum_{j=1}^{n} A_{ij}(t)\ \xi_j \qquad (i = 1, \ldots, n).$$

Let us suppose that we have found n solutions of equations (2). In order to help keep the notation in mind, we shall use Greek-letter subscripts to distinguish between different solutions and roman subscripts to distinguish between the components of a single solution. Thus the n solutions will be designated by $\xi_{i\nu}(t)$; for each fixed value of ν, the n functions $\xi_{1\nu}, \ldots, \xi_{n\nu}$ form a solution of (2), so that

$$(3)\ d\xi_{i\nu}/dt = \sum_{j=1}^{n} A_{ij}(t)\,\xi_{j\nu}(t) \quad (i, \nu = 1, \ldots, n).$$

If D is the determinant of these n solutions, the derivative of D with respect to t is found by adding n terms, the first of which is a determinant identical with D except that the first row ($\xi_{11}, \ldots, \xi_{1n}$) is replaced by ($d\xi_{11}/dt, \ldots, d\xi_{1n}/dt$), the second of which is also identical with D except that the second row ($\xi_{21}, \ldots, \xi_{2n}$) is replaced by

$$(d\xi_{21}/dt, \ldots, d\xi_{2n}/dt),$$

and so on. When the derivatives are replaced by their values as given by (3), this yields

$$
dD/dt = \begin{vmatrix} \sum_{j=1}^{n} A_{1j}\xi_{j1} & \cdots & \sum_{j=1}^{n} A_{1j}\xi_{jn} \\ \xi_{21} & \cdots & \xi_{2n} \\ \cdots & \cdots & \cdots \\ \xi_{n1} & \cdots & \xi_{nn} \end{vmatrix}
+ \cdots
+ \begin{vmatrix} \xi_{11} & \cdots & \xi_{1n} \\ \cdots & \cdots & \cdots \\ \xi_{n-1,1} & \cdots & \xi_{n-1,n} \\ \sum_{j=1}^{n} A_{nj}\xi_{j1} & \cdots & \sum_{j=1}^{n} A_{nj}\xi_{jn} \end{vmatrix}.
$$

In the first determinant, from the first row we subtract A_{12} times the second row, A_{13} times the third row, and so on; the other determinants we treat similarly. We thus find that the first determinant is $A_{11}D$, the second $A_{22}D$, and so on. Thus (4) simplifies to

$$(5) \qquad dD/dt = (A_{11} + \ldots + A_{nn})\, D.$$

The one and only solution of (5) which at a given point t_o assumes a given value $D(t_o)$ is

$$(6) \quad D(t) = D(t_o) \exp \int_{t_o}^{t} (A_{11} + \ldots + A_{nn})\, dt,$$

where exp J means the same as e^J. Since the exponential never vanishes, this implies that if D(t) ever vanishes it is identically zero.

If the determinant D is not zero, and $\xi_1, \ldots, \xi_n$ is any solution of (2), then at some one value of t, say t_0, the equations

$$(7) \qquad \xi_i(t) = \sum_{\nu=1}^{n} \xi_{i\nu}(t)\, c_\nu$$

can be solved for the constants c_ν. But then the right and left members of (7) are both solutions of (2), and coincide at $t = t_0$, so by the uniqueness theorem for solutions of differential equations, (7) holds identically. Thus every solution of (2) is a linear combination of the n solutions $\xi_{i\nu}$, with constant coefficients. In this case the n solutions are said to form a basis for solutions of the equations (2).

Since equation (7) and similar equations will recur, it is desirable to have something more than the statement that they can be solved; it is more useful to have a formula for the solution. Let $\Xi_{i\nu}$ be the cofactor of $\xi_{i\nu}$ in the determinant D(t). By a well-known theorem on determinants, if the elements of a row are multiplied by the cofactors of the same row and the products added, the result is the determinant; if they are multiplied by the cofactors of a different row and the products added, the sum is zero. The statement remains true if "row" is everywhere replaced by "column." This is conveniently formalized with the help of the "Kronecker δ" symbol, defined by

$$(8) \qquad \begin{aligned} \delta_{ij} &= 1 \text{ if } i = j, \\ \delta_{ij} &= 0 \text{ if } i \neq j. \end{aligned}$$

Then the statement about elements of rows and their cofactors takes the form

$$(9) \qquad \sum_{\nu=1}^{n} \xi_{i\nu}\, \Xi_{j\nu} = \delta_{ij} D,$$

while the statement about elements of columns and their cofactors becomes

$$\sum_{i=1}^{n} \xi_{i\nu} \Xi_{i\mu} = \delta_{\nu\mu} D. \tag{10}$$

If we introduce the new symbols

$$\lambda_{\nu i}(t) = \Xi_{i\nu}(t)/D(t), \tag{11}$$

(observe the reversal of the order of the subscripts), these two equations take the somewhat simpler form

$$\begin{aligned} \sum_{\nu=1}^{n} \xi_{i\nu}\lambda_{\nu j} &= \delta_{ij}, \\ \sum_{i=1}^{n} \lambda_{\mu i} \xi_{i\nu} &= \delta_{\mu\nu}. \end{aligned} \tag{12}$$

Let us now multiply both members of (7) by $\lambda_{\mu i}$ and sum for $i = 1, \ldots, n$. With the help of (12) and (8) we obtain

$$\begin{aligned} \sum_{i=1}^{n} \lambda_{\mu i} \xi_i &= \sum_{i,\nu=1}^{n} \lambda_{\mu i} \xi_{i\nu} c_\nu \\ &= \sum_{\nu=1}^{n} \delta_{\mu\nu} c_\nu \\ &= c_\mu. \end{aligned} \tag{13}$$

From the equations (12) we can easily find the differential equations that the $\lambda_{\mu i}$ must satisfy. In the second of equations (12) we differentiate with respect to t, obtaining

$$\sum_{i=1}^{n} (d\lambda_{\mu i}/dt)\, \xi_{i\nu} + \sum_{i=1}^{n} \lambda_{\mu i}(d\xi_{i\nu}/dt) = 0.$$

In the second sum we replace the index of summation i by k and substitute from (3), obtaining

$$\sum_{i=1}^{n} (d\lambda_{\mu i}/dt)\, \xi_{i\nu} = - \sum_{k,j=1}^{n} \lambda_{\mu k} A_{kj}\, \xi_{j\nu}.$$

In the left member we change the index of summation from i to j; then we multiply both members by $\lambda_{\nu i}$ and sum for $\nu = 1, \ldots, n$. The result, because of (12) and (8), is

$$(14) \qquad d\lambda_{\mu i}/dt = - \sum_{k=1}^{n} \lambda_{\mu k} A_{ki}.$$

In other words, each <u>row</u> of the array $\lambda_{\mu i}$ is a solution of the differential equations

$$(15) \qquad d\lambda_i/dt = - \sum_{k=1}^{n} A_{ki}\, \lambda_k.$$

Equations (15) are the system <u>adjoint</u> to (1) or (2). They differ from (2) in having the rows of the array of coefficients A_{ij} changed into columns and vice versa, and also in having all terms multiplied by - 1. Clearly the system adjoint to (15) is again the system (2).

The interrelations between the solutions of (1) and those of (15) form the basis of the methods of the next five sections. Suppose first that $\xi_i(t)$ is a solution of (1) and that $\lambda_i(t)$ is a solution of (15). From (1) and (15) we readily deduce

$$\frac{d}{dt} \sum_{i=1}^{n} \lambda_i(t)\, \xi_i(t)$$

$$(16) \quad = \sum_{i,k=1}^{n} [\, - A_{ki}\, \lambda_k\, \xi_i + \lambda_i (A_{ik}\, \xi_k + e_i)]$$

$$= \sum_{i=1}^{n} \lambda_i\, e_i.$$

Hence, by integration

$$(17)\quad \sum_{i=1}^{n} \lambda_i(t)\,\xi_i(t) = \sum_{i=1}^{n} \lambda_i(t_o)\,\xi_i(t_o) + \int_{t_o}^{t} \sum_{i=1}^{n} \lambda_i(t)\, e_i(t)\, dt.$$

Because of (12) and (8), the equations

$$(18)\quad \xi_i(t) = \sum_{j,\nu=1}^{n} \xi_{i\nu}(t)\,\lambda_{\nu j}(t)\,\xi_j(t)$$

are identities. We continue to suppose that the functions $\xi_i(t)$ satisfy (1), while we know by (14) that for each ν the functions $\lambda_{\nu j}(t)$ satisfy (15). Hence we may substitute from (17) in (18), obtaining

$$(19)\quad \xi_i(t) = \sum_{\nu=1}^{n} \xi_{i\nu}(t) \left\{ \sum_{j=1}^{n} \lambda_{\nu j}(t_o)\,\xi_j(t_o) + \int_{t_o}^{t} \sum_{j=1}^{n} \lambda_{\nu j}(t)\, e_j(t)\, dt \right\}.$$

Equation (19) is essentially a formalization of the well-known method of "variation of the parameters" for the solution of linear differential equations. It is highly important, because once we have found a basis of solutions for the homogeneous equations (2) and computed the $\lambda_{\nu j}$ by (11), we can use (19) to find the solution of (1) for any initial conditions and any disturbances $e_i(t)$ without having to integrate the system of differential equations (1); all that (19) calls for is n numerical quadratures, together with multiplications and additions.

2. Bliss' method; the adjoint system.

The first application of the adjoint system to the computation of differential effects was made by G. A. Bliss, during the First World War. We now investigate this original method of Bliss, essentially as he develops it in Mathematics for Exterior Ballistics (New York: John Wiley and Sons, Inc., 1944).

The system of equations adjoint to (VIII.3.11) is

$$
\begin{aligned}
\dot{\lambda}_1 &= 0, \\
\dot{\lambda}_2 &= -\left[h - (n-2)a_1 \right] E(\dot{x}\lambda_4 + \dot{y}\lambda_5), \\
\dot{\lambda}_3 &= 0, \\
\dot{\lambda}_4 &= -\lambda_1 + E\left[1 + (n-1)(\dot{x}^2/v^2)\right]\lambda_4 \\
&\qquad + E\left[(n-1)\dot{x}\dot{y}/v^2\right]\lambda_5, \\
\dot{\lambda}_5 &= -\lambda_2 + E\left[(n-1)\dot{x}\dot{y}/v^2\right]\lambda_4 \\
&\qquad + E\left[1 + (n-1)(\dot{y}^2/v^2)\right]\lambda_5, \\
\dot{\lambda}_6 &= -\lambda_3 + E\lambda_6,
\end{aligned}
\tag{1}
$$

where the dot denotes differentiation with respect to t. We shall now show how to compute differential effects by means of solutions of the system (1). For example, to compute the effect on x at time T of a disturbance it is necessary to find $\xi_1(T)$. Observing (1.17) we see that this calculation is indeed simple if we have available a solution of (1) for which

$$\lambda_1(T) = 1, \lambda_2(T) = \dots = \lambda_6(T) = 0,$$

for in this case (1.17) with i = 1, t = T becomes

$$\xi_1(T) = \sum_{i=1}^{6} \xi_i(t_o)\,\lambda_i(t_o) + \int_{t_o}^{T} \sum_{i=1}^{n} \lambda_i(t)\, e_i(t)\, dt.$$

In fact, any linear combination of the variations at T can be obtained if we can suitably specify the values of the λ's at T. But for any set of values of the $\lambda_i(T)$ there is a unique solution of equations (1) which assumes these values. The computation of this solution can be effected by a numerical integration using methods closely related to those described in Chapter VI; in Section 4 we shall make a few more remarks about the solution of these equations. It is then computationally feasible to specify initial values for the system (1) at t = T, and from these to obtain the values of a linear combination of the $\xi_i(T)$.

Consider now the effect on range of a disturbance q. By (VIII.3.22) this effect is

$$dx(q|y = y(T)) = \xi(T) - \cot\theta\ \eta(T).$$

We shall therefore need the solution of (1) with the initial conditions

$$(2)\quad \begin{aligned} &\lambda_1(T) = 1,\ \lambda_2(T) = -\cot\theta(T) = \cot\omega, \\ &\lambda_3(T) = \lambda_4(T) = \lambda_5(T) = \lambda_6(T) = 0. \end{aligned}$$

We then have

$$(3)\quad \begin{aligned} dx(q|y = y(T)) &= \xi(T) - (\cot\theta)\ \eta(T) \\ &= \lambda_1(T)\ \xi(T) + \lambda_2(T)\ \eta(T) \\ &\quad + \xi_3(T)\ \zeta(T) + \lambda_4(T)\ \dot{\xi}(T) \\ &\quad + \lambda_5(T)\ \dot{\eta}(T) + \lambda_6(T)\ \dot{\zeta}(T), \end{aligned}$$

if the $\lambda_i(T)$ have the values (2). It is evident that the solution of the third and sixth of equations (1) with initial values (2) is $\lambda_3(t) = \lambda_6(t) = 0$, while from the first of equations (1) and (2) we see that $\lambda_1(t)$ is identically equal to 1. Also, by (VIII.3.18) we have $e_1 = e_2 = 0$. When all these substitutions are made, (1.17) becomes

$$
(4) \quad \begin{aligned} dx(q|y = y(T)) = &\Delta x(t_o) + \lambda_2(t_o)\Delta y(t_o) \\ &+ \lambda_4(t_o)\Delta v_x(t_o) + \lambda_5(t_o)\Delta v_y(t_o) \\ &+ \int_{t_o}^{T} [\lambda_4(t)e_4(t) + \lambda_5(t)e_5(t)]dt. \end{aligned}
$$

(We have here used (VIII.3.19) with the slight notational change that the disturbance in initial values, if any, is reckoned at time t_o instead of time 0.)

Equation (4) contains in particular the differential effects on range of changes in initial conditions; these we obtain by setting $e_4 = e_5 = 0$ in (4). It also contains the formula for the differential effects of departures from standard density, temperature, etc., when initial conditions are unchanged. For then the first four terms in the right member of (4) are all zero. The integral is computed by a numerical quadrature, and its value is the differential effect of the disturbance q on the range. Thus by means of a single numerical integration of the equations (1) we have obtained the functions λ_2, λ_4 and λ_5, and no further numerical integration is needed to find the differential effects of any disturbance on the range of the projectile. At most it will be necessary to compute the functions e_4 and e_5 by means of (VIII.3.18) and then to effect the numerical quadrature in (4).

It is important to notice the special form taken by (4) when the disturbance is a departure from standard temperature, a departure from standard density or a range wind. For the first of these, by (VIII.3.18) and (1), we find

$$
(5) \quad \begin{aligned} \lambda_4 e_4 + \lambda_5 e_5 &= - En(\lambda_4 \dot{x} + \lambda_5 \dot{y}) \Delta \Theta / 2\Theta \\ &= \{ \tfrac{1}{2} n \dot{\lambda}_2 / [h - (n-2)a_1] \} \Delta \Theta / \Theta. \end{aligned}
$$

The derivative of λ_2 is known from the numerical integration which gave us the $\lambda_i(t)$. If we compute the integral

$$(6) \qquad J(t) = \int_{t_0}^{t} \{ \tfrac{1}{2}n\dot{\lambda}_2/[h - (n - 2)a_1] \} \, dt$$

for values of t from t_0 to T, by (4) and (5) the differential effect of a departure from standard temperature, which is such that $\Delta\Theta/\Theta$ is zero up to time t^* and is 1 thereafter, is the same as $J(T) - J(t^*)$. This is what we called $h(t^*)$ in Section 5 of Chapter VIII, and from it we can construct the "effect curve" in the manner described in that section. Once the "effect curve" has been found, we can proceed in either of two directions. We can find the unit effect and the weighting factor curve in the traditional way, as described in Section 5 of Chapter VIII. Or else we can break with tradition and find the norm effects and the normalized effect curves, also described in the same section. In any case, the single quadrature (6) has given us the material for finding the differential effect on range produced by any departure from standard temperature law.

Next we consider a departure ΔH from standard density. Recalling that κ is the same as $\Delta H/H$, we find from (VIII.3.18) and (1) that

$$(7) \quad \lambda_4 e_4 + \lambda_5 e_5 = \{ \dot{\lambda}_2/[h - (n - 2)a_1] \} \, \Delta H/H.$$

We can proceed as we did with the temperature, computing the integral similar to (6) but with the factor $\frac{1}{2}n$ omitted. From this we find the effect curve as in the case of departure from standard temperature, and from this in turn we find the unit effect and the weighting factor curve. The "effect curve" is the graph of a monotonic function, so there is no difference between unit effect and norm effect, nor any difference between weighting factor curve and normalized effect curve.

If the normal trajectory has been computed with constant temperature, so that $a_1 = 0$, this process is even further simplified. For then the analogue of the integral (6) is

$$(8) \quad \int_{t_0}^{t} (\dot{\lambda}_2/h)\, dt = \lambda_2(t)/h - \lambda_2(t_0)/h.$$

If the departure from standard density is such that $\Delta H/H$ is 0 before a time t^* and is 1 thereafter, the differential effect on range is

$$(9) \quad \lambda_2(T)/h - \lambda_2(t^*)/h = [\cot \omega - \lambda_2(t^*)]/h.$$

This is the quantity denoted by $h(t^*)$ in Section 5 of Chapter VIII, and from it the effect curve is found, the unit effect and weighting factor curve being found in turn from the effect curve. Thus when the normal trajectory has been computed with a standard temperature which is constant, the unit effect and weighting factor curve for effect of non-standard density on range can be found without even a quadrature.

The effect of a one per cent increase in γ is exactly the same as the effect of a constant one per cent increase in $H(y)$. This is evident from the equations of motion themselves. Or it can be seen at once from (VIII.3.18) or (VIII.4.12). Hence if differential effects of departure from standard density are known, the effect of a departure from standard value of γ is easily obtained. In fact, the differential effect of a one per cent increase in γ is the unit effect for constant one per cent increase in density. The importance of this is usually in applying it in the reverse direction; a ballistic table being prepared, it is easy to find the effect of a one per cent increase in γ, and thereby to find the unit effect of change in density. We shall make no further mention of change in reciprocal ballistic coefficient.

Whatever may be the system of computing differential effects, it must be capable of providing the effects of departures from standard density, and, as a special case of this, it must also provide the effect of a change in γ.

Finally, for a range wind w_x we find from (1) and (VIII.3.18) that

$$\lambda_4 e_4 + \lambda_5 e_5 = w_x(\dot{\lambda}_4 + 1). \tag{10}$$

The differential effect on range caused by a wind w_x which is 0 before time t^* and is 1 thereafter is given by (4) to be

$$\int_{t^*}^{T} (\dot{\lambda}_4 + 1)\, dt = T - t^* - \lambda_4(t^*). \tag{11}$$

From this information we can readily find the "effect curve," and from this in turn we can compute the unit effect and weighting factor curve for the differential effect of range wind (following wind) on range. Not even a quadrature is necessary.

An alternative derivation of equation (11) can be based on the fourth of equations (VII.4.7), in which we replace t by T and t_o by t^*. Interpreting the v_{xo} of the formula as $v_x(t^*)$, and noting that by (4) we have $[\delta x/\delta v_{xo}]_y = \lambda_4(t^*)$, we see that the fourth of equations (VII.4.7) is the same as (11).

If we wish to find the differential effects of the disturbances on the time of flight, we must perform another numerical integration of the equations (1) with different initial values. By (VIII.3.22) we see that the proper initial values to choose are

$$\begin{aligned} &\lambda_2(T) = -1/\dot{y}(T), \\ &\lambda_i(T) = 0 \qquad\qquad (i = 1, 3, 4, 5, 6). \end{aligned} \tag{12}$$

For then by (VIII.3.22) we have

$$
(13) \quad \begin{aligned}
& dt(q|y = y(T)) \\
& = [-1/\dot{y}(T)]\,\eta(T) \\
& = \lambda_1(T)\,\xi(T) + \lambda_2(T)\,\eta(T) + \lambda_3(T)\,\zeta(T) \\
& \quad + \lambda_4(T)\,\dot{\xi}(T) + \lambda_5(T)\,\dot{\eta}(T) + \lambda_6(T)\,\dot{\zeta}(T).
\end{aligned}
$$

When the corresponding functions $\lambda_i(t)$ are computed, the right member of (4) represents the differential effect $dt(q|y = y(T))$. From this point on, all the remarks about differential effects on range apply equally well to the differential effects on time of flight, it being understood that now the $\lambda_i(t)$ are those determined by the initial values (12).

The differential effects of the various disturbances on the components of striking velocity, on the speed at impact and on the angle of impact can equally well be found; all that is needed is to replace (2) or (12) by the appropriate set of initial values. But these differential effects are of minor importance, and we shall not discuss them further.

3. Differential effects on deflection.

At the end of Section 1 of Chapter IV it was pointed out that of the disturbances we are considering, only changes in $z(t_o)$ and $v_z(t_o)$, cross wind w_z and cross-acceleration a_z were capable of producing non-zero differential effects on the z-coordinate of impact, while in turn they were incapable of producing non-zero differential effects on range and time of flight. We now investigate the differential effects on the z-coordinate of impact caused by the disturbances $\Delta z(t_o)$, $\Delta v_z(t_o)$, w_z and a_z. All of these can be handled by means of the adjoint system. But it is interesting to observe that the first three of them can also be discussed without use of the adjoint system. So we now treat the first three differential effects without using the adjoint system.

If $z(t_0)$ alone is changed, the trajectory is merely translated in the direction of the z-axis by an amount $\Delta z(t_0)$, so that

(1) $dz(\Delta z_0|y) = dz(\Delta z_0|t) = dz(\Delta z_0|x) = \Delta z_0.$

If v_{zo} alone is changed, to first-order terms the effect is the same as rotating the trajectory about the z-axis through an angle $\Delta v_{zo}/v_{xo}$. (This has already been discussed in detail in Section 4 of Chapter VII.) Hence the point on the normal trajectory whose coordinates are (x, y, 0) is moved, as far as first-order terms are concerned, to the place $(x, y, x\Delta v_{zo}/v_{xo})$. From this it follows that

(2) $dx(\Delta v_{zo}|y = y(x)) = x\Delta v_{zo}/v_{xo}.$

If $w_z{}^*$ is a cross wind which is zero at all times before t^* and is 1 at all times after t^*, we can find the differential effect of $w_z{}^*$ by applying (VII.4.23) with t replaced by the time of impact T and with the beginning of the trajectory taken as $t_0 = t^*$. Then with this beginning point we have a constant wind $w_z = 1$ at all points from beginning to time T, and formula (VII.4.23) is applicable. It takes the form

(3) $dz(w_z{}^*|y = y(T)) = T - t^* - [x(T) - x(t^*)]/\dot{x}(t^*).$

This takes the place of the function $h(t^*)$ of Section 5 of Chapter VIII. From it we can find the "effect curve" by the method described in that section, and from this in turn we find the unit effect and the weighting factor curve. Formula (3) involves nothing that is not on the sheet on which the normal trajectory was computed. So this important differential effect can be found without even a quadrature.

To return to the adjoint system, we first observe that by (VIII.3.22) and (VIII.3.20) and its analogues for all other disturbances we have

(4) $dz(q|y = y(t)) = dz(q|t) = \zeta(t).$

Accordingly we wish to find the solution of the adjoint system with the initial values

$$(5) \quad \lambda_3(T) = 1, \qquad \lambda_i(T) = 0 \qquad (i = 1, 2, 4, 5, 6).$$

For with these values we have

$$(6) \quad \begin{aligned} dz(q|y = y(T)) = \lambda_1(T)\,\xi(T) + \lambda_2(T)\,\eta(T) \\ + \lambda_3(T)\,\zeta(T) + \lambda_4(T)\,\dot{\xi}(T) \\ + \lambda_5(T)\,\dot{\eta}(T) + \lambda_6(T)\,\dot{\zeta}(T). \end{aligned}$$

From (5) and the first, second, fourth and fifth of equations (2.1) we see that λ_1, λ_2, λ_4 and λ_5 are identically zero, while λ_3 is identically 1. The last of equations (2.1) is then

$$(7) \quad \dot{\lambda}_6(t) = -1 + E\lambda_6(t).$$

We multiply both members of this equation by $\dot{x}$ and recall that $E\dot{x} = -\ddot{x}$. The result is

$$(8) \quad d(\lambda_6\dot{x})/dt = -\dot{x},$$

whence, by integrating and substituting from (5),

$$(9) \quad \lambda_6(t) = [x(T) - x(t)]/\dot{x}(t).$$

With the help of the preceding equations, (1.17) takes the form

$$(10) \quad \begin{aligned} dz(q|y = y(T)) = \Delta z(t_o) + [x(T)/\dot{x}(t_o)]\Delta v_{zo} \\ + \int_{t_o}^{T} \left[\frac{x(T) - x(t)}{\dot{x}(t)}\right] e_6(t)\, dt. \end{aligned}$$

If only the initial conditions are changed, the integral in the right member vanishes, and from (10) we obtain (1) and (2) again. If the disturbance is a cross wind w_z^* which is zero up to time t^* and is 1 thereafter, by (VIII.3.18) we have $e_6 = 0$ before time t^* and $e_6 = E$ thereafter. So by (10), (9) and (7),

$$dz(w_z^* | y = y(T)) = \int_{t^*}^{T} \lambda_6 E \, dt$$

$$(11) \qquad = \int_{t^*}^{T} (\dot{\lambda}_6 + 1) \, dt$$

$$= T - t^* - \lambda_6(t^*)$$

$$= T - t^* - [x(T) - x(t^*)]/\dot{x}(t^*).$$

This is equation (3) again.

When the disturbance is a cross-acceleration a_z, as in computing the effect of the Coriolis force (see (II.1.13)) or the effect of the small aerodynamic force which causes drift (see XI.5.15), it is necessary to replace e_6 by a_z in (10) and perform the quadrature.

4. Numerical integration of the adjoint system.

The initial values for the solutions of the adjoint system are assigned at time T, which is the end of the trajectory. Consequently it is necessary to solve the equations by letting t go backwards from T to t_0 (which is usually 0). This is a slight computational annoyance, which is easily remedied by a change of variable. On the trajectory sheet, lines have been computed for certain values of t which we shall call "tabular values." Let I be the smallest tabular value greater than T, and define a new variable s by the equation

$$(1) \qquad s = I - t.$$

Then for any function f(t) of the variable t we have

$$(2) \qquad df/ds = - df/dt.$$

The initial values of the λ_i now correspond to

$$s = I - T,$$

which ordinarily will not be an integer. The integration of the adjoint system will now proceed for s increasing from I - T to I. Because of (2), the left members of equations (2.1) should be replaced by $-d\lambda_i/ds$, $i = 1, \ldots, 6$. However, the third and sixth of these equations may be discarded from the list; differential effects on deflection have already been discussed in full in the preceding section.

Also we may refrain from writing the first of the equations if we keep in mind that λ_1 is a constant. Thus equations (2.1) may be replaced by

$$
\begin{aligned}
d\lambda_2/ds &= [h - (n - 2)\, a_1]E(\dot{x}\lambda_4 + \dot{y}\lambda_5), \\
d\lambda_4/ds &= \lambda_1 - E\lambda_4 \\
&\quad - \dot{x}[(n - 1)/v^2]E(\dot{x}\lambda_4 + \dot{y}\lambda_5), \\
d\lambda_5/ds &= \lambda_2 - E\lambda_5 \\
&\quad - \dot{y}[(n - 1)/v^2]E(\dot{x}\lambda_4 + \dot{y}\lambda_5).
\end{aligned} \tag{3}
$$

Whenever I - s is a tabular value of t, the trajectory sheet furnishes us with values of the quantities E, $\dot{x}$, $\dot{y}$ and v^2 (or $v^2/100$). With the last of these we enter a table of the function $(n - 1)/v^2$, assuming that such a table is available. (For at least some drag functions both n - 1 and $(n - 1)/v^2$ have been computed and tabulated.)

On a computing sheet we tabulate either n - 1 or $(n - 1)/v^2$, according to the tables available, for each s such that I - s is a tabular value of t. It will hardly be profitable to copy the trajectory entries for E, $\dot{x}$, $\dot{y}$ and v^2, but it is desirable to tabulate t = I - s to facilitate comparison with the trajectory sheet. We provide columns for the three functions λ_2, λ_4 and λ_5, for their first derivatives, and for the first and second differences of each of these derivatives. Since we know the values of the λ_i at $s = I - T$, we can compute their derivatives by means of (3). The process of obtaining more lines is a simplification of the numerical integration process of Chapter VI. Suppose that line $s = s_{n-1}$ is complete except for a final verification of the values λ_2, λ_4 and λ_5. The second differences of the derivatives are extrapolated to the next line, $s = s_n$, and from these the values of λ_2, λ_4 and λ_5 at $s = s_n$ are computed by Simpson's rule in the form (VI.3.1). From these, the derivatives of the λ_i at $s = s_n$ are computed by means of equations (3). If these disagree excessively with the extrapolated values it will be necessary to recompute the line. When final values are reached, we will also have final values for the second differences of the derivatives at $s = s_n$. So we can verify the entries for λ_2, λ_4 and λ_5 on line $s = s_{n-1}$ by means of (VI.3.3).

The organization of the computation has of course a great degree of arbitrariness. One of the many possible ways of performing the computations with the help of a computing machine will now be described.

(A) If standard temperature is not constant, multiply the listed values of $(n - 2)/v^2$ by v^2a_1. Subtract from h, enter as $h - (n - 2)a_1$. If standard temperature is constant, so that $a_1 = 0$, omit this step.

(B) Extrapolate the second difference of $d\lambda_2/ds$, compute $\lambda_2(s_n)$ by (VI.3.1), and enter as $\lambda_2(s_n)$.

(C) Repeat (B) for λ_4 and λ_5.

(D) With the values of $\dot{x}$ and $\dot{y}$ from the trajectory sheet, compute $\dot{x}\lambda_4 + \dot{y}\lambda_5$. Transfer to keyboard, multiply by E. Transfer to keyboard, multiply by number under (A). Enter product as $d\lambda_2/ds$. Clear dials.

(E) Multiply number on keyboard (which is $E(\dot{x}\lambda_4 + \dot{y}\lambda_5)$) by $\dot{x}$, and enter in a column. Clear dials.

(F) Multiply number on keyboard by $\dot{y}$. Enter result.

(G) The two preceding steps gave us the last terms in the second and third of equations (3). Use these equations to find $d\lambda_4/ds$ and $d\lambda_5/ds$.

When the normal trajectory has been computed with constant temperature, so that $a_1 = 0$, it is customary to introduce a new variable Λ_2, defined by the equation

$$\Lambda_2 = \lambda_2/h. \tag{4}$$

Then equations (3) may be replaced by

$$\begin{aligned} d\,\Lambda_2/ds &= E(\dot{x}\lambda_4 + \dot{y}\lambda_5), \\ d\lambda_4/ds &= \lambda_1 - E\lambda_4 - \dot{x}[(n-1)/v^2]d\,\Lambda_2/ds, \\ d\lambda_5/ds &= h\,\Lambda_2 - E\lambda_5 - \dot{y}[(n-1)/v^2]d\,\Lambda_2/ds. \end{aligned} \tag{5}$$

Steps (B) to (G) of the preceding paragraph apply to these with only obvious amendments. Step (A) is omitted.

Since disturbances such as winds and departures from standard density and temperature can hardly be determined to within one per cent, it should be satisfactory to have the solutions of the adjoint system accurate to one per cent. This means that it will not be necessary to carry as many significant figures in the solution of the adjoint system as in the original solution of the normal equations. Three significant figures should be enough. As a secondary result of this, it may often be found that the second differences of the derivatives are so small that the trapezoidal rule furnishes enough accuracy in determining the λ_i from their derivatives. In any case, the integration of the adjoint equations will ordinarily be a much easier task then the solution of the normal equations.

5. Gronwall's method for integration of the adjoint system.

Suppose that we are interested in a particular solution

(1) $$x = x(t), \quad y = y(t), \quad z = 0$$

of the normal equations, and that we have found a family of solutions

(2) $$x = x(t, \varepsilon), \quad y = y(t, \varepsilon), \quad z = 0$$

of the normal equations depending in a differentiable way on a parameter ε and reducing to (1) when $\varepsilon = 0$. It is then possible to find a solution of the equations of variation (VIII.3.11) with the e_i all equal to zero. In fact, the functions

(3) $$\begin{aligned} \xi(t) &= \partial x(t, \varepsilon)/\partial\varepsilon, \\ \eta(t) &= \partial y(t, \varepsilon)/\partial\varepsilon, \\ \zeta(t) &= 0 \end{aligned}$$

evaluated for $\varepsilon = 0$ satisfy these equations. This may be shown without trouble if we recall that the normal equations are a special case of equations (VIII.2.4) and the equations of variation (VIII.3.11) are the corresponding special case of equations (VIII.2.17). In this notation, our task is to show that if the functions $y_i(t, \epsilon)$ satisfy (VIII.2.4) for all ϵ, so that the equations

(4) $$\partial y_i(t, \varepsilon)/\partial t = f_i(t, y(t, \varepsilon)) \quad (i = 1, \ldots, n)$$

are identities in t and ε, and if we define

(5) $$\eta_i(t) = \partial y_i(t, \varepsilon)/\partial\varepsilon$$

for $\varepsilon = 0$, then the functions (5) satisfy

(6) $$d\eta_i(t)/dt = \sum_{j=1}^{n} (\partial f_i/\partial y_j)\eta_j(t) \quad (i = 1, \ldots, n).$$

If we differentiate both members of (4) with respect to ε and set $\varepsilon = 0$, and make the substitution (5), we obtain (6), and the statement is established.

As a trivial application of this, we observe that if (1) is a solution of the normal equations, so is

$$(7) \qquad x = x(t) + \varepsilon, \quad y = y(t), \quad z = 0$$

for all ε. So the functions (3), which now have the special forms

$$(8) \qquad \xi(t) = 1, \quad \eta(t) = \zeta(t) = 0,$$

must be solutions of the equations (VIII.3.11) with $e_i = 0$. This is obvious from the equations. As a less trivial application, we observe that if (1) is a solution of the normal equations, so are the functions

$$(9) \qquad x = x(t + \varepsilon), \quad y = y(t + \varepsilon), \quad z = 0$$

for all values of ε. Therefore the functions (3), which now have the special forms

$$(10) \quad \xi(t) = \dot{x}(t), \qquad \eta(t) = \dot{y}(t), \qquad \zeta(t) = 0,$$

must satisfy the equations of variation (VIII.3.11) with all e_i equal to 0. But whenever the e_i are all 0, the right member of (1.17) is a constant. So by (1.17) we find

$$(11) \qquad \lambda_1\dot{x} + \lambda_2\dot{y} + \lambda_4\ddot{x} + \lambda_5\ddot{y} = k,$$

where k is a constant.

With the help of equation (11) it is possible to reduce the third-order system (4.3) to a second-order system, in any of an assortment of ways. For example, let us replace the second derivatives of x and y by their values from the normal equations; then (11) yields

$$(12) \quad \lambda_1\dot{x} + \lambda_2\dot{y} - E(\dot{x}\lambda_4 + \dot{y}\lambda_5) - g\lambda_5 = k.$$

If this is solved for λ_2 and the result substituted in (4.3), the last two of these equations determine λ_4 and λ_5. But the advantage of this procedure is meager.

However, in the special case in which standard temperature is constant, so that $a_1 = 0$, a more significant simplification can be had. With notation (4.4), (12) yields

$$(13) \qquad \lambda_5 = (\lambda_1 \dot{x} + h\,\Lambda_2 \dot{y} - d\,\Lambda_2/ds - k)/g.$$

From the first of equations (4.5),

$$(14) \qquad \lambda_4 = -(\dot{y}/\dot{x})\lambda_5 + (d\,\Lambda_2/ds)/E\dot{x}.$$

In (13) we differentiate with respect to s, recalling that because of (4.2) we have

$$d\dot{x}/ds = -\ddot{x} = E\dot{x}, \quad d\dot{y}/ds = -\ddot{y} = E\dot{y} + g.$$

In the resulting equation we first replace $d\lambda_5/ds$ by its value from (4.5) and then replace λ_5 by its value from (13). The result is

$$(15) \qquad \begin{aligned} d^2\,\Lambda_2/ds^2 &= E(2\dot{x}\lambda_1 - k) + 2hE\dot{y}\,\Lambda_2 \\ &\quad + \{\dot{y}[h + g(n-1)/v^2] - E\}d\,\Lambda_2/ds. \end{aligned}$$

For the Gâvre drag function, the function

$$h + g(n-1)/v^2$$

has been tabulated against $v^2/100$. Consequently for each s such that I - s is a tabular value of t we can compute the three coefficients

$$(16) \quad E(2\dot{x}\lambda_1 - k), \quad 2hE\dot{y}, \quad \dot{y}[h + g(n-1)/v^2] - E$$

of equation (15). The integration of equation (15) now can be effected as in Chapter VI. From the values of Λ_2 we can find the unit effects and weighting factor curves for the differential effects of departure from standard density, and also we can find either unit effects and weighting factor curves or else norm effects and normalized effect curves for the differential effects of departures from standard temperature. The procedure is the same as in Section 2 of this chapter.

If the time of firing is $t = 0$, by (2.11) the effect of a constant unit range wind is $T - \lambda_4(0)$, and after (15) is solved this can be found by use of (13) and (14) at time $t = 0$. But if a weighting factor curve for wind is desired, or any other effect is desired which requires the quadrature in (2.4), equations (13) and (14) must be applied to find λ_5 and λ_4 at a number of values of s or t spaced closely enough to permit the quadrature. The numerical integration of (15) is easier than the integration of the system (4.5), in spite of the necessity of preparing the coefficients (16). But a considerable part of this gain is dissipated in having to compute λ_4 and λ_5 by (13) and (14).

If the normal trajectory has been computed with a non-constant standard temperature law, as is in fact customary at the present time, there is no chance of comparing the method of Bliss with that of Gronwall, since Gronwall's equation (15) applies only when $a_1 = 0$. In this case Bliss' method must be applied to the equations (4.3) instead of to the slightly simpler system (4.5), but the additional labor is small.

6. The adjoint system based on slope.

So far we have discussed the adjoint system based on time; that is, the system of equations adjoint to equations (VIII.3.11). However, the use of adjoint systems is not restricted to any one system of equations. It will now be shown that there is a great gain in simplicity if the system adjoint to equations (VIII.4.11) is used instead of the system adjoint to (VIII.3.11).

According to the definition in Section 1 of this chapter, the system adjoint to (VIII.4.11) is

$$
(1)\quad
\begin{aligned}
d\lambda_1/dt &= (n + 1)E\lambda_1 - 2(\dot{y}/\dot{x})\lambda_2 - 2\lambda_3 - (1/\dot{x})\lambda_4, \\
d\lambda_2/dt &= -[h - (n - 2)a_1]E\dot{x}\lambda_1, \\
d\lambda_3/dt &= 0, \\
d\lambda_4/dt &= 0.
\end{aligned}
$$

These λ_i are of course not the same functions as in the preceding section. It is at once obvious that because of the last two equations the functions λ_3 and λ_4 are constants, so that (1) is in fact a second-order system.

If we wish to find differential effects on range, we see from the first of equations (VIII.4.16) that the initial values of the λ_i, at $t = T$, should be chosen to be

$$(2)\qquad \begin{aligned} &\lambda_1(T) = 0,\ \lambda_2(T) = -\cot\theta(T) = \cot\ \omega, \\ &\lambda_3(T) = 1,\ \lambda_4(T) = 0. \end{aligned}$$

If the differential effects on time of flight are desired, by the second of equations (VIII.4.16) we should choose

$$(3)\qquad \begin{aligned} &\lambda_1(T) = 0,\ \lambda_2(T) = -1/\dot{y}(T), \\ &\lambda_3(T) = 0,\ \lambda_4(T) = 1. \end{aligned}$$

For if the choice (2) is made, then with the help of (VIII.4.16) and (VIII.4.13), equation (1.17) reduces to

$$(4)\qquad \begin{aligned} dx(q|y = y(T)) \\ = \lambda_1(t_o)[\Delta v_{xo} - (E/g)(v_{xo}\Delta v_{yo} - v_{yo}\Delta v_{xo})] \\ + \lambda_2(t_o)[\Delta y_o + (v_{yo}/gv_{xo})(v_{xo}\Delta v_{yo} - v_{yo}\Delta v_{xo})] \\ + \lambda_3(t_o)[\Delta x_o + (1/g)(v_{xo}\Delta v_{yo} - v_{yo}\Delta v_{xo})] \\ + \lambda_4(t_o)[(1/gv_{xo})(v_{xo}\Delta v_{yo} - v_{yo}\Delta v_{xo})] \\ + \int_{t_o}^{T}[\lambda_1(t)e_1 + \lambda_2(t)e_2 + \lambda_3 e_3 + \lambda_4 e_4]\,dt. \end{aligned}$$

With the choice of initial values (3), the right member of (4) represents $dt(q|y = y(T))$.

Formula (4) looks cumbersome, but in specialization to the most important instances it undergoes notable simplification. First let us consider a departure $\Delta\Theta^*$ from standard temperature which is zero up to time t^* and is equal to Θ thereafter.

By (4) and (VIII.4.12),

$$(5)\quad dx(\Delta\,\Theta^{*}|\,y = y(T)) = \int_{t^*}^{T} \lambda_1(t)\ E\dot{x}[(n - 2)/2]\, dt.$$

This is the quantity called $h(t^*)$ in Section 5 of Chapter VIII. From it we can find either unit effect and weighting factor curve, or norm effect and normalized effect curve. The factors $\lambda_1(t)$ and $E\dot{x}$ should be available from the sheet on which equations (1) were integrated, and the same sheet should also contain the values either of $n - 2$ or of $n + 1$, from which the factor $(n - 2)/2$ is readily found. If the $\lambda_1(t)$ has been found by solving equations (1) with initial values (3) the right member of (5) will furnish the differential effect of the disturbance on the time of flight, instead of on the range.

Next we investigate the differential effects of a departure from standard density. If the departure ΔH^* from standard density has value zero before time t^* and value H thereafter, by (4) and (VIII.4.12) we find that

$$(6)\qquad dx(\Delta H^*|\,y = y(T)) = -\int_{t^*}^{T} \lambda_1(t)\ E\dot{x}\ dt,$$

when the initial values are those in (2). If they were those in (3), the right member of (6) would represent the differential effect on time of flight. Since $E\dot{x}$ has been found in solving (1), this is an easy quadrature. From it we find the effect curve for differential effect of non-standard density, and from this in turn we obtain weighting factor curves and unit effects.

In the special case in which the normal trajectory has been computed with constant temperature, the integrand in (6) differs only by the constant factor h from the right-hand member of the second of equations (1). So the quadrature in (6) was in effect performed while equations (1) were being integrated.

Equation (6) takes the form

$$(7) \qquad dx(\Delta H^*|y = y(T)) = [\lambda_2(T) - \lambda_2(t^*)]/h.$$

The differential effects of range winds can most easily be found with the help of equation (VII.4.8). Let us suppose that the range wind w_x^* is zero before time t^* and is 1 thereafter. Clearly there is no effect on the trajectory before time t^*. We regard t^* as the start of the trajectory; then at all times after this initial time we have $w_x^* = 1$. Equation (VII.4.8) then furnishes us with

$$(8) \quad dx(w_x^*|y = y(T)) = T - t^* - [\delta x/\delta v_{xo}]_{y=y(T)},$$

wherein v_{xo} is understood to be the horizontal component at time t^*. With (2) and (4), this yields

$$(9) \quad \begin{aligned} dx(w_x^*|y = y(T)) = T - t^* &- \lambda_1(t^*)[1 + E\dot{y}(t^*)/g] \\ &+ \lambda_2(t^*)[\dot{y}(t^*)]^2/g\dot{x}(t^*) \\ &+ \dot{y}(t^*)/g. \end{aligned}$$

The quantity $g + E\dot{y}$ is available from the trajectory sheet, and likewise $\dot{y}$; the value of $\dot{y}/\dot{x}$ is available from the solution of (1). So the computation of the right member of (9) is not very difficult. It could be effected for example by the following sequence of steps: multiply $\lambda_2(t^*)$ by $\dot{y}/\dot{x}$; add 1; multiply the sum by $\dot{y}$; add $\lambda_1(t^*)\ddot{y}(t^*)$; divide by g; add $T - t^*$.

The integration of equations (1) is slightly facilitated by making the substitution

$$(10) \qquad \Lambda_2(t) = \lambda_2(t)/h.$$

Then the first two of equations (1) take the form

$$(11) \quad \begin{aligned} d\lambda_1/dt &= (n + 1)E\lambda_1 - 2(h\dot{y}/\dot{x})\Lambda_2 - 2\lambda_3 - (1/\dot{x})\lambda_4, \\ d\Lambda_2/dt &= -[1 - (n - 2)a_1/h]E\dot{x}\lambda_1. \end{aligned}$$

For finding differential effects on range, we use the initial values (2), which now become

$$\begin{aligned} &\lambda_1(T) = \lambda_4(T) = 0, \\ (12) \qquad &\Lambda_2(T) = \cot\,\omega/h, \\ &\lambda_3(T) = 1. \end{aligned}$$

For finding differential effects on time of flight we use the initial values (3), which in the present notation are

$$\begin{aligned} &\lambda_1(T) = \lambda_3(T) = 0, \\ (13) \qquad &\Lambda_2(T) = -\,1/h\dot{y}(T), \\ &\lambda_4(T) = 1. \end{aligned}$$

As in both previous methods, it is advisable to change variable from t to $s = I - t$, where I is the first tabular value of t above T. The left members of (11) are thereby replaced by $-\,d\lambda_1/ds$ and $-\,d\,\Lambda_2/ds$ respectively. For each s such that $I - s$ is a tabular value of t, we look up the value of $n - 2$ in the appropriate table and enter it on a computing sheet. If a_1 is not zero, we compute and enter $1 - (n - 2)a_1/h$; if a_1 is zero, we omit this step. From the trajectory sheet we compute $2h\dot{y}/\dot{x}$ and enter it; and if differential effects on time of flight are desired, we also compute and enter the function $1/\dot{x}$. On the computing sheet we provide columns for λ_1, Λ_2, their derivatives, and the first and second differences of the derivative of λ_1. If a_1 is not zero, we also provide a column for the values of $E\dot{x}\lambda_1$.

The integration process differs only trivially from that described in Section 4. First the derivative of λ_1 is extrapolated, and by Simpson's rule in the form (VI.3.1) the value of λ_1 is computed. This is multiplied by $E\dot{x}$ and entered in the column so headed if $a_1 \neq 0$ and in the column $d\,\Lambda_2/ds$ if $a_1 = 0$. In the former case we multiply $E\dot{x}\lambda_1$ by the coefficient $1 - (n - 2)a_1/h$ and enter the product as $d\,\Lambda_2/ds$.

Again we apply (VI.3.1) to find Λ_2, and then compute $d\lambda_1/ds$ by the first of equations (11). If this disagrees excessively with the extrapolated value it will be necessary to re-compute the line.

The alterations in previous formulas due to the substitution (10) are simple ones. In (9), the term involving λ_2 should be replaced by

$$\Lambda_2(t^*)\ (2h\dot{y}/\dot{x})\ (\dot{y}/2g).$$

The second factor was computed in the process of integrating the differential equations. Also, in the case $a_1 = 0$ we can use equation (7), which in the present notation takes the slightly simpler form

$$(14) \qquad dx(\Delta H^* | y = y(T)) = \Lambda_2(T) - \Lambda_2(t^*).$$

Equations (11) are noticeably easier to handle than those of Section 2, whether by Bliss' method or that of Gronwall. They form a second-order system, as in Gronwall's method, but they are not restricted to the special case $a_1 = 0$, and they do not require auxiliary calculations after the integration is complete, as in (5.13, 14).

As a rather easy example of a use of the method of this section (which however could equally well be treated by either of the preceding methods) we shall show that the relation

$$(15) \quad \text{maximum ordinate} = (\text{time of flight})^2/8g,$$

which is exact for ground-to-ground trajectories in vacuo, has an error which is of second order in γ. Thus we may anticipate that (15) will furnish a good estimate for the maximum ordinate when time of flight is known. Examination of a collection of artillery trajectories will show that this is indeed the case; the error in (15) is surprisingly small. Let us choose the origin at the summit of the trajectory.

As normal trajectory we select the trajectory in vacuo with initial velocity v_0 and angle of departure 0; the time is taken to be 0 at the summit. Then the equations of the trajectory are

$$(16) \qquad x = v_0 t, \; y = -gt^2/2.$$

We first find the differential effects on time at the level $y = Y = -gT^2/2$ on the descending branch of the trajectory. Since $E = 0$, the functions λ_2, λ_3 and λ_4 are all constant, as is evident from (1); and since the initial values are given by (3), we have

$$(17) \qquad \lambda_2 = 1/gT, \quad \lambda_3 = 0, \quad \lambda_4 = 1.$$

It is now easy to solve the first of equations (1) with the initial value $\lambda_1(T) = 0$; the solution is

$$(18) \qquad \lambda_1(t) = (t^2 - tT)/v_x T,$$

wherein we should recall that v_x is a constant.

Now let us suppose that a range wind with speed w_x is blowing; that the velocity at the summit changes from v_0 to $v_0 + \Delta v_0$; and that the drag deceleration changes from the value 0 on the normal trajectory to the value $(\Delta E)v$ on the disturbed trajectory. Then the added acceleration has components $-(\Delta E)\dot{x}$, $-(\Delta E)\dot{y}$, which are the a_x and a_y in (VIII.4.12). An important property of ΔE is that it depends only on the velocity and on the altitude, since density and temperature are determined by the altitude. Since velocity and altitude are the same at time $-t$ as at time t, we have

$$(19) \qquad \Delta E(-t) = \Delta E(t).$$

Recalling that on the normal trajectory E is 0, from (VIII.4.12) we obtain

$$(20) \qquad e_1 = -(\Delta E)\dot{x}, \quad e_2 = e_3 = e_4 = 0.$$

The right member of (4) is the differential effect on time at $y = Y$ caused by the disturbances, since initial conditions (3) were used. Since the only initial condition changed is v_{xo} and $\lambda_1(0) = 0$, we thus find

$$(21)\quad dt(q|y = Y) = -\int_0^T [(\Delta E)\dot{x}(t^2 - tT)/v_x T]\,dt,$$

where q is an abbreviation for the aggregate of all the disturbances acting on the trajectory.

In the same way, in order to find the differential effects at level $y = Y = -gT^2/2$ on the ascending branch, we solve equations (1) with initial conditions (3) holding at time $t = -T$. This time we obtain

$$(22)\quad \begin{aligned} &\lambda_1(t) = -(t^2 + tT)/v_x T, \\ &\lambda_2 = -1/gT, \quad \lambda_3 = 0, \quad \lambda_4 = 1. \end{aligned}$$

Corresponding to (21) we obtain for the differential effect of the disturbances at altitude Y on the ascending branch

$$(23)\quad dt(q|y = Y) = +\int_0^{-T} [(\Delta E)\dot{x}(t^2 + tT)/v_x T]\,dt.$$

If we change variable from t to − t, recalling that ΔE and $\dot{x}$ have the same values at − t as at t, we find that this is the same as (21). That is, to first order the change in time of passage of altitude Y on the ascending branch is the same as the change in time of passage of altitude Y on the descending branch. Therefore the time of flight, which is the time-interval between the two passages of altitude Y, is to first order left unchanged, as we wished to prove.

7. Moulton's method.

Since the computation of differential effects by the methods of Sections 2, 5 and 6 requires integrating the adjoint system of equations from time T backward, it is evident that it requires some modification before it can be effectively applied to trajectories in which, mathematically speaking, there is no end point. In Section 1 it was mentioned that anti-aircraft trajectories and bomb trajectories are of this type. This difficulty can be circumvented in either of two ways which superficially seem quite different. The first possibility is to find a basis of solutions of the adjoint system. In terms of this basis we can express every solution of the adjoint system. So when we select any particular point at which we desire to find differential effects, say on range, we can form that combination of the solutions in the basis which has values (2.2) at the point where the effects are wanted. From this point we can proceed as in Section 2. The second possibility is to find a basis of solutions of the homogeneous equations of variation, consisting say of equations (VIII.3.11) with all e_i set equal to 0. Then the solution of the non-homogeneous equations (VIII.3.11) can be found from this basis by means of equation (1.19). The former of these methods was suggested by Bliss in his original papers on the adjoint system; the latter is proposed by Moulton in his book <u>New Methods in Exterior Ballistics</u> (Chicago: The University of Chicago Press, 1926). It is interesting to observe that the two methods are not really as distinct from each other as a first glance might suggest. If, for example, we select the second method, in order to use equations (1.19) we must compute the functions $\lambda_{\nu j}$. But these functions form a basis of solutions of the adjoint system. On the other hand, if we select the first method we integrate the adjoint system to find a basis of solutions of the adjoint system. For a given T we wish to find the combination of these functions whose values at T are given by (2.2) or by (2.12) if differential effects on time are desired. This means that we must solve equations like (1.7) with the $\lambda_{\nu j}$ in place of the $\xi_{i\nu}$. But the method of solving these equations explained in the sentences following (1.7) would lead us to functions $\xi_{i\nu}$ which form a basis of solutions of the homogeneous equations of variation.

Since the differential effects on deflection have already been discussed, we need only consider the first, second, fourth and fifth of equations (VIII.3.11). These may be written in the form:

$$
\begin{aligned}
d\xi/dt &= \dot{\xi}, \\
d\eta/dt &= \dot{\eta}, \\
d\dot{\xi}/dt &= [h - (n - 2)a_1]E\dot{x}\eta - E\dot{\xi} \\
&\quad - E\dot{x}[(n - 1)/v^2](\dot{x}\dot{\xi} + \dot{y}\dot{\eta}) + e_4, \\
d\dot{\eta}/dt &= [h - (n - 2)]E\dot{y} - E\dot{\eta} \\
&\quad - E\dot{y}[(n - 1)/v^2](\dot{x}\dot{\xi} + \dot{y}\dot{\eta}) + e_5.
\end{aligned}
\tag{1}
$$

We seek a basis of the homogenized equations (1), that is of equations (1) with e_i set equal to 0. For this we need four sets of functions

(2) $\xi_\nu(t), \eta_\nu(t), \dot{\xi}_\nu(t), \dot{\eta}_\nu(t) \quad (\nu = 1, 2, 3, 4)$

each of which satisfies (1) with $e_i = 0$ and whose determinant is not zero. By (5.8) one solution is

(3) $\xi_1(t) = 1,\ \eta_1(t) = 0,\ \dot{\xi}_1(t) = 0,\ \dot{\eta}_1(t) = 0.$

By (5.10) another solution is

$$
\begin{aligned}
&\xi_2(t) = \dot{x}(t),\ \eta_2(t) = \dot{y}(t), \\
&\dot{\xi}_2(t) = \ddot{x}(t) = -E\dot{x}(t), \\
&\dot{\eta}_2(t) = \ddot{y}(t) = -E\dot{y} - g.
\end{aligned}
\tag{4}
$$

It remains to find two more solutions by numerical integration. The initial values, at time $t = 0$, of these remaining two solutions is immaterial, apart from the requirement that the determinant

$$(5) \qquad D(t) = \begin{vmatrix} \xi_1(t) & \xi_2(t) & \xi_3(t) & \xi_4(t) \\ \eta_1(t) & \eta_2(t) & \eta_3(t) & \eta_4(t) \\ \dot{\xi}_1(t) & \dot{\xi}_2(t) & \dot{\xi}_3(t) & \dot{\xi}_4(t) \\ \dot{\eta}_1(t) & \dot{\eta}_2(t) & \dot{\eta}_3(t) & \dot{\eta}_4(t) \end{vmatrix}$$

be not zero. However, as Moulton points out, for firing tables it is necessary to know the change of range with change of initial velocity and also with change of angle of departure. Hence it is convenient to choose initial values of the last two sets of solutions in such a way that one of them corresponds to change of initial velocity alone, the other to change of angle of departure alone. Since

$$(6) \qquad \dot{x}(0) = v_o \cos \theta_o, \qquad \dot{y}(0) = v_o \sin \theta_o,$$

corresponding to a change Δv_o in initial velocity we have

$$(7) \quad \Delta v_{xo} = \Delta v_o \cos \theta_o, \quad \Delta v_{yo} = \Delta v_o \sin \theta_o.$$

So for the initial values of the third solution of the equations of variation we select

$$(8) \qquad \begin{array}{ll} \xi_3(0) = 0, & \eta_3(0) = 0, \\ \dot{\xi}_3(0) = \cos \theta_o, & \dot{\eta}_3(0) = \sin \theta_o. \end{array}$$

Corresponding to a change $\Delta \theta_o$ in angle of departure we have, to first-order terms,

$$(9) \qquad \begin{aligned} \Delta v_{xo} &= - v_o \sin \theta_o \Delta \theta_o = - v_{yo} \Delta \theta_o, \\ \Delta v_{yo} &= v_o \cos \theta_o \Delta \theta_o = v_{xo} \Delta \theta_o. \end{aligned}$$

So for the fourth solution we shall choose, except in the case $v_{yo} = 0$, the initial values

$$(10) \qquad \begin{aligned} &\xi_4(0) = 0, && \eta_4(0) = 0, \\ &\dot{\xi}_4(0) = -v_{yo}, && \dot{\eta}_4(0) = v_{xo}. \end{aligned}$$

With these initial values of the solutions, determinant D has at t = 0 the value

$$(11)\ D(0) = \begin{vmatrix} 1 & v_{xo} & 0 & 0 \\ 0 & v_{yo} & 0 & 0 \\ 0 & -Ev_{xo} & \cos\theta_o & -v_{yo} \\ 0 & -Ev_{yo} - g & \sin\theta_o & v_{xo} \end{vmatrix} = v_o v_{yo}.$$

Unless launching is horizontal, as it is in the case of bomb trajectories, this is different from zero, and the four solutions form a basis. In the case of horizontal launching the four functions fail to form a basis; so instead of (10) we would use, for example, the initial values

$$(12) \quad \xi_4(0) = 0, \ \eta_4(0) = 1, \ \dot{\xi}_4(0) = 0, \ \dot{\eta}_4(0) = 0.$$

If these are put in place of the last column in (11) we find $D(0) = g \cos\theta_o \neq 0$, so this set of four solutions forms a basis.

In the case $v_{yo} \neq 0$ Moulton suggests replacing the initial values (10) by their ratios to $v_o v_{yo}$, so that $D(0) = 1$. This results in a slight computational saving in finding $D(t)$.

After the third and fourth solutions have been found by numerical integration of equations (1) with $e_i = 0$, it would be possible to compute the determinant $D(t)$ from equation (5). But computationally this is quite undesirable. It is much easier to compute $D(t)$ with the help of (1.6). Since in (1) there are four equations, the integrand in (1.6) is the sum of four terms;

the first is the coefficient of ξ in the first equation, which is 0; the second is the coefficient of η in the second equation, which is also 0; the third is the coefficient of $\dot{\xi}$ in the third equation, which is

$$- E - E\dot{x}^2(n - 1)/v^2;$$

and the fourth is the coefficient of $\dot{\eta}$ in the fourth of the equations, which is

$$- E\dot{y}^2(n - 1)/v^2 - E.$$

The sum of the four is $-(n + 1)E$, so by (1.6) we have

$$D(t) = D(0) \exp \int_0^t [-(n + 1)E]\, dt. \tag{13}$$

Since E is known from the trajectory sheet, and n + 1 can easily be found from quantities entered on the sheet on which the third and fourth solutions of (1) were computed, the integrand $(n + 1)E$ can be found without difficulty. Then, by a numerical quadrature and a table of the exponential function, D(t) can be found.

Next we need the four-by-four array of functions $\lambda_{\nu i}(t)$ defined by equations (1.11). The first column is

$$\lambda_{11}(t) = 1,\ \lambda_{21} = \lambda_{31} = \lambda_{41} = 0. \tag{14}$$

The second column is

$$\begin{aligned} \lambda_{12}(t) &= -\dot{x}\,\lambda_{22}(t) - \xi_3\,\lambda_{32}(t) - \xi_4\,\lambda_{42}(t), \\ \lambda_{22}(t) &= [\dot{\xi}_3\dot{\eta}_4 - \dot{\xi}_4\dot{\eta}_3]/D, \\ \lambda_{32}(t) &= [E\dot{x}\dot{\eta}_4 - (g + E\dot{y})\dot{\xi}_4]/D, \\ \lambda_{42}(t) &= [(g + E\dot{y})\dot{\xi}_3 - E\dot{x}\dot{\eta}_3]/D. \end{aligned} \tag{15}$$

It will be seen that only the values at $t = t_0$ of the functions (15) enter the computations, so it is sufficient to compute them for this single value of t.

Moreover, even the values at $t = t_0$ enter only in finding the differential effects of change of location of the muzzle, and unless these effects are wanted we may omit the four functions (15) completely. The third column of the array is

$$
(16)\quad
\begin{aligned}
\lambda_{13}(t) &= [-\dot{x}(\dot{\eta}_3\eta_4 - \eta_3\dot{\eta}_4) \\
&\qquad - \xi_3(\dot{y}\dot{\eta}_4 + (g + E\dot{y})\eta_4) \\
&\qquad + \xi_4(\dot{y}\dot{\eta}_3 + (g + E\dot{y})\eta_3)]/D, \\
\lambda_{23}(t) &= [\dot{\eta}_3\eta_4 - \eta_3\dot{\eta}_4]/D, \\
\lambda_{33}(t) &= [\dot{y}\dot{\eta}_4 + (g + E\dot{y})\eta_4]/D, \\
\lambda_{43}(t) &= -[\dot{y}\dot{\eta}_3 + (g + E\dot{y})\eta_3]/D.
\end{aligned}
$$

It might be a help in computation to observe that the first of these is the sum of $-\dot{x}$ times the second, $-\xi_3$ times the third and $-\xi_4$ times the fourth. The fourth column of the array is

$$
(17)\quad
\begin{aligned}
\lambda_{14}(t) &= [-\dot{x}(\eta_3\dot{\xi}_4 - \eta_4\dot{\xi}_3) \\
&\qquad + \xi_3(\dot{y}\dot{\xi}_4 + E\dot{x}\eta_4) \\
&\qquad - \xi_4(\dot{y}\dot{\xi}_3 + E\dot{x}\eta_3)]/D, \\
\lambda_{24}(t) &= \eta_3\dot{\xi}_4 - \eta_4\dot{\xi}_3, \\
\lambda_{34}(t) &= -\dot{y}\dot{\xi}_4 - E\dot{x}\eta_4, \\
\lambda_{44}(t) &= \dot{y}\dot{\xi}_3 + E\dot{x}\eta_3.
\end{aligned}
$$

The remark after (16) applies here also.

We are now ready to use equations (1.19). However, we must observe that the quantities which in Section 1 were denoted by the symbols e_1, e_2, e_3, e_4 are here designated as 0, 0, e_4, e_5, because we have omitted the third of equations (VIII.3.11). Equations (1.19) can be put in the form

$$\xi(t) = \sum_{\nu=1}^{4} \xi_\nu(t)\, c_\nu(t),$$

(18)

$$\eta(t) = \sum_{\nu=1}^{4} \eta_\nu(t)\, c_\nu(t),$$

with the corresponding equations for the derivatives of ξ and η, the coefficients $c_\nu(t)$ being defined by

(19)
$$c_\nu(t) = c_\nu(t_o) + \int_{t_o}^{t} [\lambda_{\nu 3}(t)\, e_4(t) + \lambda_{\nu 4}(t)\, e_5(t)]\, dt,$$

where the initial values of the c_ν are

(20)
$$c_\nu(t_o) = \lambda_{\nu 1}(t_o) \Delta x_o + \lambda_{\nu 2}(t_o) \Delta y_o + \lambda_{\nu 3}(t_o) \Delta v_{xo} + \lambda_{\nu 4}(t_o) \Delta v_{yo}.$$

(Here it is understood that initial values for the trajectory are given at t_o and that Δx_o, etc., are the changes in these initial values.)

If we wish to find the effects of a change in initial conditions, it would be possible to compute the functions in (14), (15), (16) and (17) at $t = t_o$, compute the $c_\nu(t_o)$ by (20), observe that by (19) the $c_\nu(t)$ are constants because the e_i vanish identically, and finally compute $\xi(t)$ and $\eta(t)$ by (18). These are the differential effects on x and y at fixed t, due to the disturbance in initial conditions. From them we can deduce the differential effects on time and range at fixed y or on time and altitude at fixed x, if these are desired; we have only to apply (VII.1.10) or its special case (VIII.3.22). But because of the way in which we have selected the initial values of the second and third of the solutions of (1), we already have the differential effects of the most important changes in initial conditions; these are the functions ξ_3, η_3 and ξ_4, η_4 themselves.

If initial values are unchanged, but there is some disturbance which causes the e_i to be different from zero, we first compute the corresponding e_4 and e_5. The functions in (16) and (17) having been prepared, we form the integrands in the four integrals (19) and perform quadratures so that the $c_\nu(t)$ are known for the entire interval of values of t for which the normal trajectory was computed. Then equations (18) furnish us with the differential effects of the disturbance on x and on y at all times t from the start to the end of the computed trajectory. From these we may find the effects on range and time of flight at fixed y or on time of flight and altitude for fixed x if we so desire.

The process of computing a weighting factor curve is closely analogous to that in preceding sections. Suppose, for example, that we are studying the differential effects of a departure from standard temperature. The differential effects are desired at a point of the trajectory where time is T and y is y(T). Let t^* be a time intermediate between t_0 and T, and suppose that $\Delta\Theta$ is zero at times before t^* and is equal to Θ at all times after t^*. By (VIII.3.18) and (19) and (20), we see that

$$(21) \qquad c_\nu(T) = -\int_{t^*}^{T} \tfrac{1}{2}En(\dot{x}\lambda_{\nu 3} + \dot{y}\lambda_{\nu 4})\,dt.$$

With these coefficients we compute $\xi(T)$ and $\eta(T)$ by (18). If we wish the differential effect on x at fixed y, we apply (VIII.3.21) to obtain

$$(22) \qquad dx(\Delta\Theta \mid y = y(T)) = \xi(T) - \cot\theta\,\eta(T).$$

This is the quantity called $h(t^*)$ in Section 5 of Chapter VIII. From it we deduce either unit effect and weighting factor curve, or else norm effect and normalized effect curve. If the curves are wanted for several different values of T, the materials are at hand;

it is easy to change the upper limit in (21) and make the corresponding changes in (18).

The weighting factor curves and unit effects for differential effects of range winds can be obtained without performing the quadratures in (19) by using (VII.4.7, 8). Also there is a device by which the weighting factor curves and unit effects for the differential effects of departures from standard density can be found without performing these quadratures. The details will not be written out here. They will be found in the next section, in connection with the method of computation of differential effects explained there. The authors are of the opinion that the method in the next section is computationally preferable to the one just explained, and will leave further details of specific applications of the present method to any reader who may be interested in its use.

8. Differential corrections based on slope.

For trajectories having no specific end point, it is possible to compute the differential effects of disturbances at a smaller cost in time and effort than is required by the method of the preceding section. The method now to be described has as its central feature the matching of points of equal slope on normal and disturbed trajectories. Before beginning it two remarks of a general nature should be made. First, the method is equally applicable to anti-aircraft trajectories and to bomb trajectories. In the latter, however, it is customary to choose the y-axis positive downward. The adaptation of our formulas to this system of coordinates is quite easy; all that is needed is to replace h, g and a_1 everywhere by $-h$, $-g$ and $-a_1$ respectively. Second on bomb trajectories it will often be found that for large values of t the trajectory has been computed with a number of significant figures in $\dot{x}$ and $\ddot{x}$ inadequate for present purposes. However, E has a sufficient number of significant figures, so we can find improved values of $\dot{x}$ by integration of the equation $\ddot{x} = -E\dot{x}$.

From this we obtain

$$(1) \qquad \dot{x}(t) = \dot{x}(t_o) \exp \int_{t_o}^{t} (-E)\, dt.$$

In the system of equations (VIII.4.11) the first two can be solved independently of the last two; the last two can be solved afterward by a quadrature. We isolate the first two and write them in the homogeneous form, with $e_1 = 0$, obtaining

$$(2) \qquad \begin{aligned} d\rho/dt &= -(n+1)E\rho + [h - (n-2)a_1]E\dot{x}\eta, \\ d\eta/dt &= (2\dot{y}/\dot{x})\rho. \end{aligned}$$

From the trajectory sheet we can find $v^2/100$ for each tabular t, and with this and a table of n we can find n + 1 and n - 2. The trajectory sheet also furnishes us with E and $\dot{y}$, and $\dot{x}$ we obtain either from the trajectory sheet or if necessary from (1). So for each tabular value of t we can compute the three coefficients in (2). We suppose these prepared and tabulated. The numerical integration of (2) with given initial values is a fairly easy task. One reasonable procedure would be to extrapolate the second difference of $d\rho/dt$, compute ρ on the new line by Simpson's rule in the form (VI.3.1), compute $d\eta/dt$ by the second of equations (2), compute η by Simpson's rule in form (VI.3.1), and then compute $d\rho/dt$ by the first of equations (2). If this disagrees excessively with the original extrapolation the line must be re-computed.

We shall need two independent solutions of (2), in order to have a basis of solutions. Any two will be satisfactory. Usually it will be convenient to choose (1, 0) and (0, 1) as the two sets of initial values. Thus if, as usual, the trajectory begins at time t = 0, we select initial values

$$(3) \qquad \rho_1(0) = 1, \quad \eta_1(0) = 0; \quad \rho_2(0) = 0, \quad \eta_2(0) = 1.$$

As a result the determinant

$$(4) \qquad D(t) = \begin{vmatrix} \rho_1(t) & \rho_2(t) \\ \eta_1(t) & \eta_2(t) \end{vmatrix}$$

will have the initial value

$$(5) \qquad D(0) = 1.$$

However, this is not essential; it is merely somewhat convenient. The determinant D(t) could be computed without much trouble from the formula (4) after the two solutions of (2) have been found. However, by (1.6) we have

$$(6) \qquad D(t) = D(0) \exp \int_0^t [-(n+1)E]\,dt.$$

The integrand is already available, being one of the coefficients in (2). So the computation of D(t) by (6) is easy, especially if the initial values (3) are used so that D(0) = 1. Having the independent computation (6) for D(t) furnishes us with a valuable computational check on the numerical integration of (2).

After the four functions ρ_1, η_1, ρ_2, η_2 have been found, we can compute by quadratures the four functions

$$(7) \qquad \begin{aligned} \xi_1(t) &= \int_0^t 2\rho_1(t)\,dt, \\ \tau_1(t) &= \int_0^t (1/\dot{x})\,\rho_1(t)\,dt, \\ \xi_2(t) &= \int_0^t 2\rho_2(t)\,dt, \\ \tau_2(t) &= \int_0^t (1/\dot{x})\,\rho_2(t)\,dt. \end{aligned}$$

We tabulate these four functions. It is possible to proceed without computing them, but they are useful in simplifying the computation of differential effects of range winds and departures from standard density.

Since there are two equations in the system (2), equations (1.11) define four functions, namely,

$$(8) \qquad \begin{aligned} \lambda_{11}(t) &= \eta_2(t)/D(t), \\ \lambda_{12}(t) &= -\rho_2(t)/D(t), \\ \lambda_{21}(t) &= -\eta_1(t)/D(t), \\ \lambda_{22}(t) &= \rho_1(t)/D(t). \end{aligned}$$

Suppose now that there are disturbances which produce changes in initial conditions and also disturbances along the trajectory. The corresponding functions e_1, e_2, e_3, e_4 can be calculated by means of (VIII.4.12), while the initial values of the functions $\rho(t)$, etc., are furnished by (VIII.4.13). Then the differential effects at equal values of the slope m are given by the functions $\rho(t)$, etc. By (1.19) and (VIII.4.11) these functions have the values

$$(9) \qquad \begin{aligned} \rho(t) &= c_1(t)\,\rho_1(t) + c_2(t)\,\rho_2(t), \\ \eta(t) &= c_1(t)\,\eta_1(t) + c_2(t)\,\eta_2(t), \\ \xi(t) &= \xi(t_o) + \int_{t_o}^{t} (2\rho + e_3)\,dt, \\ \tau(t) &= \tau(t_o) + \int_{t_o}^{t} (\rho/\dot{x} + e_4)\,dt, \end{aligned}$$

wherein the two coefficients c_1, c_2 are defined by

$$c_1(t) = [\eta_2(t_o)\rho(t_o) - \rho_2(t_o)\eta(t_o)]/D(t_o) + \int_{t_o}^{t} [(\eta_2 e_1 - \rho_2 e_2)/D]\, dt,$$

$$(10)\qquad c_2(t) = [-\eta_1(t_o)\rho(t_o) + \rho_1(t_o)\eta(t_o)]/D(t_o) + \int_{t_o}^{t} [(-\eta_1 e_1 + \rho_1 e_2)/D]\, dt.$$

In these equations it is not necessary that t_o be 0. If it is, the first terms in the right members of (10) are somewhat simplified.

To find the effects of a change in initial conditions, we take $t_o = 0$ and all $e_i = 0$ in these equations. If the initial values of the two solutions of (2) are given by (3), by (3), (10) and (VIII.4.13) the c_i are the constants

$$(11)\qquad \begin{aligned} c_1 &= \rho(t_o) = \Delta v_{xo} - (E/g)(v_{xo}\Delta v_{yo} - v_{yo}\Delta v_{xo}), \\ c_2 &= \eta(t_o) = \Delta y_o + (v_{yo}/g v_{xo})(v_{xo}\Delta v_{yo} - v_{yo}\Delta v_{xo}). \end{aligned}$$

Then by (9), (7) and (VIII.4.13),

$$\rho(t) = c_1\rho_1(t) + c_2\rho_2(t),$$

$$\eta(t) = c_1\eta_1(t) + c_2\eta_2(t),$$

$$(12)\qquad \xi(t) = \Delta x_o + (1/g)(v_{xo}\Delta v_{yo} - v_{yo}\Delta v_{xo}) + c_1\xi_1(t) + c_2\xi_2(t),$$

$$\tau(t) = (1/v_{xo})(v_{xo}\Delta v_{yo} - v_{yo}\Delta v_{xo}) + c_1\tau_1(t) + c_2\tau_2(t).$$

From these the differential effects at fixed values of y can be found with the help of equations (VIII.4.16).

Consider next the differential effects of a disturbance which does not affect initial values, but produces terms e_i not all zero. We first compute the e_i by (VIII.4.12), and then by numerical quadrature we compute and tabulate the integrals

$$
(13)\qquad
\begin{aligned}
J_1(t) &= \int_0^t [(\eta_2 e_1 - \rho_2 e_2)/D]\,dt,\\
J_2(t) &= \int_0^t [(-\eta_1 e_1 + \rho_1 e_1)/D]\,dt,\\
J_3(t) &= \int_0^t [2(J_1\rho_1 + J_2\rho_2) + e_3]\,dt,\\
J_4(t) &= \int_0^t [(1/\dot{x})(J_1\rho_1 + J_2\rho_2) + e_4]\,dt.
\end{aligned}
$$

By (10), the first two of these are respectively $c_1(t)$ and $c_2(t)$. Hence by the first two of equations (9)

$$
(14)\qquad
\begin{aligned}
\rho(t) &= J_1(t)\,\rho_1(t) + J_2(t)\,\rho_2(t),\\
\eta(t) &= J_1(t)\,\eta_1(t) + J_2(t)\,\eta_2(t).
\end{aligned}
$$

This and the last two of equations (9) show that the last two functions in (13) are respectively $\xi(t)$ and $\tau(t)$. If q stands for the disturbance producing the functions e_i, by (VIII.4.16) and the preceding equations we have

$$
(15)\qquad
\begin{aligned}
&dx(q|y = y(T))\\
&\quad = J_3(T) - [\dot{x}(T)/\dot{y}(T)][J_1(T)\eta_1(T) + J_2(T)\eta_2(T)],\\
&dt(q|y = y(T))\\
&\quad = J_4(T) - [1/\dot{y}(T)][J_1(T)\eta_1(T) + J_2(T)\eta_2(T)],
\end{aligned}
$$

together with two equations for the differential effects on components of velocity which we shall omit.

If we were interested only in a single value of T, this method could not compete with that of Section 6. To obtain the quantities in (15) two numerical integrations of equations (2) were needed, plus the four quadratures (13). If only the differential effect on range is desired, this could be attained by the method of Section 6 at the cost of one numerical integration of equations (6.1), which is about as much work as one integration of (2), followed by a single quadrature (6.5). If differential effects on both range and time of flight are desired, the advantage of the method of Section 6 is much diminished, for then two numerical integrations of (6.1) and two quadratures would be needed. But if differential effects both on range and on time of flight are desired at as many as two different values of T, the four integrations and four quadratures called for by the method of Section 6 would require more work than is called for by the present method; and the advantage of this method increases with each increase in the number of values of T at which the differential effects are to be found.

If we compare the method of this section with that of Section 7, we notice first that the two integrations of (2) are considerably easier than the two integrations of (7.1), which amount to two integrations of a third-order system followed by two quadratures. Equations (2) are of second order, and have simpler coefficients than (7.1). The quadrature (6) is a trifle easier than the identical computation (7.13) because the integrand in (6) is already tabulated as one of the coefficients in (2). The four quadratures (13) are of the same order of difficulty as the four quadratures (7.21). But the remaining numerical work, consisting mostly of (15), is far less laborious than the computations (7.15, 16, 17, 18).

If the disturbance is of a type for which weighting factor curves are needed, we can find these curves by a modification of the method just described, although we shall see later that for the important cases of range wind, departure from standard density and departure from standard temperature easier methods can be found. Whatever be the type of disturbance, when the disturbance q has unit value, the functions e_i will have certain values given by (VIII.4.12). With these e_i we compute and tabulate the four integrals (13). Now let t^* be any time between 0 and the last tabular value of t, and let q^* be a disturbance of the type being discussed whose value is 0 at all times prior to t^* and is 1 at all times thereafter. The corresponding e_i are those for which the integrals (13) were computed when $t > t^*$; for $t < t^*$ the e_i are zero. By (10),

$$(16)\quad c_1(t) = J_1(t) - J_1(t^*), \quad c_2(t) = J_2(t) - J_2(t^*).$$

Then $\rho(t)$ and $\eta(t)$ are given by the first two of equations (9), while

$$(17)\quad \begin{aligned} \xi(t) &= \int_{t^*}^{t} (2\rho + e_3)\, dt \\ &= J_3(t) - J_3(t^*) - J_1(t^*)[\xi_1(t) - \xi_1(t^*)] \\ &\qquad - J_2(t^*)[\xi_2(t) - \xi_2(t^*)], \\ \tau(t) &= \int_{t^*}^{t} (\rho/\dot{x} + e_4)\, dt \\ &= J_4(t) - J_4(t^*) - J_1(t^*)[\tau_1(t) - \tau_1(t^*)] \\ &\qquad - J_2(t^*)[\tau_2(t) - \tau_2(t^*)]. \end{aligned}$$

By means of (VIII.4.16), from the functions $\eta(t)$ given by (9) and (16) and the functions $\xi(t)$ and $\tau(t)$ given by (17) we form the differentials $dx(q^*|y = y(T))$ and $dt(q^*|y = y(T))$ for any desired value of T. From these functions of t^* we can find the effect curves as in Section 5 of Chapter VIII, and from these effect curves we can compute either unit effects and weighting factor curves, or norm effects and normalized effect curves.

It is fortunate that for the important special cases of range wind, density and temperature effects a simpler procedure can be devised. Consider first the case of a range wind. Let w_x^* be a range wind which at all times before t^* has the value 0 and at all times thereafter has the value 1. We regard t^* as the initial time; no disturbance has taken place previously, and by regarding t^* as initial time and confining our attention to subsequent times we may consider that there is a wind of constant strength 1 at all times from initial time on. If the initial conditions (at time t^*) are changed by amounts

(18) $$\Delta x = \Delta y = \Delta v_y = 0, \quad \Delta v_x = 1,$$

by (VIII.4.13) the functions $\rho(t)$, etc., have the initial values

(19)
$$\begin{aligned} \rho(t^*) &= 1 + E\dot{y}(t^*)/g, \\ \eta(t^*) &= -[\dot{y}(t^*)]^2/g\dot{x}(t^*), \\ \xi(t^*) &= -\dot{y}(t^*)/g, \\ \tau(t^*) &= -\dot{y}(t^*)/g\dot{x}(t^*). \end{aligned}$$

To find the differential effects at a time t later than t^* produced by the change in initial values (18) at time t^*, we first apply (10) with the e_i set equal to zero and with t_0 chosen as t^*. The c_i are then constants, with values

(20)
$$\begin{aligned} c_1 &= \{\eta_2(t^*)[1 + E\dot{y}(t^*)/g] \\ &\quad + \rho_2(t^*)[\dot{y}(t^*)]^2/g\dot{x}(t^*)\}/D(t^*), \\ c_2 &= -\{\eta_1(t^*)[1 + E\dot{y}(t^*)/g] \\ &\quad + \rho_1(t^*)[\dot{y}(t^*)]^2/g\dot{x}(t^*)\}/D(t^*). \end{aligned}$$

The functions $\rho(t)$, etc., defined by (9) with these coefficients are the differential effects on v_x, etc., at equal values of m, caused by a change of 1 in $v_x(t^*)$. So by (VII.3.10) these are the quantities denoted by $[\delta v_x/\delta v_x(t^*)]_{m=m(T)}$, etc. With this notation equations (9) become

$$
(21)\quad
\begin{aligned}
&[\delta v_x/\delta v_x(t^*)]_{m(T)} = c_1\rho_1(T) + c_2\rho_2(T), \\
&[\delta y/\delta v_x(t^*)]_{m(T)} = c_1\eta_1(T) + c_2\eta_2(T), \\
&[\delta x/\delta v_x(t^*)]_{m(T)} \\
&\qquad = -\dot{y}(t^*)/g + c_1[\xi_1(T) - \xi_1(t^*)] \\
&\qquad\qquad + c_2[\xi_2(T) - \xi_2(t^*)], \\
&[\delta t/\delta v_x(t^*)]_{m(T)} \\
&\qquad = -\dot{y}(t^*)/g\dot{x}(t^*) + c_1[\tau_1(T) - \tau_1(t^*)] \\
&\qquad\qquad + c_2[\tau_2(T) - \tau_2(t^*)].
\end{aligned}
$$

Combining these equations with (VIII.4.16) and (VII.4.8) yields

$$
(22)\quad
\begin{aligned}
&dt(w_x^*|y = y(T)) \\
&\qquad = \dot{y}(t^*)/g\dot{x}(t^*) \\
&\qquad\quad - c_1[\tau_1(T) - \tau_1(t^*) - \eta_1(T)/\dot{y}(T)] \\
&\qquad\quad - c_2[\tau_2(T) - \tau_2(t^*) - \eta_2(T)/\dot{y}(T)], \\
&dx(w_x^*|y = y(T)) \\
&\qquad = T - t^* + \dot{y}(t^*)/g \\
&\qquad\quad - c_1[\xi_1(T) - \xi_1(t^*) - (\dot{x}(T)/\dot{y}(T))\eta_1(T)] \\
&\qquad\quad - c_2[\xi_2(T) - \xi_2(t^*) - (\dot{x}(T)/\dot{y}(T))\eta_2(T)],
\end{aligned}
$$

together with two equations for the differential effects on the components of velocity which we shall not exhibit.

Equations (22) have been used in the following way. For an appropriate selection of values of t^* (the selection $t^* = 2$, 4, and all multiples of 8 seconds was found serviceable) the coefficients c_1 and c_2 were computed by (20). The effects (22) were desired at certain values of Y which were exact multiples of 1000 feet;

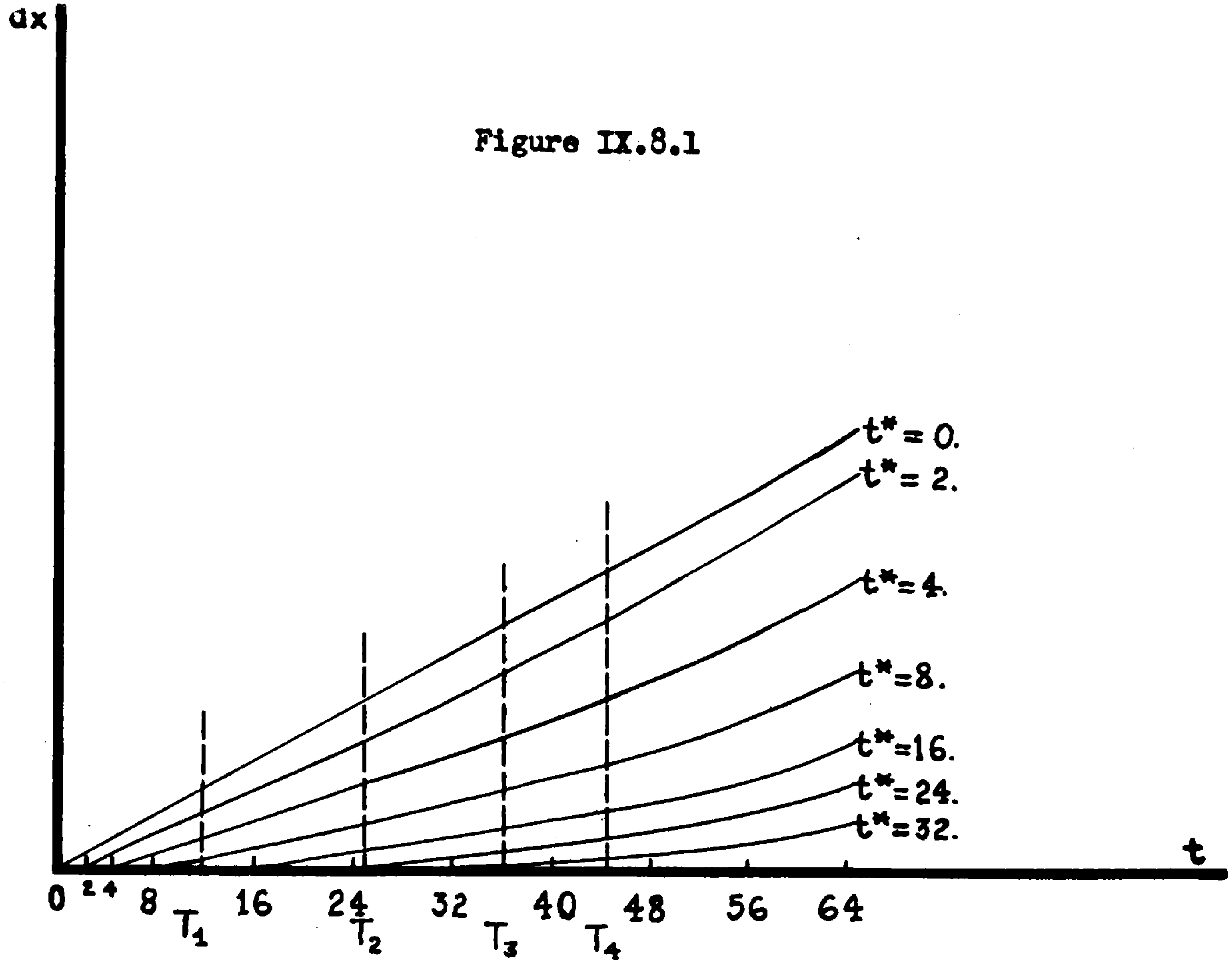
these corresponded to non-integral values of t, say T_1, T_2, ... Instead of interpolating for the values of the functions corresponding to these T_i, a graph of each function (22) was drawn, showing it as a function of T for each fixed selected t^*. Points on the graph were found for tabular values of t near the T_i, so that the graphs would be accurate at the T_i. The resulting graphs had the appearance of Figure 1. Now for each selected T_i the graphs furnish the values of $dx(w_x^* | y = y(T_i))$ and $dt(w_x^* | y = y(T_i))$ for the selected values of t^*. Thus for each fixed T_i the differential effects are known as functions of t^*. These are the same as the functions $h(t^*)$ of Section 5 of Chapter VIII from which the unit effects are weighting factor curves may be deduced.

There are three of the more important disturbances, namely, departure from standard density, departure from standard temperature and change of G, which share the property that $e_2 = e_3 = e_4 = 0$, as is shown by (VIII.4.12). It is possible to utilize this common property in such a way as to avoid the quadratures (13), or rather to replace these quadratures by a much simpler computation. Let $\varepsilon_1(t)$ be a function continuous for $t_0 \leq t \leq T$. To be specific, we shall suppose that we are interested in the effect on range at height $y = y(T)$; a similar discussion applies to effect on time of flight. If a disturbance (say in density) is absent until time t^* and from t^* to T is present in such magnitude as to produce $e_1 = \varepsilon_1(t)$, it will have a certain differential effect, which we will call $f(t^*, T)$. By (6.4) we know that there is a certain function $\lambda_1(t)$ such that

$$(23) \qquad f(t^*, T) = \int_{t^*}^{T} \lambda_1(t)\, \varepsilon_1(t)\, dt;$$

and we shall make use of this. However, the real point of the present discussion is the avoidance of the actual computation of this and related integrals.

Figure IX.8.1



Next we suppose that $q(t)$ is a unit disturbance of any one of the three types mentioned in the first sentence of the preceding paragraph. To this disturbance corresponds a continuous function $e_1(t)$; as already stated, $e_2 = e_3 = e_4 = 0$. Let $q_{t*}(t)$ be a similar disturbance which is 0 before time t^* and is 1 thereafter. We denote by $h(t^*, T)$ the differential effect of q_{t^*} on range x at height $y = y(T)$:

$$h(t^*, T) = dx(q_{t*} | y = y(T)).$$

This is the same as the $h(t^*)$ of Section 5 of Chapter VIII. Again by (6.4),

$$h(t^*, T) = \int_{t^*}^{T} e_1(t)\, \lambda_1(t)\, dt. \tag{24}$$

Now let $t_1 \leq t \leq t_2$ be an interval on which neither ε_1 nor λ_1 is zero. Let b and B denote the least and greatest values of the ratio $e_1(t)/\varepsilon_1(t)$ for the interval $t_1 \leq t \leq t_2$. Then on this interval we have

$$b\varepsilon_1(t)\lambda_1(t) \leq e_1(t)\lambda_1(t) \leq B\varepsilon_1(t)\lambda_1(t) \tag{25}$$

if $\varepsilon_1 \cdot \lambda_1 > 0$, the inequalities being reversed if $\varepsilon_1 \cdot \lambda_1 < 0$. By integrating from t_1 to t_2,

$$\begin{aligned} b[f(t_2, T) - f(t_1, T)] &\leq h(t_2, T) - h(t_1, T) \\ &\leq B[f(t_2, T) - f(t_1, T)], \end{aligned} \tag{26}$$

the inequalities being reversed if $\varepsilon_1 \cdot \lambda_1 < 0$. In any case, $h(t_2, T) - h(t_1, T)$ is the product of

$$[f(t_2, T) - f(t_1, T)]$$

by a number between b and B, which by the continuity of e_1/ε_1 is the value of $e_1(\bar{t})/\ _1(\bar{t})$ at some $\bar{t}$ between t_1 and t_2. We have thus shown that if $\varepsilon_1 \cdot \lambda_1 \neq 0$ between t_1 and t_2, <u>there is a number</u> $\bar{t}$ <u>between</u> t_1 <u>and</u> t_2 <u>such that</u>

$$\begin{aligned} &h(t_2, T) - h(t_1, T) \\ &\quad = [e_1(\bar{t})/\varepsilon_1(\bar{t})] \cdot [f(t_2, T) - f(t_1, T)]. \end{aligned} \tag{27}$$

This statement attains usefulness by way of a judicious choice of $\varepsilon_1(t)$. We select

$$\varepsilon_1(t) = - E\dot{x}[h - (n - 2)a_1].$$

If we substitute this in place of e_1 in (VIII.4.11), along with $e_2 = e_3 = e_4 = 0$, the resulting equations are

$$(28)\quad \begin{aligned} d\rho/dt &= -(n + 1)E\rho + [h - (n - 2)a_1]E\dot{x}(\eta - 1), \\ d\eta/dt &= 2\dot{y}\rho/\dot{x}, \\ d\xi/dt &= 2\rho, \\ d\tau/dt &= (1/\dot{x}) \end{aligned}$$

Thus we see that from time t^* on, the four functions $\rho(t)$, $\eta(t) - 1$, $\xi(t)$, $\tau(t)$ satisfy the homogeneous equations (VIII.4.11). At time t^* the values of x, y, $\dot{x}$ and $\dot{y}$ are unaffected by the disturbance, so by (VIII.4.13) all four functions $\rho(t^*)$, $\eta(t^*)$, $\xi(t^*)$, $\tau(t^*)$ are zero. Since $\rho(t)$ and $\eta(t) - 1$ satisfy (2), they must be linear combinations of the two pairs of functions in the basis, so that there are constants c_1, c_2 such that

$$(29)\quad \begin{aligned} \rho(t) &= c_1\rho_1(t) + c_2\rho_2(t), \\ \eta(t) - 1 &= c_1\eta_1(t) + c_2\eta_2(t). \end{aligned}$$

The coefficients can be found by setting $t = t^*$; they are

$$(30)\quad c_1 = \rho_2(t^*)/D(t^*),\ c_2 = -\rho_1(t^*)/D(t^*).$$

From (28) and the fact that ξ and τ vanish at t^* we find

$$(31)\quad \begin{aligned} \xi(t) &= c_1[\xi_1(t) - \xi_1(t^*)] + c_2[\xi_2(t) - \xi_2(t^*)], \\ \tau(t) &= c_1[\tau_1(t) - \tau_1(t^*)] + c_2[\tau_2(t) - \tau_2(t^*)]. \end{aligned}$$

Thus at altitude $y = y(T)$ the disturbance $\varepsilon_1(t)$ has the differential effects (recalling (VIII.4.16))

$$dx(\varepsilon_1 | y = y(T))$$

$$= -(\dot{x}(T)/\dot{y}(T))$$

$$+ \{\rho_2(t^*)[\xi_1(T) - (\dot{x}(T)/\dot{y}(T))\eta_1(T) - \xi_1(t^*)]$$

$$- \rho_1(t^*)[\xi_2(T) - (\dot{x}(T)/\dot{y}(T))\eta_2(T) - \xi_2(t^*)]\}/D(t^*),$$

(32)

$$dt(\varepsilon_1 | y = y(T))$$

$$= -1/\dot{y}(T)$$

$$+ \{\rho_2(t^*)[\tau_1(T) - (1/\dot{y}(T))\eta_1(T) - \tau_1(t^*)]$$

$$- \rho_1(t^*)[\tau_2(T) - (1/\dot{y}(T))\eta_2(T) - \tau_2(t^*)]\}/D(t^*),$$

together with expressions for differential effects on components of velocity which we shall not write.

To be specific, let us suppose that we are trying to find differential effects on range; differential effects on time of flight can be treated in the same way, using the second of equations (32) instead of the first. Let the left member of the first of equations (32) be denoted by $f(t^*, T)$. Let us select any one of the several values of T at which the differential effects are wanted. For this fixed T we compute $f(t^*, T)$ for a collection of values of t^*. This is somewhat easier than might be suspected from a first glance at (32). The first step would be to compute the three numbers

$$\dot{x}(T)/\dot{y}(T),$$

(33)

$$\xi_1(T) - (\dot{x}(T)/\dot{y}(T))\eta_1(T),$$

$$\xi_2(T) - (\dot{x}(T)/\dot{y}(T))\eta_2(T).$$

If t^* is any tabular value of t between 0 and T, we can compute $f(t^*, T)$ on a computing machine without writing any intermediate steps. First we set $\rho_2(t^*)$ on the keyboard, multiply it by the second of numbers (33) and then by $-\xi_1(t^*)$. The result is left on the dials, and the keyboard re-set to the number $\rho_1(t^*)$. This is multiplied by the negative of the last of the numbers (33) and by $\xi_2(t^*)$. The long row of dials now shows the quantity in braces in the first of equations (32). This is divided by $D(t^*)$. The quotient is set into the keyboard, and the first of numbers (33) added to it to furnish $f(t^*, T)$.

Now let us apply this to find the effects of a departure from standard density. A unit change in density ratio is one for which $\Delta H = H$ for all y. The corresponding e_1, by (VIII.4.12), is

(34) $$e_1 = -E\dot{x}.$$

At any particular T, the differential effect on range produced by a departure from standard density such that $\Delta H = 0$ before time t^* and $\Delta H = H$ thereafter is $h(t^*, T)$. If the interval from 0 to T is subdivided into small intervals by points

$$t_o = 0, t_1, t_2, \ldots, T,$$

each small enough so that $\varepsilon_1\lambda_1$ does not change sign on it (note that ε_1 is positive), then (27) holds. Although $\bar{t}$ is unknown, a good approximation can be had by replacing $e_1(\bar{t})/\varepsilon_1(\bar{t})$ by the arithmetic mean of its values at the beginning and end of the small interval. But

(35) $$e_1/\varepsilon_1 = 1/[h - (n - 2)\, a_1].$$

Hence, approximately,

(36) $$h(t_i, T) - h(t_{i-1}, T) = (Av(35))[f(t_i, T) - f(t_{i-1}, T)],$$

where (Av (35)) means the average of the values of the quantity in (35) at t_{i-1} and at t_i. Since $h(T, T) = 0$, by addition of the differences in (36) we can find $h(t_i, T)$ for each t_i.

In case $a_1 = 0$, the process is a little simpler, for then the quantity (35) is always $1/h$, and therefore

(37) $$h(t, T) = f(t, T)/h.$$

In either case, from $h(t, T)$ we proceed as in Section 5 of Chapter VIII to compute either unit effects and weighting factor curves, or else norm effects and normalized effect curves, completing the task.

Next consider a departure from standard temperature such that $\Delta \Theta/\Theta = 1$. By (VIII.4.12), the corresponding e_1 is

(38) $$e_1(t) = E\dot{x}(n - 2)/2.$$

Therefore

(39) $$e_1(t)/\varepsilon_1(t) = -(n - 2)/2[h - (n - 2)a_1].$$

As before, let $h(t^*, T)$ be the differential effect on range produced by a departure from standard temperature such that $\Delta \Theta = 0$ before time t^* and $\Delta\Theta = \Theta$ after time t^*. Then by (27) we have approximately

$$h(t_i, T) - h(t_{i-1}, T) = (\text{Av}(39))[f(t_i, T) - f(t_{i-1}, T)],$$

where Av(39) is the average of the values of the quantity in (39) at t_{i-1} and at t_i. Since $h(T, T) = 0$, this permits us to find by addition the values of $h(t_i, T)$, which as usual can be used to compute either unit effect and weighting factor curve, or else norm effect and normalized effect curve.

A similar procedure can be applied to a change in G, but in this case only the differential effect is desired; no weighting factor curves are required.

9. Non-linear effects of large temperature changes.

In the process of finding the differential effects of disturbances, there was in essence a double linearization. In the first step, equations (VIII.2.17) were shown to provide linear functionals of the Δf_i which approximated the effects of the disturbances Δf_i with an error which vanished to the second order with the norm of the disturbances. In the second step, the Δf_i themselves were replaced by approximations linear in the several variables appearing as parameters in the f_i, which in the ballistic applications were the density, relative sound velocity, etc. This second step in linearization leads to no additional error in the case of density, since E is already linear in H. But the temperature Θ, expressed in degrees Kelvin, enters through the relative sound velocity

$$(1) \qquad a = \sqrt{\Theta/288},$$

which occurs in the equation

$$(2) \qquad E = \gamma \, H(y) \, a \, G(u/a).$$

The dependence of E on Θ is therefore by no means exactly linear. We may anticipate, as a result, that the differential effect of a fairly large departure from standard temperature will be a less accurate approximation to the actual effect than will the differential effect of a comparably large departure from standard density. This is in fact the case. Differential corrections are adequately accurate for small departures from standard temperatures, but leave rather excessive residual errors when the departure from standard temperature is as much as 50 degrees or 60 degrees Centigrade.

Insofar as these departures from standard temperature are caused by fluctuations, not much can be done about them. But at high altitudes there is not much fluctuation, and it is possible to keep the departures from standard rather small by judicious choice of standard temperature law. Unfortunately, in the case of the trajectories of bombs we are handicapped in the choice of temperature law by the need of using a constant standard temperature. It was pointed out in Section 2 of Chapter IV that by assuming $a_1 = 0$, it was made possible to eliminate one parameter and thereby greatly reduce the number of trajectory computations needed to prepare a set of bomb ballistic tables. Although the assumption of constant temperature of 59 degrees Fahrenheit is rather far from the truth, it leads to no serious trouble when the altitude of release does not exceed some 25,000 feet. For then in the upper portions of the trajectory $\Delta \Theta$ may be quite large, but the bomb has not yet reached velocities near the velocity of sound, so E is not yet large and the factor n - 2 is also not much different from zero. In the lower parts of the trajectory (n - 2)E is larger, but these parts of the trajectory are in the warmer levels of the atmosphere near the ground, and so $\Delta \Theta$ is not so large. However, when the release altitude exceeds 25,000 feet the effect of departure from standard temperature becomes more significant. But it is hardly feasible to make a differential correction for departure from standard temperature. At stratospheric levels the departure from the constant standard of 59 degrees Fahrenheit is normally over 100 degrees Fahrenheit, and the differential effect of such a departure from standard temperature may be a very poor approximation to the actual effect.

Let us then envisage ourselves in the following situation. A number of trajectories have been computed, with an assortment of values of v_0 and γ_s (or C_s), each running through a range of values of y beginning with 0 and ascending to a bound greater than any reasonable bombing altitude. Each of these is computed on the assumption that standard sound velocity ratio a is constantly equal to 1. From them a set of ballistic tables has been prepared, showing range and time of flight as functions of release altitude Y, initial velocity v_0 and reciprocal ballistic coefficient γ.

Also, unit effects and weighting factor curves are prepared for several types of disturbance, including departure from standard density. Now we choose a new standard relative sound velocity function $a^*(y)$ which is a better approximation to average conditions than the original standard $a = 1$. We wish to find adequately accurate answers to two questions:

(A). Suppose that a bomb of reciprocal ballistic coefficient γ is dropped from altitude Y with speed v_0, all conditions being standard including the relative sound velocity, which is the new standard a^*. The bomb will have a certain range X and a certain time of flight T. If in the ballistic tables we find the reciprocal ballistic coefficient of a bomb whose range is X when release altitude is Y and initial velocity is v_0, the value we find will not be γ, but will be some other value $\gamma_{b,X}$, because the ballistic tables were prepared under the assumption $a = 1$. (The subscripts b, X are intended to signify that the γ was found from the range by using the ballistic tables.) Similarly, from the ballistic tables we find that if a bomb has time of flight T when dropped from altitude Y at speed v_0, its ballistic coefficient must be $\gamma_{b,T}$, which will be different from γ because of the assumption $a = 1$ underlying the ballistic tables. How can we determine $\gamma_{b,X}$ from γ, and conversely; and how can we determine $\gamma_{b,T}$ from γ, and conversely?

(B). On any given day, the temperature will not be the same as standard. How can we find the differential correction for the effect of the difference between actual temperature and that corresponding to the new standard relative sound velocity function a^*?

If the drag function on which the tables are based is not greatly different from the actual drag function of a given bomb, after corrections have been made for all departures from standard conditions we should anticipate that experimental range bombings from various altitudes will provide nearly equal values for the reciprocal ballistic coefficient. But if the tables have been based on constant 59 degree Fahrenheit temperature, and no correction is made for departure from standard temperature, it is to be expected that experiments at different altitudes will lead to different estimates for γ, the variability being especially marked when the release altitude is great. This makes extrapolation to altitudes and speeds beyond those of the experiments both more difficult and more doubtful. If we can answer (A) and (B) with satisfactory accuracy, we will be able to correct for effects of temperature and thus (presumably) to obtain more nearly constant values of γ from the experiments, with consequent gain in ease and trustworthiness of extrapolation.

Since we are interested in effects of temperature changes and not in effects of change of initial conditions, we shall assume that the bomb is launched horizontally with speed v_0 at height Y. We can choose the origin vertically below the point of release and integrate the equations of motion with initial values $x(0) = 0$, $y(0) = Y$, $\dot{x}(0) = v_0$, $\dot{y}(0) = 0$. At the point at which $y = 0$ the x-coordinate will be the range X and the time will be the time of flight T. It is of course not usual to choose the axes in this way, since the axis system in Section 2 of Chapter IV is more convenient in practice. Nevertheless the present choice is logically sound and is more easily fitted into the discussion about to be presented.

The range X will depend on the reciprocal ballistic coefficient γ, on the function H which specifies relative air density as a function of altitude y, and on the function a which specifies relative sound velocity as a function of altitude y. In the terminology of Section 1 of Chapter VIII, X is a functional of these two functions; and in accordance with the notation we have been using for functionals, we write

$$X = \mathfrak{X}[\gamma, H(\ ), a(\ )]. \tag{3}$$

Likewise the time of flight T also depends on these same variables, so that

$$(4) \qquad T = T[\gamma, H(\ ), a(\ )].$$

The dependence on Y and v_0 and g will be omitted from the notation, since these quantities are not to be changed in this discussion. If the standard density function is denoted by $H^*(y)$, the ranges and times of flight in the ballistic tables, for initial velocity v_0 and altitude Y, are the quantities

$$(5) \qquad X = X[\gamma, H^*(\ ), 1], \quad T = T[\gamma, H^*(\ ), 1].$$

Suppose now that at some time the relative sound velocity function is a(y). For given values Y, v_0 and γ_b of altitude, initial velocity and reciprocal ballistic coefficient the normal trajectory is the solution of the equations

$$(6) \qquad \begin{aligned} \ddot{x} &= -\gamma_b H^*(y)\, G(v)\, \dot{x}, \\ \ddot{y} &= -\gamma_b H^*(y)\, G(v)\, \dot{y} - g. \end{aligned}$$

For each y we can find the corresponding v and compute the ratio

$$(7) \qquad R(y) = G(v)/a(y)G(v/a(y)).$$

In (6) we substitute the value of G(v) from (7), obtaining

$$(8) \qquad \begin{aligned} \ddot{x} &= -\gamma_b [R(y)H^*(y)]\, a(y)\, G(v/a(y))\, \dot{x}, \\ \ddot{y} &= -\gamma_b [R(y)H^*(y)]\, a(y)\, G(v/a(y))\, \dot{y} - g. \end{aligned}$$

For the given v_0 and Y the solutions of (6) and (8) are identical, since the equations are in fact identical. Hence at y = 0 the solutions of (6) have the same values of x and t as the solutions of (8).

In the notation (5) and (4)

$$(9)\qquad \begin{aligned} X[\Upsilon_b, R(\ )H^*(\ ), a(\ )] &= X[\Upsilon_b, H^*(\ ), 1], \\ T[\Upsilon_b, R(\ )H^*(\ ), a(\ )] &= T[\Upsilon_b, H^*(\ ), 1]. \end{aligned}$$

If density is normal, or (more realistically) if effects of departure from normal density have been removed, the quantities of interest to us are the range and time of flight corresponding to density law H^* and relative sound velocity $a(y)$. If the reciprocal ballistic coefficient of a bomb is Υ, and it is dropped at initial velocity v_0 from height Y, and all conditions are standard except that the sound velocity function is $a(y)$, the range and time of flight will be

$$(10)\ X = X[\Upsilon, H^*(\ ), a(\ )],\ T = T[\Upsilon, H^*(\ ), a(\ )].$$

With the given altitude and initial velocity, the ballistic tables will show the range X as that corresponding to a certain reciprocal ballistic coefficient Υ_b. This is what we called $\Upsilon_{b,X}$ in (A); temporarily we are omitting the subscript X for ease in printing. So by (5),

$$(11)\qquad X[\Upsilon, H^*(\ ), a(\ )] = X[\Upsilon_b, H^*(\ ), 1].$$

A similar equation would hold for time of flight, but the Υ_b would then be what was termed $\Upsilon_{b,T}$ in (A). By (9) and (11),

$$(12)\ \ X[\Upsilon, H^*(\ ), a(\ )] = X[\Upsilon_b, R(\ )H^*(\ ), a(\ )].$$

Our approximate evaluation of the terms in (12) will be based on two assumptions. The first is that the difference between the right member of (12) and the corresponding term with the second argument replaced by the standard $H^*(y)$ can be adequately estimated by means of the differential effect of change of density on range. We may feel confident of this, because many numerical trials have shown that the linear approximation to effect of change of density is quite accurate over a large range of density change. The second assumption is that the differential effect of replacing, say, $X[\gamma_b, H^*(\ ), a^*(\ )]$ by $X[\gamma_b, R^*(\ )H^*(\ ), a^*(\ )]$, can be adequately estimated by using the weighting factor curve for effect of change of density on range which was computed for the normal trajectory with the same γ_b, v_o and Y. This is probably a satisfactory assumption, since the weighting factor curves for effect of change of density on X or T at a given Y are seen by inspection to be insensitive to change of γ or v_o. So the weighting factor curve which we should use, namely the one corresponding to the trajectory with the same γ_b, v_o and Y, standard density H^* and standard relative sound velocity a^*, is presumably not much different from the one with the same γ_b, v_o and Y, and the same H^*, but with $a = 1$; and the latter has the virtue of being available, while the former is not.

Let p(k) be the function whose graph is the weighting factor curve for differential effect of departure from standard density, determined for altitude Y with reciprocal ballistic coefficient γ and initial velocity v_o. At altitude kY ($0 \leq k \leq 1$) the density used in computing $X[\gamma_b, R(\ )H^*(\ ), a(\ )]$ is R(kY) times standard density at that altitude. The ballistic density (for range) defined as in (VIII.5.12) is then

$$(13)\quad B_b = B_b[\gamma_b, v_o, Y, a(\ )] = \int_0^1 R(kY)\, dp(k).$$

Then to first-order terms the equation

$$(14)\quad X[\gamma_b, R(\ )H^*(\ ), a(\ )] = X[\gamma_b, B_bH^*(\ ), a(\ )]$$

is satisfied, by the definition of ballistic density, or ballistic density-excess. But it is evident from (6) that multiplying H^* by the constant B_b has the same effect as multiplying γ by B_b. Hence (14) implies

$$(15)\quad X[\gamma_b, R(\ )H^*(\ ), a(\ )] = X[\gamma_bB_b, H^*(\ ), a(\ )].$$

For given Y, v_o, given density law H^* and given relative sound velocity $a(y)$, different values of γ correspond to different ranges. By (12) and (15), γ and $\gamma_b B_b$ correspond to the same range, and therefore must be equal:

$$(16) \qquad \gamma = \gamma_b B_b[\gamma_b, v_o, Y, a(\ )].$$

In particular, if the relative sound velocity function is the new standard $a^*(y)$ we shall use the abbreviation

$$(17) \quad B_b = B_b[\gamma_b, v_o, Y] = B_b[\gamma_b, v_o, Y, a^*(\ )].$$

By use of (13) we can compute the values of this function for a collection of values of γ_b, v_o and Y. This furnishes us with the answer to question (A). If the relative sound velocity function at the time of a range bombing were $a^*(y)$, other conditions being standard, and its reciprocal ballistic coefficient deduced from range is γ_b according to the ballistic tables, then its reciprocal ballistic coefficient is actually $\gamma = \gamma_b B_b[\gamma_b, v_o, Y]$. Conversely, given γ, the range of a bomb dropped under new standard conditions can be found from the ballistic tables by solving (16) for γ_b and then looking up the range determined by v_o, Y and γ_b according to the ballistic tables. This latter process can be made a bit easier by first re-tabulating B_b as a function of γ, v_o and Y. Having determined the function (17), for each tabular v_o and Y the value of γ_b determines both γ and B_b. The latter is tabulated against the former, and by interpolation B_b can be found for equally spaced values of γ. In this form we denote it by

$$(18) \qquad B_b = B[\gamma, v_o, Y],$$

the subscript b being omitted from the functional symbol B to remind us that it is also absent from the γ inside the brackets. In terms of this new function, (16) becomes

$$(19) \qquad \gamma_b = \gamma / B[\gamma, v_o, Y].$$

The foregoing discussion has been in terms of range, merely for brevity. A similar discussion applies to time of flight also. For this we would use in (13) the weighting factor curve for effect of departure from standard density on time of flight. To distinguish the γ_b of the preceding paragraphs from that when the weighting factor curve for effect on time of flight is used, we shall add a subscript X to the γ_b and B_b of the preceding paragraphs, and add a subscript T when the weighting factor curve for effect on time of flight is used. Thus under the new standard conditions, (16), (19) and their analogues for time of flight take the form

$$
\begin{aligned}
\gamma &= \gamma_{bX} B_{bX}[\gamma_{bX}, v_o, Y] \\
&= \gamma_{bT} B_{bT}[\gamma_{bT}, v_o, Y], \\
\gamma_{bX} &= \gamma / B_X[\gamma, v_o, Y], \\
\gamma_{bT} &= \gamma / B_T[\gamma, v_o, Y].
\end{aligned}
\tag{20}
$$

The computation in (13) is a rather straightforward one. For a set of tabular values of t on the normal trajectory the corresponding altitude is found, and from this the new standard $a^*(y)$ is found. From the trajectory sheet we read $v^2/100$, divide it by $[a^*(y)]^2$, and use the G-table to find $G(v/a^*)$. We now compute R by (7). Knowing the altitude y we compute $k = y/Y$ and read $p(k)$ from the weighting factor curve. For each pair of consecutive values of k we multiply the difference of the values of $p(k)$ by the average of the values of R. The result is approximately the integral in (13).

Next we take up the problem of making corrections for departures from the new standard relative sound velocity law. Suppose that the relative sound velocity $a(y)$ exceeds the standard $a^*(y)$ by an amount $\Delta a(y)$. If all the other conditions are standard, the range will be $X[\gamma, H^*(\ ), a(\ )]$, while if temperature also had been (new) standard, the range would have been $X[\gamma, H^*(\ ), a^*(\ )]$. The effect of the departure from standard temperature is the former of these minus the latter. That is,

$$(21)\qquad \Delta X(\Delta a \mid y = Y) = X[\gamma, H^*(\ ), a(\ )] - X[\gamma, H^*(\ ), a^*(\ )].$$

By (11) the first term on the right is $X[\gamma_b, H^*(\ ), 1]$. We define γ_b^* by (11) with a^* in place of a. Then (21) becomes

$$(22)\qquad \Delta X(\Delta a \mid y = Y) = X[\gamma_b, H^*(\ ), 1] - X[\gamma_b^*, H^*(\ ), 1].$$

To first-order terms, the right member is the product of $\partial X/\partial\gamma$ evaluated at γ_b by the difference $\gamma_b - \gamma_b^*$. Temporarily we use R^* for the quantity defined by (7) with a^* in place of a, and B_b^* for the quantity defined by (13) with R^* in place of R. Then by (16)

$$(23)\qquad \gamma_b^* B_b^* = \gamma_b B_b,$$

whence

$$\begin{aligned} \gamma_b - \gamma_b^* &= -(\gamma_b^*/B_b)(B_b - B_b^*) \\ (24)\qquad &= (\gamma_b^*/B_b) \int_0^1 \left\{ \frac{G(v)}{a^*(kY)G(v/a^*(kY))} - \frac{G(v)}{a(kY)G(v/a(kY))} \right\} dp(k). \end{aligned}$$

If we multiply both members by $\partial X/\partial\gamma$, evaluated at γ_b, we obtain the first-order estimate of ΔX. If we replace a by $a^* + \Delta a$ and expand to first-order terms, we further obtain

$$(25)\qquad \Delta X(\Delta a \mid y = Y) = -(\gamma_b^*/B_b)[\partial X/\partial\gamma]_{\gamma b} \int_0^1 \frac{\Delta\Theta}{2\Theta^*}(n-2)R^*(kY)\, dp(k).$$

Here we have replaced $\Delta a/a^*$ by $\Delta \Theta/2\Theta^*$, which is correct to first-order terms. To first order, we identify γ_b^* and γ_b in the right member of (25). The right member of (25) is the linear (principal) part of the left member, and is therefore the same as $dx(\Delta a | y = Y)$ or $dx(\Delta \Theta | y = Y)$. The computation of the right member for a given disturbance $\Delta \Theta$ can best be done by preparing normalized effect curves. To do this we choose a set of tabular values of k, and for each one of these we compute the right member for the particular $\Delta \Theta$ which is 0 for greater values of k and is Θ for smaller values of k. That is, we compute the usual finite sum which approximates

$$(26) \qquad \int_0^k [(n - 2)/2] R^*(kY) \, dp(k)$$

for each tabular k, and we multiply this by the coefficient of the integral in (25). The product is a function of k whose graph is the "effect curve." From this we can deduce the norm effect and the normalized effect curve, or if preferred we can deduce the unit effect and the weighting factor curve. As an alternative procedure we could compute the integral (26) and refrain from multiplying it by the coefficient of the integral in (25). If this resulting function of k is graphed, the result is the "effect curve" for the differential effect of departure from new standard temperature on the coefficient B_b. For this differential effect we could compute norm effects and normalized effect curves, and for any given departure from new standard temperature we could use these to find the effect on B_b. With the corrected B_b we would then use (16) to compute γ. The advantage of this procedure would be that (26) is more nearly constant than (25), so that interpolation would be easier.

Chapter X

B O M B I N G

1. The bombing problem.

It is our purpose here to consider certain problems which are not part of the general problem of exterior ballistics, the problem of predicting the motion of a projectile, but which are closely related to this. We suppose that the principal problem of exterior ballistics for bombing has been solved, and that trajectories, computed for each bomb for all usable combinations of air speed and altitude of release, are available. The question is, in what form is the ballistic data most readily usable by the bombardier, and what will his procedures be?

We first list some of the assumptions under which the operation of bombing is usually performed. It is assumed that the atmosphere has a standard density and temperature structure, that the acceleration due to gravity has a standard value, that the target is at sea-level, and that the launching is perfect, there being no initial angular velocity and the axis of the bomb pointing in the direction of the initial velocity. The earth is supposed flat and the motion of the aircraft is presumed to be straight-line motion in a horizontal plane at a uniform speed. However, it is not assumed that there is no wind. The vertical component of the wind is supposed to be zero, and the horizontal component is supposed constant, independent of the altitude. Even under these restrictive assumptions the bombing problem is not trivial.

First, let us consider the motion of the bomb from the point of view of the man doing the bombing. The bomb is released in the bomb bay and is subject to two major forces, the force of gravity, and the drag. Since the first of these is directed downward and the second is initially opposite in direction to the motion of the plane, the bomb must appear to the bombardier to fall down and back from the airplane. The bomb, in fact, will never leave the vertical plane which is attached to the airplane and contains the velocity vector of the aircraft. The most natural description of the trajectory, from the point of view of the man in the airplane, would be to list the distance of the bomb behind the airplane and the depth of the bomb beneath the airplane against the time from release. It is this fact which motivates the definition of <u>trail</u> which we give.

(1) <u>Definition</u>. Suppose that a bomb is dropped from an aircraft moving horizontally with constant velocity. The <u>trail</u>, r, is the horizontal component of the vector from the position of the aircraft at the impact of the bomb to the position of impact. The symbol T (time of flight) is used for the time from release of the bomb to impact.

Both r and T depend on the air speed, u, the altitude at release, Y, and on the particular bomb. Under standard bombing table conditions there is no other dependence.

The condition that a bomb dropped be a hit can now be stated concisely. The bombardier must fly along such a straight line and drop his bomb at such a time that T seconds later the target lies r feet directly behind him. Thus the bombing table consists primarily of trail and time of flight, or quantities equivalent to these, listed against air speed and altitude. Both trail and time of flight must be considered as important, in contrast to the situation as regards artillery firing against fixed targets where only one func-

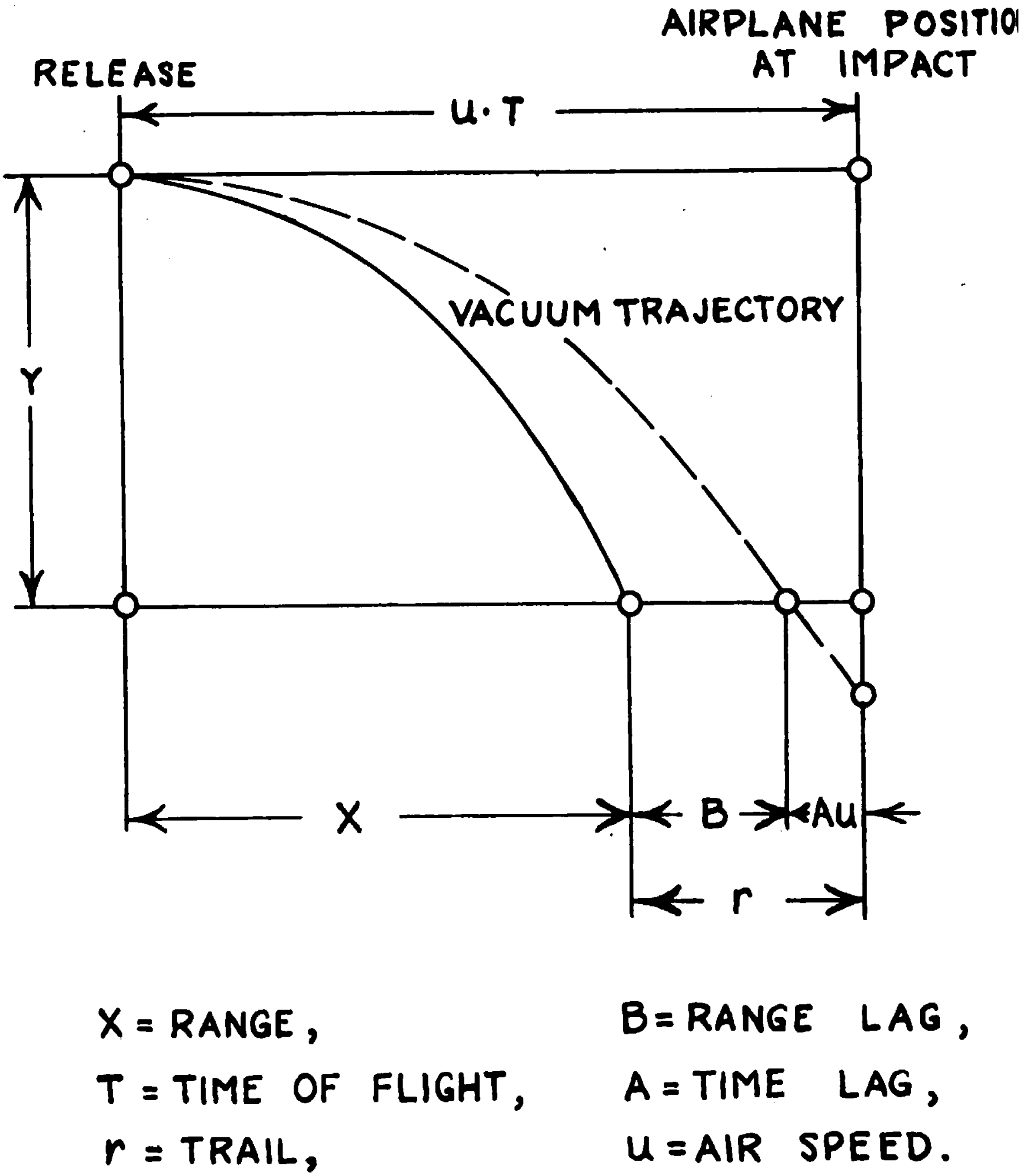


Figure X.1.1

Elements of a Bomb Trajectory
in the Absence of Wind

tion, the angle of elevation, is of primary importance. This fact is due to the presence of wind. In the absence of wind, or to be more precise, in the absence of relative motion of the air and the target, a single function, the range, would furnish sufficient information to the bombardier to permit him to make a hit. There is, however, an advantage in having a mechanism which is independent of the wind. A motion of the target will be treated precisely by the same mechanism. For example, the problem of bombing a ship which is steaming north at 20 knots when there is no wind is precisely the same as the problem of bombing the ship when it is motionless with respect to the water, but when there is a wind of 20 knots blowing to the south. The pertinent data are the motion of the aircraft relative to the air (because this information is easily available to the bombardier) and the motion of the target relative to the air. Motion of the target relative to the ground and motion of the ground relative to the air are quite irrelevant. Thus, any device which may be used to get a hit on a target when there is wind may also be used to get a hit on a target which is in uniform straight-line motion. For convenience, we shall speak only of wind, but it will be understood that target motion is automatically covered.

It is convenient to define here certain terms and symbols which will be used in the remainder of the chapter. First, let us consider a trajectory when there is no wind. (See Figure 1.) Under these conditions we define the following nomenclature and symbols.

(2) Definitions. The range, X, is the length of the horizontal component of the vector from the point of release to the point of impact. The difference between the range in vacuo and the range for the same initial conditions is the range lag, B. The difference between the time of flight, T, and the time of flight in vacuo is the time lag, A.

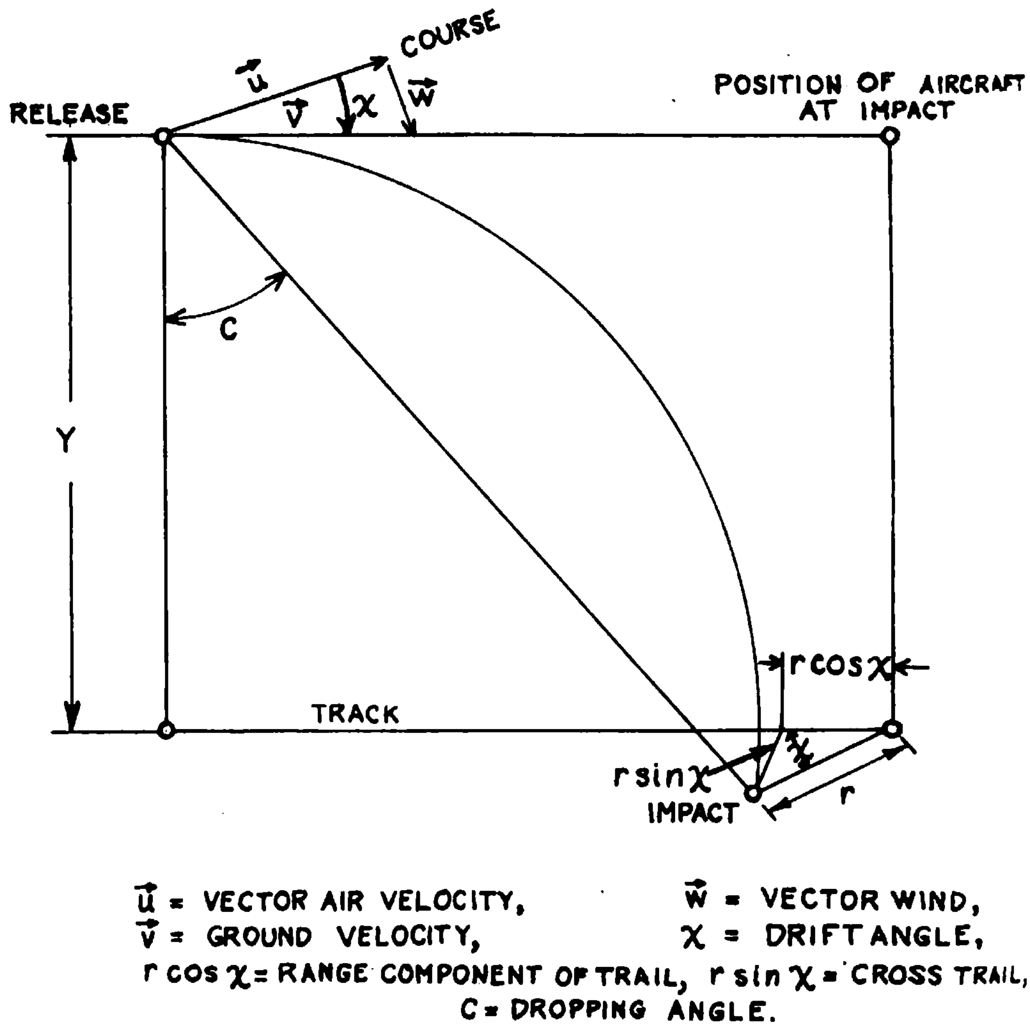


Figure X.1.2

Elements of a Bomb Trajectory
in Ground Coordinates

The bomb ballistic tables give the quantities A and B as functions of u, Y and the reciprocal ballistic coefficient, $\gamma = 1/C$. The quantities A and B are smaller and change much more slowly with changes in u, Y and γ than do the range, X, and the time of flight, T, and interpolation is correspondingly simplified. The relation between A, B and r is easy to see. A bomb falling in vacuo would remain directly under the bombing aircraft. At the end of the time of flight in vacuo the aircraft would therefore be B feet ahead of the (eventual) position of impact of the bomb. In the remaining time, A seconds, before the actual impact the airplane would travel uA feet. Thus

$$(3) \qquad r = B + uA.$$

In the presence of wind the situation is more complicated. The following terms are used. (See Figure 2).

(4) Definitions. The course, sometimes called the heading, of an aircraft is the direction of the vector velocity of the airplane with respect to the air. The track is the vertical projection of the path of the airplane. Its direction is the direction of the vector velocity of the aircraft with respect to the ground. The drift angle χ is the angle, measured positively to the right, from the course to the direction of the track. The wind speed is denoted w, the component, the range wind, along the track being w_x and the component, the cross wind, perpendicular to the track being w_z. A tail wind is a positive range wind. The range component of trail is the projection of the trail on the track, and the cross trail is the projection of the trail perpendicular to the track. The dropping angle is the angle measured at release from the vertical to the aircraft-target line.

We note that the trail is measured from the foot of the vertical from the aircraft at impact, backwards in the direction opposite the course. That this is the correct direction is clear from the argument

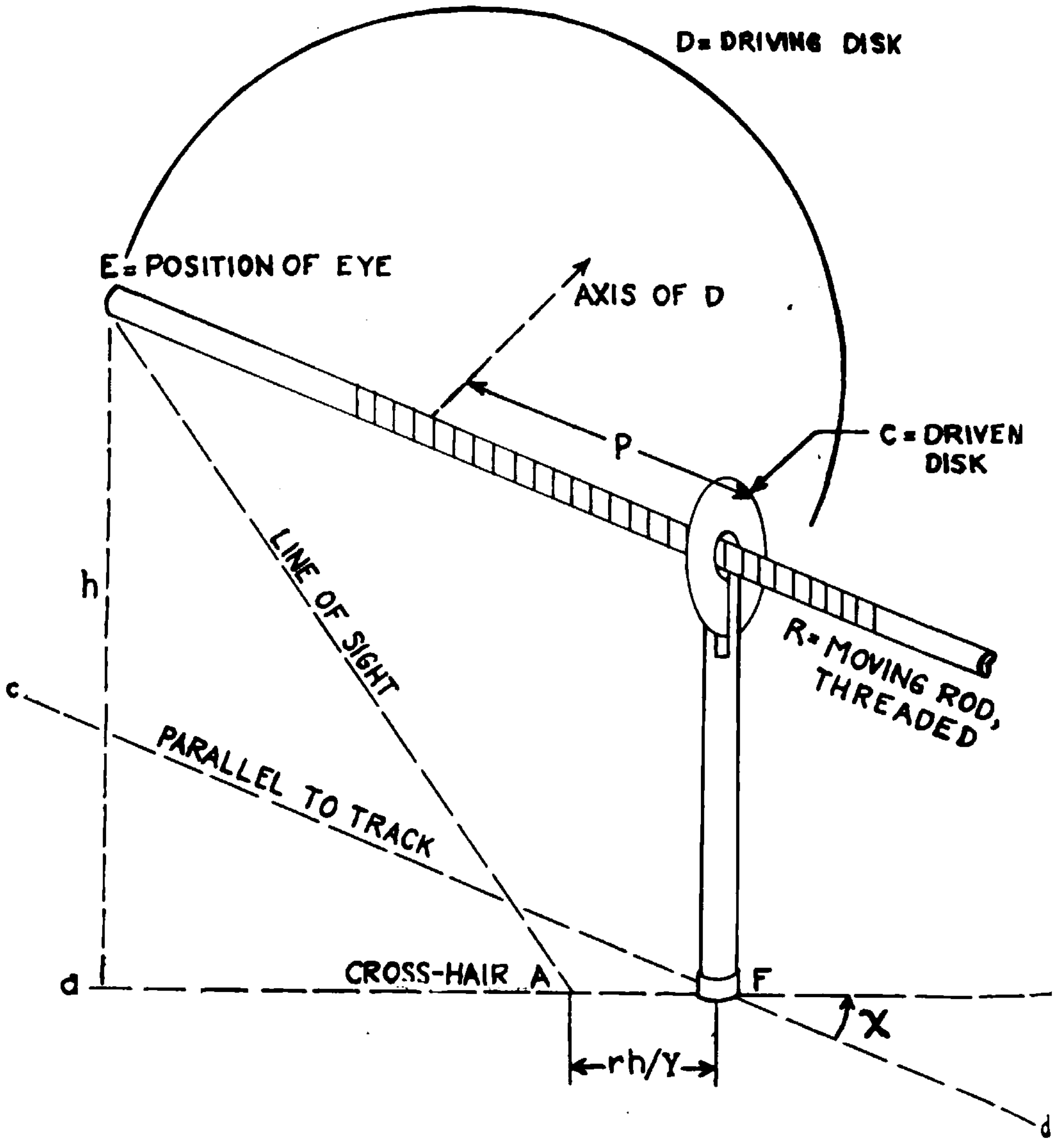


Figure X.2.1

Schematic Bombsight

presented earlier, showing that with constant wind the bomb always falls directly back from the airplane.

2. An hypothetical bombsight.

In this section we show how knowledge of the trail and time of flight permits a bombardier to obtain a hit. The bombsight we describe is suggestive of some of Rube Goldberg's creations, but circumstances do not permit discussion of the sights currently used. However, the bombsight will have enough of the features common to several German, British, and American sights to give a basis for discussing certain problems which occur in bombing.

The bombsight is shown in Figure 1. The line ab is fixed in the plane of symmetry of the aircraft— its direction is supposed to be the course. The distance AF is supposed adjustable, and the entire remaining mechanism of the sight pivots about F. In a position h units vertically above F is the "driven" disc, C. The inner bearing surface of C is supposed to be threaded, so that the rod R, which is slotted to prevent rotation, is driven forward or backward by rotation of C. The disc C is driven by contact with the drive disc D, and the distance, p, from C to the axis of D is supposed to be adjustable, so that the rate is controllable.

The bombardier's procedure is, very roughly, as follows. After noting the air speed u and the altitude Y the bombardier finds in a table the value of r/Y, called the trail ratio, and the value of the reciprocal of the time of flight, 1/T, for the particular bomb. He then begins the process known as synchronization. First, he attempts to adjust the distance p, and thereby the rate of motion of the rod, so that the cross-hair remains on the target. If the target moves, say to the left, relative to the cross-hair, the aircraft is turned to the left beyond the target and the angle χ is increased slightly. Successive adjustment eventually reaches the stage where, without further change by the bombardier, the line of sight remains on the target. At this stage the relative motion of the target with respect to the aircraft is determined and the bombing problem is solved. We have actually set up a duplicate of the Figure 1.2 in the bombsight, and a little computation will indicate at what point the release must be made to get a hit. The rod R is moving at a rate a proportional to the distance p multiplied by the angular rate of D, which in turn is proportional to 1/T. Let us suppose that the constant of proportionality is one. Then $a = p/T$. We assert that the bomb should be released when the position of the eyepiece E reaches the axis of the driving disc D. For, T seconds later, since $a = p/T$, the eyepiece would reach the disc C, the bomb would hit the ground and the line of sight would run from C through A. The line of sight would then run through a point on the ground which is

$$(hr/Y)Y/h = r$$

feet behind the aircraft. This is precisely the position of the bomb.

The bombsight has, in the process of synchronization, actually found the ground speed of the plane (to be more precise, the relative velocity of the plane with respect to the target). Using straight proportionality, we must have

$$a/h = p/Th = v/Y.$$

Thus
$$v = Yp/Th$$

and these quantities are known after synchronization.

Finally, it should be remarked that a bombsight operating on this principle has certain advantageous features. The problem solved by the bombsight is purely geometric and no ballistic data are "built into" the sight. If different bombs are employed, different sight settings are used, but a complete redesign of the entire series of bombs used would not require redesign of the sight.

3. Differential corrections.

The operation of bombing must be performed in a very short time. The bombing run, during which the bombardier makes instrument settings, synchronizes and releases his bomb, must be made as short as possible since this is the time when the aircraft is most vulnerable to anti-aircraft fire. Anti-aircraft directors are at their best in tracking and predicting the motion of an aircraft in the straight-line uniform motion of the bombing run, and the risk in making a run over a well-protected target goes up tremendously after the first twenty seconds. These facts require that the operation of bombing be kept as simple as possible. The magnitude of the dispersion at present inherent in bombing, together with this requirement, makes the inclusion of a large number of differential effects impractical. Save for a few general remarks our discussion will therefore be limited to an analysis of differential ballistic wind.

Any deviation of the actual bombing conditions from standard will in general result in a change in both trail and time of flight. In order to be more easily usable the effects are frequently listed as an effect on trail alone, following the argument we now give. Suppose the correct trail to use in a given situation if r, and the correct time of flight, T. We shall compute the error resulting from the use of an incorrect trail r' and an incorrect time of flight T'. The error will be broken into components along and perpendicular to the track, the first being referred to as <u>range error</u> and the second as <u>deflection error</u>. At time T' after release, since the bombsight is unaware of the deception practiced on it, the aircraft will be $r' \cos \chi$ ahead of the target where χ is the drift angle. Therefore, at T after release the aircraft is $r' \cos \chi + (T - T')v$ ahead of the target. However, in order to secure a hit, at time T the aircraft should have been $r \cos \chi$ ahead and the range error is accordingly $(r - r') \cos \chi + (T - T')v$. On the other hand, the aircraft will fly a distance $r' \sin \chi$ to one side of a line parallel to the track and passing through the target. Since its correct position is $r \sin \chi$ to one side, the deflection error is $(r - r') \sin \chi$. Summarizing:

(1) <u>If the correct trail and time of flight are</u> r <u>and</u> T, <u>and</u> r' <u>and</u> T' <u>are used in bombing, the range error is</u>

$$(r - r') \cos \chi + (T - T')v$$

<u>and the deflection error is</u>

$$(r - r') \sin \chi.$$

The first of these formulae suggests strongly the advisability of making an error in both trail and time of flight, if an error is to be made. For example, suppose the values r_0 and T_0 correspond to standard bombing conditions and the correct values for the particular conditions are $r = r_0 + \Delta r$ and $T = T_0 + \Delta T$. Instead of listing Δr and ΔT it is desired to list a single correction on trail. The incorrect value $T' = T_0$ for time is therefore used. We then decide to use an incorrect trail, r', so that the range error will be zero if there is no wind; i.e.,

$$(r_0 + \Delta r - r') + (T_0 + \Delta T - T_0)u = 0$$

and hence $(r' - r_0) = \Delta r + u\Delta T$. The quantity $(r' - r_0)$ is listed as the correction to trail. If there is a wind the error can be computed from (1). The range error is

$$\begin{aligned}(r - r') \cos \chi + (T - T')v \\ &= -\Delta T\, u \cos \chi + \Delta T\, v \\ &= \Delta T\, w_x\end{aligned}$$

where w_x is the range wind. The deflection error will be

$$(r - r') \sin \chi = u \Delta T \sin \chi$$
$$= \Delta T \; w_z$$

where w_z is the cross wind. These errors may, if ΔT is not large, be quite tolerable.

4. Differential ballistic wind.

Of the many oversimplifications made in the assumptions which comprise the standard bombing conditions one of the most serious is the assumption of a constant wind. In actual practice the wind is found to vary markedly with altitude, resulting in errors which, for certain combinations of bomb and atmospheric conditions, are too large to ignore. As a result it has become necessary to devise some sort of procedure to correct for the effects of a variable wind structure. In order to do this it is convenient to make use of the concept of "ballistic wind," as defined in Section 5 of Chapter VIII. Let us first define the "differential wind" at altitude y to be the vector difference (wind at altitude y minus wind at bombing altitude Y). The effect on range caused by this differential wind is not strictly a linear functional of the differential wind. But there is a strictly linear functional which approximates the actual effect to within a small error (an infinitesimal of order higher order than 1 in the "norm" of the disturbance.) This linear approximation is the differential effect on range produced by the variable differential wind. There is a certain constant differential wind, having the same value at all levels from target to just below bomb bay, which would produce the same differential effect. This is called the differential ballistic wind, abbreviated DBW. It should be noted that the exact effects of the variable differential wind and the DBW may not be the same. Their differential effects are equal; their full effects may differ by a higher order term.

The DBW is computed from the variable differential wind by use of the appropriate weighting factor curves, and thus depends on the true air speed of the airplane, the altitude of release and the ballistic coefficient of the bomb, as well as on the wind structure.

The differential wind at altitude y is the velocity of the air at altitude y with respect to axes moving with the air at bombing altitude Y. Consequently it is desirable to refer positions to such a set of axes. We choose a set of axes fixed with respect to the air at altitude Y, with origin vertically under the release point, y-axis vertically upward, x'-axis in the direction of the course, and z'-axis perpendicularly to the right. These are obtained, by rotation through the angle χ, from the xyz-axes, where x is along the track. If the vector differential wind $\delta \mathbf{w}(y)$ at altitude y is resolved into components $\delta w_{x'}$, $\delta w_{z'}$ along the x'- and z'-axes, the ballistic wind along the x'-axis is found by use of the weighting factor curves for effect of range wind on range, and we shall therefore denote it by $W_x[\delta w_{x'}]$; the ballistic wind along the z'-axis is found by use of weighting factor curves for effect of cross wind on deflection, and we shall therefore denote it by $W_z[\delta w_{z'}]$. The differential effect on x' produced by the component $\delta w_{x'}$ is the same as the differential effect of a constant component $W_x[\delta w_{x'}]$. It is given by (VII.4.8). In the present notation, $t - t_o$ is T, or $T_{vac} + A$, while x at y = 0 is X, or $T_{vac}u - B$, and v_{xo} is u, the true air speed of the airplane. So (VII.4.8) becomes

$$dx'(\delta w_{x'}|y = 0) = (T - \partial X/\partial u)W_x[\delta w_{x'}] = (A + \partial B/\partial u)W_x[\delta w_{x'}] . \quad (1)$$

The A, B, X, T refer of course to time lag, etc., in absence of differential wind. In a similar way, (VII.4.23) yields

$$(2) \qquad dz'(\delta w_{z'} \mid y = 0) = (T - X/u)W_z[\delta w_{z'}]$$
$$= (A + B/u)W_z[\delta w_{z'}].$$

The differential effect of $\delta w_{z'}$ on time of flight is 0. The differential effect of $\delta w_{x'}$ on time oı flight is obtained by computing the corresponding ballistic wind and multiplying by unit effect. Since the weighting factor curve used is that for effect of range wind on time, we denote it by $W_t[\delta w_{x'}]$. By (VII.4.8),

$$(3) \qquad dt(\delta w_{x'} \mid y = 0) = -(\partial T/\partial u)W_t[\delta w_{x'}]$$
$$= -(\partial A/\partial u)W_t[\delta w_{x'}].$$

Inspection of the bomb ballistic tables snows that B is not far from linear as a function of u, so that $(A + \partial B/\partial u)$ and $(A + B/u)$ are not widely different. However, differential winds are necessarily 0 at altitude Y and are greater near the ground. For a wind δw which is large near $y = 0$ and small near $y = Y$, it will be found that $W_z[\delta w]$ is greater than $W_x[\delta w]$, while $(A + \partial B/\partial u)$ is greater than $(A + B/u)$. The result is that $(A + \partial B/\partial u)W_x[\delta w]$ and $(A + B/u)W_z[\delta w]$ differ by a significantly smaller amount than a comparison of the unit effects would indicate, usually roughly half as much. In addition, it is tactically desirable to approach the target nearly down-wind when the wind is large, so that the cross wind effect is not very great. Therefore we shall henceforth use the quantity $(A + \partial B/\partial u)W_x[\delta w_{z'}]$ instead of the correct value $(A + B/u)W_z[\partial w_{z'}]$ for the differential effect of cross-course wind on the coordinate z' at impact. Now, to this approximation, the differential effect of the differential wind is a vector with components

$$(4) \qquad (A + \partial B/\partial u)W_x[\delta w_{x'}],\ (A + \partial B/\partial u)W_x[\delta w_{z'}].$$

This is the same as though in computing the differential ballistic wind, the vector zone-wind $\delta \mathbf{w}(y)$ had been multiplied by the zone-weight, and the products added. In vector notation, to the approximation we are using, the differential effect in the coordinate-system moving with the air at altitude Y is the vector

(5) $$(A + \partial B/\partial u)W_x[\delta \mathbf{w}].$$

In the absence of differential wind, impact would occur at a point $x' = X$, $z' = 0$ at time $T(u)$. In the presence of the differential wind, impact occurs at

(6) $$x' = X + (A + \partial B/\partial u)W_x[\delta w_{x'}],$$
$$z' = (A + \partial B/\partial u)W_x[\delta w_{z'}],$$

approximately, and at time $T(u) - (\partial A/\partial u)W_t[\delta w_{x'}]$. The delay in impact is $-(\partial A/\partial u)W_t[\delta w_{x'}]$, and during this time the (x', y, z')-system has moved a vector amount $-\mathbf{w}(Y)(\partial A/\partial u)W_t[\delta w_{x'}]$ with respect to the (x, y, z)-system. So on the ground the impact has been displaced by the vector amount

(7) $$(A + \partial B/\partial u)W_x[\delta \mathbf{w}] - \mathbf{w}(Y)(\partial A/\partial u)W_t[\delta w_{x'}].$$

The second term is far smaller then the first; for example, at $Y = 30{,}000$ ft, $\gamma = 0.5$, $u = 500$ mi/hr, even with $\delta \mathbf{w}$ in the x'-direction and $w(Y) = 100$ mi/hr, the second term is about .04 times the first. It will be ignored. The result is:

(8) <u>Except for an error term which is negligible unless the air speed at release and the differential winds are very great, the displacement of the impact point caused by the differential wind</u> $\delta \mathbf{w}(y)$ <u>is the vector</u>

$$(A + \partial B/\partial u)W_x[\delta \mathbf{w}].$$

<u>5. Correcting for effects of DBW.</u>

A discussion of differential wind essentially as in the preceding section was made by one of the authors before the Second World War, and a method of correcting for the effect was proposed. Since at the Aberdeen Proving Ground the wind structure is measured frequently, it is feasible to furnish a bombardier with the DBW. The quantity (4.8) is divided by Y/1000 to convert to mils and plotted as a vector on the usual navigational computer, to resolve into trail and deflection effects. The trail effect can be set on the sight; the deflection effect must be allowed for by estimation. Trials during range bombings, by Col. S. C. Smink and bombardiers under his command, showed that the method is feasible when the wind structure is known. However, there are obvious difficulties in attempting to use such a procedure operationally. It is necessary by some means or other to determine the wind structure over the target area, so as to determine the DBW. Two methods suggest themselves. First, a prediction of the winds over the target can be made by meteorologists. However, this prediction is of necessity based on weather data taken at a distance from the target, a distance which in certain theatres was of the order of a thousand miles. Such predictions are of course subject to large errors. A second possibility is to measure the wind over the target, or near the target, shortly before the attack. This is feasible, but requires that aircraft fly at several altitudes over enemy territory for periods long enough to make a ground speed determination. Moreover, while in this undesirable situation the navigator is required to perform certain computations. So the data obtained may well be inaccurate—even if they are brought back. Therefore the procedure under discussion, while theoretically sound, is seldom a practicable one under service conditions.

Chapter XI

ANGULAR MOTION OF A PROJECTILE

1. Requirements for a theory of motion of a projectile.

This chapter is devoted to the solution of the equations of motion of a projectile when it is not assumed that drag is the only aerodynamic force. The equations considered are those derived in Chapter II, which include a complete system of aerodynamic forces and torques. Before undertaking this rather involved computation it seems proper to explain why it is necessary.

Suppose that all observations are interpreted on the basis of drag alone. If the experimental data consist of range bombing from 2000 feet altitude, which is part of the range bombing of every bomb, serious difficulties may result. The value of the ballistic coefficient which results from the measurement of range may differ by a factor of two or three from the value obtained from the measurement of time of flight. This is utterly inexplicable on the basis of drag alone. Worse, if the altitude of release is 1000 feet, the range of the bomb may be more than the range in vacuo, giving a negative ballistic coefficient, while the time of flight may give a small ballistic coefficient. The reason for this phenomenon was given in a qualitative way by one of the authors several years ago. This chapter will contain a later precise analysis. Of course, the result of interpreting data such as these on the basis of drag alone is that wind effects, etc., are totally incorrect. Another extreme example of this sort of thing occurred in the range firing of a mortar. The ballistic coefficient changed violently with angle of elevation — in itself an indication of the inadequacy of the theory applied. When the effect on range of a head wind was computed, it was found that a head wind increased the range! It is surely necessary to have a theory to predict what sort of "non-particle" effects to expect, and to interpret and properly reduce range firing data.

In many cases, rather extreme extrapolations must be made in constructing a firing table. For example, aircraft firing tables must predict the trajectories of bullets fired under densities as low as one fifth normal, at temperatures down to - 55 degrees F., with initial yaw as high as 10 degrees. It is quite impossible to conduct range firings which cover in an adequate fashion the tremendous variety of initial conditions. In addition to density and temperature variations, one has to consider variations in aircraft speed, in azimuth and elevation of the fire. It actually comes down to the following situation: we must construct a theory which is adequate to predict, from firings conducted on a range in the basement of the Ballistic Research Laboratory, the trajectory of a bullet fired sideways from an aircraft travelling at 300 miles per hour at an altitude of 25,000 feet. It would be foolhardy (and wrong) to assume that the drag coefficient alone was sufficient information on which to base this prediction. The solution of the equations of motion must tell us what the important factors in this prediction are, and must indicate a method of measuring these.

Finally, the principal assumption of the normal trajectory, that the yaw is small enough to be negligible, certainly requires a careful examination. This is the problem of stability. It is necessary to have a reasonable mathematical basis for predicting when a shell will travel with small yaw. The theory must be capable of being used to devise experiments to test the stability of shell and to indicate in the light of such tests what measures are necessary to improve the stability.

Since the force system on a projectile has been analyzed in Chapter II the principal problem of the present chapter is simply that of obtaining a sufficiently accurate approximate solution to the equations (II.7.14). This solution must then be utilized to answer the questions we have raised. The presentation given here differs from those previously given in two ways. No assumption will be made concerning the magnitude of the spin of the projectile, so that the analysis is simultaneously valid for bombs and for shell. This added generality is utilized in the discussion of stability, and a criterion for stable motion is derived which is new. This criterion explains in a rather satisfactory way the observed instability of certain spinning bombs. Further, the theory explains and gives a basis for the treatment of the rather remarkable experimental results on range bombing. Finally, we obtain, with somewhat more precision than the earlier authors, the necessary formulas for computation of non-drag effects in sideways fire from aircraft.

Some rather involved mathematical calculations are used in obtaining the solution of the equations of motion, but the method used is intrinsically simple. It is a method of successive approximation. It is first assumed that the motion of the center of mass is given by the solution of the normal equations. On this basis a solution for the yawing motion is obtained. Using this solution for the yawing motion, a second solution for the motion of the center of mass is made. The equations are "almost" linear and this fact is used strongly in obtaining a solution. A number of simplifying approximations are made, and of course the final test of their validity lies in the accuracy with which the solution obtained fits experimental data. In the next chapter we will give some discussion of the adequacy of the analysis.

2. Simplification of the equations of motion.

We now begin the task of simplifying the equations of motion which comprise (II.7.14). Because of the large number of terms this is extremely tedious. The procedure of this section is the following. We first re-state the equations and adopt a notation which is a first step in a general scheme to reduce the equations to a dimensionless form. It is then assumed temporarily that the normal equations give an adequate approximation to the motion of the center of mass. On the basis of this assumption, we then change independent variable from time to arc length measured in calibers, which gives a dimensionless independent variable. Finally, we change dependent variables, in two ways. First, instead of the cross velocity, the cross velocity divided by axial velocity is used so that the tangent of the yaw instead of cross velocity is the dependent variable. We then change the coordinate frame, which was attached to the projectile, to a non-spinning frame. To aid in following the argument we break it into short subsections.

a. The equations.

For convenience, the equations from (II.7.14) which will be needed are re-stated. The notation used is that of Section II.7.

$$m(\dot{\xi} - iu_1\eta + i\omega_1\xi) = c_1\xi + c_2\eta - mg\mathcal{Y},$$

$$B\dot{\eta} + (B - A)i\omega_1\eta = c_3\xi + c_4\eta,$$

$$(1)\quad m[\dot{u}_1 - (\xi\bar{\eta} - \bar{\xi}\eta)i/2] = F_1 - mgy_1,$$

$$A\dot{\omega}_1 = G_1, \qquad \dot{y}_1 = (\mathcal{Y}\bar{\eta} - \bar{\mathcal{Y}}\eta)i/2,$$

$$\dot{\mathcal{Y}} = iy_1\eta - i\omega_1\mathcal{Y}, \dot{Y} = u_1y_1 + (\xi\bar{\mathcal{Y}} + \bar{\xi}\mathcal{Y})/2.$$

The values of c_1, c_2, c_3, c_4, F_1 and G_1 in terms of the aerodynamic coefficients of the projectile are given by (II.4.6).

b. New notation.

To simplify notation we make the following convention:

(2) Convention: The symbol J with a subscript is defined to be the corresponding aerodynamic coefficient multiplied by the "density factor" $\rho d^3/m$. For example, $J_M = \rho d^3 K_M/m$. The J's are dimensionless. We further use spin per caliber of travel instead of spin. Thus

(3) $$\nu = \omega_1 d/u_1.$$

Then the c's and G_1 can be written in the following form:

(4) $$\begin{aligned} c_1 &= (-J_N + i\nu J_F)mu_1/d, \\ c_2 &= (\nu J_{XF} + iJ_S)mu_1, \\ c_3 &= (-\nu J_T - iJ_M)mu_1, \\ c_4 &= (-J_H + i\nu J_{XT})mdu_1, \\ G_1 &= -J_A \nu u_1^2 m. \end{aligned}$$

c. Approximation by means of the normal equations.

We now take the normal equations as a first approximation to the motion of the center of mass. If θ is the angle measured from the horizontal to the tangent to the trajectory, the normal equations have the form (II.8.3):

(5) $$\begin{aligned} \dot{u} &= -\rho d^2 u^2 K_D/m - g \sin\theta, \\ &= -u^2 J_D/d - g\sin\theta, \\ \dot{\theta} &= -g\cos\theta/u, \\ \dot{X} &= u\cos\theta, \quad \dot{Y} = u\sin\theta. \end{aligned}$$

The assumption which will be made in order to obtain the approximate yawing motion is the following.

(6) <u>Assumption</u>. The equations (5) are supposed solved, and it is presumed that u_1 may be replaced by the solution U of these equations.

In terms of the J notation the first of equations (5) can be written:

$$\dot{U} = -J_D U^2/d - g \sin \theta. \tag{7}$$

d. Change of independent variable.

The new dimensionless independent variable can now be defined:

$$p = \int_0^t (u_1/d)\, dt = \int_0^t (U/d)\, dt. \tag{8}$$

This is simply the arc length measured along the trajectory in calibers. The reason for adopting this parameter is that by this means the equations determining the yawing motion will become essentially independent of the size of the projectile. This will imply, for example, that a large bomb has the same period of yaw, measured in calibers of travel, as a smaller model. Returning to the calculation, the derivative of any quantity with respect to t is simply its derivative with respect to p multiplied by

$$dp/dt = u_1/d = U/d.$$

The derivatives with respect to p will be denoted by primes. Equations (1) then must be modified by replacing $\dot{\xi}$, $\dot{\eta}$, $\dot{\omega}_1$, $\dot{y}$, $\dot{\mathcal{Y}}$ by $\xi' U/d$, $\eta' U/d$, etc. It will be convenient to write out explicitly the equations for U', θ', ν'. From (7) it follows at once that

$$\begin{aligned} U' &= -UJ_D - gd \sin \theta/U, \\ \theta' &= -gd \cos \theta/U^2. \end{aligned} \tag{9}$$

To compute ν, from the fourth of equations (1), $\omega_1' = dG_1/AU$. Replacing G_1 by its value in terms of aerodynamic coefficients from (4) gives

$$\omega_1' = - J_A \nu\, mdU/A.$$

Hence

$$\nu' = (\omega_1 d/U)' = \omega_1' d/U - \omega_1 dU'/U^2$$
$$= - J_A \nu\, md^2/A + \nu J_D + \nu\, gdU^{-2} \sin\theta,$$

and we may write

(10) $$\nu' = \nu(J_D - J_A md^2/A + gdU^{-2} \sin\theta).$$

e. Change of dependent variable.

We now transform the dependent variables ξ, η, g_1 and $\mathcal{Y}$ by the formulas:*

(11)
$$\lambda = (\xi/U) \exp\left[i \int_0^p \nu\, dp \right],$$
$$\mu = (\eta d/U) \exp\left[i \int_0^p \nu\, dp \right],$$
$$\gamma = g \mathcal{Y} \exp\left[i \int_0^p \nu\, dp \right],$$
$$g_1 = gy_1.$$

Geometrically, these equations can be interpreted as follows. The quantity ξ/U is the "vector" yaw. It has the magnitude of the sine of the angle of yaw.

---

*Recall that exp ( ) is the notation for $e^{(\quad)}$, where e is the base for natural logarithms.

The factor

$$\exp\left[\dot{i}\int_0^p \nu\, dp\right] = \exp\left[i\int_0^t \omega_1\, dt\right]$$

changes the coordinate system from one which rotates with the projectile to one which does not rotate about the x_1-axis. Explicitly, λ is the complex yaw measured from the axis of the projectile to the trajectory, on a coordinate system with one axis along the axis of the shell, and which is not spinning about this axis. Similarly, μ, γ, g_1 have interpretations in this system.

The coordinate system thus described is determined up to a constant angular rotation about the x_1-axis. For convenience we describe a particular coordinate system which meets the specifications rather closely except for high-angle fire.

(12) The x_2'-axis is supposed to lie in a vertical plane containing the x_1-axis and points downward. The x_3'-axis is therefore perpendicular to this plane and points to the left.

We shall not use this specification immediately. It will not be needed until Section 5, where the yaw of repose is discussed. However, with a view to future convenience we make a further slight digression. The aerodynamic force perpendicular to the axis of the shell has the complex number representation, in coordinates fixed in the shell,

$$\mathfrak{F} = (-J_N + i\,\nu J_F)mU\,\xi/d + (\nu J_{XF} + iJ_S)mU\,\eta .$$

In a non-rotating coordinate system the force is this quantity multiplied by

$$\exp\left[i\int_0^p \nu\, dp\right]$$

which is

$$\mathfrak{F}' = (-J_N + i\,\nu J_F)mU^2\lambda/d + (\nu J_{XF} + iJ_S)mU^2\mu/d.$$

Suppose we now consider a coordinate system with X_2''-axis perpendicular to the trajectory and pointing downward, and X_3''-axis horizontal and pointing to the left. The aerodynamic force acting on the projectile would have the complex number representation (to first-order terms), $\mathcal{F}'$, together with that component of the axial drag which is in the $X_2''X_3''$-plane. This component is $\lambda J_D U^2 m/d$, so that we may state:

(13) <u>The aerodynamic force perpendicular to the trajectory has the complex number representation</u>

$$[(J_D - J_N + i\nu J_F)\lambda + (\nu J_{XF} + iJ_S)\mu]\, mU^2/d.$$

<u>The real axis is perpendicular to the trajectory pointing downward, and the imaginary axis is horizontal pointing to the left. The quantities λ and μ are the cross velocity divided by the axial velocity U and the cross angular velocity multiplied by d/U.</u>

We now return to the main line of argument.

f. Final form of equations.

The remaining task is simply to transform equations (1) to equations in λ, μ, γ and g_1, with p as independent variable. Computing:

$$\dot{\xi} = \xi' U/d = \left\{ \lambda U \exp\left[-i \int_o^p \nu \, dp \right] \right\}' U/d$$

$$= [\lambda' U + \lambda(-J_D - gdU^{-2} \sin\theta)U - i\nu\lambda U]$$

$$\cdot \left\{ \exp\left[-i \int_o^p \nu \, dp \right] \right\} U/d$$

$$= [\lambda' - i\nu\lambda - \lambda(J_D + gdU^{-2} \sin\theta)]$$

$$\cdot \left\{ (U^2/d) \exp\left[-i \int_o^p \nu \, dp \right] \right\}.$$

Similarly,

$$\dot{\eta} = [\mu' - i\nu\mu - \mu(J_D + gdU^{-2}\sin\theta)] \cdot \left\{ (U^2/d^2) \exp\left[- i \int_0^p \nu \, dp \right] \right\},$$

$$g\dot{\mathcal{Y}} = (\gamma' - i\nu\gamma)(U/d) \exp\left[- i \int_0^p \nu \, dp \right].$$

Substituting these values in (1) leads to the equations:

$$\begin{aligned} m\,[\lambda' - \lambda(J_D + gdU^{-2}\sin\theta) - i\mu] &= (c_1\xi + c_2\eta - mg\mathcal{Y}) \cdot \left\{ (d/U^2) \exp\left[+ i \int_0^p \nu \, dp \right] \right\} \\ &= c_1\lambda(d/U) + c_2\mu(1/U) - m\gamma(d/U^2), \\ B[\mu' - \mu(J_D + gdU^{-2}\sin\theta)] - iA\nu\mu &= c_3\lambda(d^2/U) + c_4\mu(d/U) \\ \gamma' &= ig_1\mu, \quad g_1' = (\gamma\bar{\mu} - \gamma\mu)i/2. \end{aligned} \tag{14}$$

We now exhibit these equations with the values of the c's substituted from (4). For ease of writing, one more abbreviation is made. We define

$$k^2 = B/md^2 = (\text{transverse radius of gyration in calibers})^2. \tag{15}$$

From (14) and (4), the final equations are then formed.

$$\begin{aligned} \lambda' &= (J_D - J_N + i\nu J_F + gdU^{-2}\sin\theta)\lambda + (\nu J_{XF} + iJ_S + i)\mu - \gamma dU^{-2}, \\ \mu' &= (-\nu J_T - iJ_M)\lambda k^{-2} + (J_D - k^{-2}J_H + i\nu k^{-2}J_{XT} + gdU^{-2}\sin\theta + iA\nu/B)\mu, \\ \gamma' &= ig_1\mu, \qquad g_1' = (\gamma\bar{\mu} - \bar{\gamma}\mu)i/2. \end{aligned} \tag{16}$$

3. Solutions of the equations of motion.

In this section we shall find an approximate solution of the "homogeneous" part of equations (2.16). But for reasons that will be clearer to the reader if he glances ahead to equations (5.5), we group together the pair of terms in the first of equations (2.16) which contain a factor g: one of these visibly depends on λ, but the other is less conspicuously dependent on λ in such a way that the sum of the two is very nearly a function of p alone. With this grouping the first two of equations (2.16) become the following pair, whose homogeneous part (omitting the last term in the first equation) we shall try to solve:

$$
\begin{aligned}
\lambda' &= (J_D - J_N + i\nu J_F)\lambda + (\nu J_{XF} + iJ_S + i)\mu \\
&\qquad + (g\lambda \sin\theta - \gamma)dU^2, \\
\mu' &= (-\nu J_T - iJ_M)k^{-2}\lambda \\
&\qquad + (J_D - k^{-2}J_H + i\nu k^{-2}J_{XT} + gdU^{-2}\sin\theta + iA\nu/B)\mu.
\end{aligned} \tag{1}
$$

It is necessary to make some approximations, and to do this some estimates of the order of magnitude of the various terms in (1) must be made. Each J-term contains a factor $\rho d^3/m$, which will be quite small. For a hundred-pound bomb, with a diameter of 8 inches, this factor is less than 1/6000, and for a 3-inch shell, weighing about 17 pounds, the factor is less than 1/14000. It is therefore very reasonable to neglect J-terms in comparison with terms of the order of magnitude of A/B, which is usually about 1/10 and is at least 1/20. Further, gd/U^2 is less than 1/1000 if d is less than 1 foot and U is greater than 180 feet per second, and it will therefore be assumed that this is of the same order of magnitude as a J-term. Finally, the derivatives of J-terms will be neglected. These terms are functions of the density ρ and the Mach number, and except in the immediate neighborhood of the velocity of sound this treatment should be wholly justified.

The equations (1) are rather cumbersome and it will be convenient to abbreviate them as follows:

$$(2)\qquad \begin{aligned} \lambda' &= a_1\lambda + a_2\mu + b, \\ \mu' &= a_3\lambda + a_4\mu, \end{aligned}$$

where, recalling the remarks about the order of magnitude of J-terms,

$$(3)\qquad \begin{aligned} a_1 &= J_D - J_N + i\,\nu J_F, \\ a_2 &= \nu J_{XF} + i, \\ a_3 &= (-\nu J_T - iJ_M)k^{-2}, \\ a_4 &= J_D - k^{-2} J_H + gdU^{-2}\sin\theta + iA\nu/B, \\ b &= (g\lambda\sin\theta - \gamma)dU^{-2}. \end{aligned}$$

We now proceed to eliminate μ from the equations (2) by solving the first equation for μ and substituting in the second. The result of this computation, which is performed in a perfectly straightforward manner, is

$$(4)\qquad \begin{aligned} &\lambda'' + \lambda'(-a_1 - a_4 - a_2'/a_2) \\ &\quad + \lambda(a_1a_4 - a_2a_3 + a_1a_2'/a_2 - a_1') \\ &\quad + b(a_2'/a_2 + a_4 - b'/b) = 0. \end{aligned}$$

Under our assumptions, in computing the derivative of an a, it is only necessary to examine the terms in ν. For convenience we recall equation (2.10):

$$(5)\qquad \nu' = (-J_A md^2/A + J_D + gdU^{-2}\sin\theta)\,\nu.$$

Examining the equation (4) we find that in each parenthesis the real and imaginary parts contain terms of the order of J at least, whereas a_2'/a_2 and a_1' are of the order of J^2. It is therefore permissible to replace (4) with the equation

$$(6)\qquad \begin{aligned} &\lambda'' + \lambda'(-a_1 - a_4) \\ &\quad + \lambda(a_1a_4 - a_2a_3) + b(a_4 - b'/b) = 0. \end{aligned}$$

For the remainder of this section we consider only the homogeneous part of (6), deferring the finding of a particular solution to Section 5.

There is a standard way of changing the dependent variable in such a second-order linear equation in order to eliminate the first-derivative term. This is done by defining q by the equation

$$\text{(7)} \qquad \lambda = q \exp \tfrac{1}{2} \int_0^p (a_1 + a_4)\, dp.$$

Computing,

$$\lambda' = [\, q' + \tfrac{1}{2}(a_1 + a_4)q \,] \exp \tfrac{1}{2} \int_0^p (a_1 + a_4)\, dp,$$

$$\lambda'' = [\, q'' + (a_1 + a_4)q' + \tfrac{1}{4}(a_1 + a_4)^2 q + \tfrac{1}{2}(a_1 + a_4)' q] \cdot \exp \tfrac{1}{2} \int_0^p (a_1 + a_4)\, dp.$$

This substitution of these expressions in (6) leads to the equation

$$\text{(8)} \qquad q'' - r^2 q = 0,$$

where

$$r^2 = \tfrac{1}{4}[\, (a_1 + a_4)^2 - 4(a_1 a_4 - a_2 a_3) - 2(a_1 + a_4)'] = \tfrac{1}{4}[\, (a_1 - a_4)^2 + 4a_2 a_3 - 2(a_1 + a_4)' \,] .$$

We are now faced with the problem* of finding an approximate solution to the equation $q'' - r^2 q = 0$, where r is a slowly varying function of the independent variable p. This equation is "almost" a familiar type.

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*Approximate solutions of equations of this type have been obtained by H. Jeffreys, Proceedings of the London Mathematical Society (2) Vol. 23 (1923), p. 438, and by Wentzel, Kramers, Brillouin ("WKB method"). See, for example, G. Wentzel, Zeitschrift für Physik, Vol. 38 (1926), p. 518. The solution we obtain will be essentially the same as the solution obtained by these methods.

If the substitution

(9) $z = (\log q)'$, ($\log$ = log to the base e),

is made, the resulting equation is

(10) $$z' + z^2 - r^2 = 0.$$

If r were actually constant, this equation would have the solution $z = \pm r$. Since r is nearly constant, the equation may be expected to have solutions nearly equal to r and $-r$ respectively. Consider first the solution which is nearly equal to r. Writing it in the form

(11) $$z = r + \varepsilon$$

where ε is small, the equation (10) becomes

(12) $$\varepsilon' + 2r\varepsilon + \varepsilon^2 + r' = 0.$$

Thus if we define*

$$\varepsilon = - r'/2r,$$

the differential equation is satisfied except for terms ε' and ε^2. The error thus is in the ratio $(\varepsilon' + \varepsilon^2)/r^2$ to one of the terms of the equation, and we can verify that for the problem at hand this is negligible. Referring to (11), the solution for z is $z = r - r'/2r$, and hence for q, from (9),

(13) $$q = \exp \int_0^p (r - r'/2r)\, dp.$$

Of course, the term $r'/2r$ can be integrated at once. With notable restraint we refrain from doing this integration. Use of $-r$ instead of r in the above argument leads to another solution, so that we have the pair of solutions

(14) $$q = \exp \int_0^p [(-r'/2r) \pm r]\, dp.$$

*We omit discussion of the case where r may be zero.

Since the equations are linear, every solution is a linear combination of these.

Returning to the variable λ, by the definition (7) of q, the solutions for λ are

$$(15) \quad \lambda = \exp \int_0^p [(a_1 + a_4 \quad r'/r)/2 \pm r]\, dp.$$

We must now put this solution for the yaw λ in terms of the original combinations of aerodynamic coefficients. This simply requires use of the definitions (3) and (8) of the a's and of r. Keeping in mind the relative magnitude of the various terms, we compute:

$$(16) \quad a_1 + a_4 = 2J_D - J_N - k^{-2} J_H + gdU^{-2} \sin \theta + iA\nu/B$$

and

$$4r^2 = (a_1 - a_4)^2 + 4a_2a_3 - 2(a_1 + a_4)'$$

$$= (-J_N + k^{-2}J_H - gdU^{-2} \sin \theta - iA\nu/B)^2$$

$$+ 4(\nu J_{XF} + i)(-\nu J_T - iJ_M)k^{-2} - 2iA\nu'/B,$$

or, referring to the expression (5) for ν',

$$4r^2 = -A^2 \nu^2/B^2 + 4k^{-2}J_M$$

$$(17) \quad + [J_N - J_D - k^{-2}J_H$$

$$- (2J_T - J_A)md^2/A]\, 2iA\nu/B.$$

A further approximation will be applied in computing r'/r. Certain terms in the expression for $4r^2$ are frequently useful later and have special nomenclature. Namely we define

$$(18) \quad s = \underline{\text{stability factor}} = A^2 \nu^2/4Bk^{-2}J_M,$$

$$\sigma = \sqrt{1 - 1/s}.$$

The real part of $4r^2$ can then be written $-A^2\nu^2\sigma^2/B^2$. In computing r'/r we shall ignore the imaginary part of $4r^2$. In a sense this is quite justifiable. If r'/r is computed precisely and the denominator multiplied by its conjugate, our usual conventions concerning magnitudes would lead us to precisely the same form unless r is extremely small. When r is small the form of solution is no longer valid; to avoid complication we shall omit discussion of this special case. Computing on this basis,

$$
\begin{aligned}
r'/r &= (4r^2)'/2(4r^2) \\
&= (\nu^2\sigma^2)'/2\nu^2\sigma^2 \\
&= \nu'/\nu + \sigma'/\sigma \\
&= J_D - J_A md^2/A + gdU^{-2}\sin\theta + \sigma'/\sigma .
\end{aligned}
\tag{19}
$$

Making use of this expression leads at once to the form of solution needed for the discussion of stability.

(20) The yawing motion is a combination of the solutions

$$
\begin{aligned}
\lambda = \sqrt{\sigma_0/\sigma}\,\exp \tfrac{1}{2}\int_0^p \Big[& (J_D - J_N - k^{-2}J_H + J_A md^2/A + i\nu A/B) \\
& \pm \{ -A^2\nu^2/B^2 + 4k^{-2}J_M \\
& \quad + [J_N - J_D - k^{-2}J_H \\
& \quad\quad - (2J_T - J_A)md^2/A]\, 2i\nu A/B\}^{\frac{1}{2}} \Big]\, dp.
\end{aligned}
$$

For stable projectiles it will turn out that there is a simpler form which is an adequate approximation. For all adequately stable projectiles, spinning or not, the ratio of the imaginary part of r^2 to the real part is less than 1/20, so that the binomial theorem may be used to approximate the square root, and thus obtain a simple expression for r.

According to the binomial theorem, if $|a| > |b|$,

$$\sqrt{a+b} = \sqrt{a}\,(1 + b/a)^{\frac{1}{2}}$$
$$= \sqrt{a}\,(1 + b/2a - b^2/8a^2 + \ldots),$$

Thus if the ratio $|b/a|$ is less than 1/20, the error in approximating $\sqrt{a+b}$ by

$$\sqrt{a}\,(1 + b/2a) = \sqrt{a} + b/2\sqrt{a}$$

is less than 1/2000. Applying this to the expression for $4r^2$ shows that

$$2r = iA\nu\sigma/B + [J_N - J_D - k^{-2}J_H - (2J_T - J_A)md^2/A]/\sigma. \qquad (21)$$

Applying this approximation to equation (20) leads to the conclusion*:

(22) <u>If a projectile is adequately stable, its yawing motion is given by an arbitrary combination of the two solutions:</u>

$$\lambda = \sqrt{\sigma_o/\sigma}\,\exp \tfrac{1}{2}\int_o^p \{ J_D - J_N - k^{-2}J_H + J_A md^2/A \pm [J_N - J_D - k^{-2}J_H - (2J_T - J_A)md^2/A]/\sigma + i\nu A(1 \pm \sigma)/B\}\, dp.$$

---

*The authors are grateful to Mr. B. Fallon for pointing out an error, an omission of a "gd" term, in their earlier work. It is to be understood that g is the acceleration due to gravity and d is the diameter of the projectile.

This is the form of solution which has been used at Aberdeen in the reduction of spark range data. We refer to Chapter XIII for a geometric description of this function.

An alternate form of the solution (22) may also be noted. In computing r'/r, referring to (17), we may write:

$$r'/r = (4r^2)'/2(4r^2)$$

$$= - A^2 \nu \nu'/B^2(- A^2 \nu^2\sigma^2/B^2)$$

$$= (J_D - J_A md^2/A)/\sigma^2.$$

The equations (15), (16), (17) and (21) then lead to the following form.

(23) <u>An alternate form of the solutions</u> (22) <u>is</u>:

$$\lambda = \exp \tfrac{1}{2} \int_0^p \{ 2J_D - J_N - k^{-2}J_H + gdU^{-2} \sin \theta$$

$$- (J_D - md^2J_A/A + gdU^{-2} \sin \theta)/\sigma^2$$

$$\pm [J_N - J_D - k^{-2}J_H$$

$$- (2J_T - J_A)md^2/A]/\sigma$$

$$+ [i \nu A(1 \pm \sigma)B] \} dp.$$

We intend to base an analysis of spark range data on the form (23), instead of, as is now done, on (22). Of course the validity of (22) and (23) depends on σ being at least 20 times as large as a J-term. We now turn to a discussion of stability, which will show that for adequately stable shell this assumption is justified.

4. Stability.

We are now prepared to discuss the question of stability*. One might define a projectile as stable provided the yaw decreased, tending toward zero as p, the arc length in calibers, increased. The definition of stability which we wish to adopt is not, however, of this sort. We will say a projectile is stable if small disturbances have no permanent effect, that is, if the yawing motion does not, in the limit, depend on the initial yaw and the initial yawing motion (the initial cross spin). To be precise, the yawing motion of the projectile is given as a linear combination of the solutions (15) of the previous section and a particular solution. The projectile is stable if the solutions (3.15) decrease in magnitude, so that in the limit only the particular solution remains. This does not always mean that the yaw will become and remain small, for the particular solution may give a large value of the yaw. For example, if the spin given a shell is excessive, its rate of precession is slow, and the direction of its axis changes only slowly. Since the trajectory curves, the yaw becomes large. Nevertheless, the projectile may be stable in the sense in which we use the word. In order to assure that the yaw be small it is necessary to know that the projectile is stable and that the particular integral is small. Actually, except in the case of high-angle fire, the particular integral will be small, so that if the projectile is stable its yaw will become and remain small.

---

*The "first stability condition" (see (9)) is quite old, and is probably due to George Greenhill. The criteria we derive are new. For shell with normal spin, this stability condition, in a somewhat less precise form, was contained implicitly in the work of Fowler, Gallop, Lock and Richmond, as was stated by R. H. Kent.

The question to be decided in this section is then the following: what are the necessary and sufficient conditions that the solutions given by (3.20) approach zero as the arc length p increases? Except for the trivial case where r is extremely small (in which case the solution must be replaced by another simpler form), it is easy to see that the yaw will decrease if and only if the real part of the first bracket under the integral is negative, and in absolute value greater than the real part of the second. That is, the projectile is stable if and only if*

$$a > \left| R \sqrt{b + ic} \right| , \tag{1}$$

where R denotes "real part of" and

$$a = (- J_D + J_N + k^{-2}J_H - J_A md^2/A),$$

$$b = - (A^2 v^2/B^2) + 4k^{-2}J_M,$$

$$c = 2Av \left[J_N - J_D - k^{-2}J_H - (2J_T - J_A)md^2/A\right]/B.$$

*To be precise, we should consider stability in the special case where r is very near zero. However, such an investigation would be of no practical interest, since, for military use a projectile must not only be stable, but projectiles with aerodynamic characteristics differing somewhat from the design must be stable. (Production items frequently show fairly substantial differences.) It seems quite improbable then that stability attained for r = 0 could give satisfactory performance on production items. Consequently, when in this chapter we state "necessary and sufficient" conditions for stability, we mean in this practical sense and not in the strict mathematical sense.

The quantity b + ic can be written as

$$\sqrt{b^2 + c^2}\,(\cos P + i \sin P),$$

where

$$\cos P = b/\sqrt{b^2 + c^2}$$

and

$$\sin P = c/\sqrt{b^2 + c^2}.$$

Since

$$\cos^2(P/2) = (1 + \cos P)/2,$$

it follows from De Moivre's theorem that

$$(2) \quad \begin{aligned} |R\sqrt{b + ic}| &= \sqrt[4]{b^2 + c^2}\sqrt{(1 + b/\sqrt{b^2 + c^2})/2} \\ &= \sqrt{(b + \sqrt{b^2 + c^2})/2}. \end{aligned}$$

The inequality (2) is therefore equivalent to one of the form

$$(3) \qquad a > \sqrt{(b + \sqrt{b^2 + c^2})/2},$$

where a is the negative of the real part of $a_1 + a_4$, and b and c are respectively the real and imaginary parts of $4r^2$. We now perform a few elementary algebraic manipulations with the inequality (3). Each line in the following list is equivalent to the preceding and to the inequality (3).

$$a^2 > (b + \sqrt{b^2 + c^2})/2 \text{ and } a > 0,$$

$$2a^2 > (b + \sqrt{b^2 + c^2}) \text{ and } a > 0,$$

$$(2a^2 - b) > \sqrt{b^2 + c^2} \text{ and } a > 0,$$

$$(4) \quad (2a^2 - b)^2 > b^2 + c^2 \text{ and } a > 0$$

$$\text{(since always } |b| \leq \sqrt{b^2 + c^2}),$$

$$-4a^2b > c^2 - 4a^4 \text{ and } a > 0,$$

$$-b > -a^2 + (c^2/4a^2) \text{ and } a > 0.$$

The condition (4) is then necessary and sufficient for stability. From the expressions (1) it is easy to see that a^2 is of the order of J^2, while $c^2/4a^2$ and b are of the order of one, so that the a^2-term may be neglected. For convenience, let us denote the part of c which is inside the bracket by f, so that $c = (f2a\nu/B)$. Then the inequality (4) can be written

$$(A^2 \nu^2/B^2) - 4k^{-2}J_M > (f2A\nu/B)^2/4a^2,$$

(5)
$$1 - (4B^2k^{-2}J_M/A^2\nu^2) > f^2/a^2,$$

$$(4B^2k^{-2}J_M/A^2\nu^2) < 1 - (f^2/a^2) = (a + f)(a - f)/a^2.$$

We now state the principal result of this section, defining the stability factors s_1, s_2 and s_3 as a, a + f and a - f respectively.

(6) <u>In order that a projectile be stable it is necessary and sufficient that the following inequalities be satisfied.</u>

$$1/s = 4B^2k^{-2}J_M/A^2\nu^2 < 1 - \left[\frac{J_N - J_D - k^{-2}J_H - (2J_T - J_A)md^2/A}{J_N + k^{-2}J_H - J_D - J_A md^2/A}\right]^2,$$

<u>and</u>

$$J_N + k^{-2}J_H - J_D - J_A md^2/A > 0.$$

(7) <u>An equivalent form is:</u>

$$1/s < s_2 s_3/s_1^2 \text{ and } s_1 > 0,$$

<u>where</u>

$$s_1 = J_N + k^{-2}J_H - J_D - J_A md^2/A,$$

$$s_2 = 2J_N - 2J_D + 2J_T md^2/A,$$

$$s_3 = 2k^{-2}J_H + (2J_T - 2J_A)md^2/A.$$

The notable feature of either form is that the spin v occurs only as a factor on one side of the inequality so that it is extremely simple to discover the effect on stability of varying the spin. We now consider separately the stability of spin-stabilized and fin-stabilized projectiles.

Case I. Spin-stabilized projectile: $K_M > 0$.

We first remark that the classical stability condition, $s > 1$, is shown to be necessary by (6). Of course this condition ($s > 1$) is by no means sufficient. Inspection of the equations shows that if s is greater than one, by an amount which is large relative to the J-terms, say by 0.5, then the yawing motion of the projectile will be epicyclic, and there will be many maxima of yaw. (See Chapter XIII.) However, the successive values of the yaw at maxima may increase without bound. On the other hand, if $s < 1$ the yaw increases steadily, and the motion is very similar to the falling-down motion of a top when its spin becomes too small.

A remarkable fact obtained from these inequalities is that it is quite possible that shell exist which are incapable of spin stabilization. First, the projectile is surely unstable if s_1 is negative. This is actually rather improbable, for J_N and J_H are usually positive and of the order of ten times the magnitude of J_D. However, the possibility of J_H being negative cannot be ruled out, in view of the formula (II.5.7) which gives the change in K_H for a change in center of mass position. Even if s_1 is positive, it may not be possible to stabilize the shell, for, referring to (6), the bracketed quantity may be greater than 1, whereas the quantity on the left is always positive. If $s_1 > 0$, the bracketed quantity will be less than one if and only if the denominator $\pm$ the numerator are both positive, i. e., if s_2 and s_3 are positive. On the other hand, if s_2 and s_3 are positive, their sum, which is $2s_1$, is positive.

Thus we may state:

(8) <u>A projectile with K_M positive is capable of spin stabilization if and only if s_2 and s_3 are positive. If the projectile is capable of spin stabilization the amount of spin required is given by</u>

$$s > s_1^2/s_2 s_3.$$

We remark that this can be considered as a sharpening of the classical stability condition; instead of requiring that $s > 1$ we require that s be greater than a certain combination of aerodynamic coefficients which is, by reason of its form, always greater than 1.

There is ample experimental evidence of the importance of the complete stability conditions. For example, A. C. Charters has fired rounds in the aerodynamic range at Aberdeen which, although having a stability factor s of 1.5, were unstable. Further, models have been fired which were unstable, and yet a model having the same exterior contour but a center of mass farther <u>back</u> was stable. That is, by a change in distribution of mass, which reduced the factor s, stability was attained. In this case the cause of the instability was the excessive magnitude of J_T. As we have remarked, J_T is usually negative and in this case the factor s_3 was therefore negative. By moving the center of mass to the rear the magnitude of J_T was decreased and consequently the shell became stable.

Too little is known about the stability characteristics of shell to be able to draw any conclusions about design criteria. It is known, in a general way, that very short shell seldom have stability troubles of this sort. Unfortunately, very short shell are unsuitable for most ballistic purposes because of their high drag. The question of stability is now being studied experimentally in some detail.

Case II. Fin-stabilized projectiles: $K_M < 0$.

For fin-stabilized projectiles the implications of the stability criteria are quite different. If s_1 is positive and the spin is sufficiently small the projectile, in view of (7), is surely stable. If, further, $s_2 s_3$ is positive the projectile is stable regardless of spin. (In this case, since $2s_1 = s_2 + s_3$, both s_2 and s_3 are positive.) On the other hand, if either s_2 or s_3 is negative then the condition (7) limits the amount of spin possible before instability ensues. Summing up:

(9) For a projectile with K_M negative it is necessary that $s_2 + s_3$ be positive in order that the projectile be stable. If both s_2 and s_3 are positive the projectile is stable regardless of the amount of spin. If $s_2 + s_3$ is positive but either s_2 or s_3 is negative, then a necessary and sufficient condition that the projectile be stable is that the square of the spin be less than

$$| 4B^2 J_M k^{-2} s_1{}^2 / A^2 s_2 s_3 | .$$

It would be rather exceptional if for a fin-stabilized projectile the quantity

$$s_2 + s_3 = 2s_1$$

were negative. The normal force coefficient for a bomb is usually of the order of twice the coefficient for a shell at the same Mach number. The damping coefficient K_H is usually larger by an even greater factor, and for some bombs this coefficient is as large as 30. These are, of course, natural effects of the guiding surfaces. On the other hand, there is no particular reason to expect that one of the numbers s_2 and s_3 should not, for some designs, be negative. That this is the case is indicated by the fact that a number of bombs, particularly those of marginal stability, have been observed to go into a "flat spin" when, due to launching or to misalignment of fins, some axial spin had been induced.

Actually, the analysis which we have given does not apply to most bombs. In defining the aerodynamic coefficients a factor

$$\omega_1 = \text{axial component of angular velocity}$$

was removed from the Magnus forces and torques on the grounds that, if the projectile has a plane of mirror symmetry these forces and torques are odd functions of ω_1. Most bombs do not have a plane of mirror symmetry. The edges of the fins are usually turned over to give the fins additional rigidity. This means that when the bomb is yawing the fins act like the vanes on an anemometer, inducing a spin on the bomb. Further, in this instance there is no reason to suppose that even with spin zero the force perpendicular to the plane of yaw, which we call Magnus force, vanishes. The form chosen for the aerodynamic coefficients is then not suitable, since it cannot be supposed that they are almost constant — in fact, there is reason to think that for such a projectile the Magnus force coefficient as defined here would have an infinite discontinuity at $\omega_1 = 0$. A further analysis thus seems necessary to get adequate criteria for this sort of fin-stabilized projectile. This analysis will not be carried out here; presumably criteria similar to (6) and (7) would result.

5. A particular integral; the yaw of repbse.

The solutions so far obtained are solutions of the homogeneous equations (3.2). For stable projectiles these solutions are transients, that is λ and μ decrease exponentially along the trajectory and eventually become negligibly small. The importance of these solutions lies in the fact that they enable us to analyze the initial motion of a projectile and to compute the effects of a particular set of launching conditions. We now turn our attention to a particular solution.

This solution is essentially unique, in that it is the non-transient part of any complete solution. The expression for λ we obtain is called the yaw of repose, since it is that yaw which a projectile will have after the transients have died out.

It might at first seem strange that the "repose" position of a projectile is not that of zero yaw, but a little reflection serves to show that this must be the case. If a projectile moves with no yaw, the fact that the trajectory is curved means that the projectile must have a non-zero angular velocity,

$$\dot{\theta} = - g \cos \theta/U.$$

This angular velocity changes along the trajectory, decreasing steadily along the upward branch. The question is, what torque causes this change? If the yaw is zero, the only possible torque is the damping torque, depending on K_H, and if K_H is positive this is of the wrong sign. One must conclude that in general the rest position of a projectile is not a position of zero yaw. We now proceed to compute this yaw, the corresponding angular velocity and the effects of these.

It will be convenient to obtain the particular solution for λ and μ in terms of γ . We return to the notation of Section 3, and recall that the equations, a particular solution of which is desired, are:

$$\text{(1)} \qquad \begin{aligned} \lambda' &= a_1\lambda + a_2\mu + b, \\ \mu' &= a_3\lambda + a_4\mu, \end{aligned}$$

where

$$a_1 = J_D - J_N + i\nu J_F,$$

$$a_2 = \nu J_{XF} + i,$$

$$a_3 = (- \nu J_T - iJ_M)k^{-2},$$

$$a_4 = J_D - k^{-2}J_H + gdU^{-2} \sin \theta + iA\nu/B,$$

$$b = (g\lambda \sin \theta - \gamma)dU^{-2}.$$

Further, the equation which is obtained by eliminating μ from this system is equation (3.6), which is

$$\begin{aligned}(2)\qquad & \lambda'' + \lambda'(-a_1 - a_4) \\ & + \lambda(a_1a_4 - a_2a_3) + b(a_4 - b'/b) = 0.\end{aligned}$$

The solution which would exist if the coefficients of this equation were constant instead of slowly varying functions of the arc length p actually furnishes an adequately accurate approximation. We define the yaw of repose, λ_r, by

$$(3)\qquad \lambda_r = -b(a_4 - b'/b)/(a_1a_4 - a_2a_3).$$

If the values of the a's and b in terms of the aerodynamic coefficients are written, as they will be below, it can be verified that λ_r' is of the order of J times ba_4, and λ_r'' is even smaller. Thus λ_r furnishes a particular solution of the equations (1).

We now compute the expression for λ_r in terms of the definition (1) of the a's and b. If, as before, J^2-terms are neglected,

$$\begin{aligned}(4)\qquad a_1a_4 - a_2a_3 &= (J_D - J_N + i\nu J_F)i\nu A/B \\ &\quad + ik^{-2}(\nu J_T + iJ_M).\end{aligned}$$

We now choose that coordinate system described just before (2.13). In this system we make the following approximations, whose justification is given in a note at the end of this chapter:

$$\begin{aligned}(5)\qquad \gamma - g\lambda\sin\theta &= -g\cos\theta, \\ b &= gdU^{-2}\cos\theta.\end{aligned}$$

The expressions (2.9) for U' and θ' may be used to compute b'/b.

$$\begin{aligned}b'/b &= (U^{-2}\cos\theta)'/U^{-2}\cos\theta \\ &= -2U'/U - \theta'\sin\theta/\cos\theta \\ &= -2(-J_D - gdU^{-2}\sin\theta) + gdU^{-2}\sin\theta \\ &= 2J_D + 3gdU^{-2}\sin\theta.\end{aligned}$$

Thus,

$$a_4 - b'/b = -J_D - k^{-2}J_H - 2gdU^{-2}\sin\theta + i\nu A/B).$$

We therefore have the following expression for the yaw of repose:

$$(6) \qquad \lambda_r = \frac{-gdU^{-2}\cos\theta(-J_D - k^{-2}J_H - 2gdU^{-2}\sin\theta + i\nu A/B)}{(J_D - J_N + i\nu J_P)i\nu A/B + ik^{-2}(\nu J_T + iJ_M)}.$$

This expression is actually of a different order of magnitude depending on whether the projectile is spin-stabilized or fin-stabilized. We therefore consider separately the two cases. First, if $\nu = 0$, we have at once:

(7) <u>If the spin</u> ν <u>is zero the yaw of repose is given by</u>

$$\lambda_r = gdU^{-2}\cos\theta(-J_D - k^{-2}J_H - 2gdU^{-2}\sin\theta)/k^{-2}J_M.$$

Since for a bomb the coefficient J_M is negative, the yaw of repose is positive real, which means that the trajectory is below the axis of the projectile. Thus the "repose" position of a fin-stabilized projectile is such that the nose of the projectile is above the trajectory. This, as we shall see, causes a "lift effect," so that the range of a bomb is greater than would be the case if drag alone were acting. Further, the time of flight is greater than would be the case on the basis of drag alone.

The other special case we wish to consider is that of a spin-stabilized projectile with normal spin; i. e., $\nu A/B$ is of the order of 1/50. In this case the first part of the denominator of the expression for λ_r is much smaller than the second part. Thus, if both numerator and denominator are multiplied by the conjugate of the denominator, the resulting denominator can be approximated by $k^{-4}(\nu^2 J_T^2 + J_M^2)$.

The numerator is then

$$- (gdU^{-2} \cos \theta)(- J_D - k^{-2}J_H - 2gdU^{-2} \sin \theta + i \nu A/B)$$
$$\cdot [- k^{-2}J_M - A \nu^2 J_F/B$$
$$- i \nu A(J_D - J_N + J_T md^2/A)/B].$$

Under the assumption on the order of magnitude of $\nu A/B$ this can be replaced by

$$- (gdU^{-2} \cos \theta)(- k^{-2}J_M - J_F \nu^2 A/B) i \nu A/B.$$

Thus the yaw of repose can be approximated by

$$\text{(8)} \quad \lambda_r = (gdU^{-2} \cos \theta)(i \nu A) \cdot (k^{-2}J_M + J_F \nu^2 A/B)/Bk^{-4}(\nu^2 J_T^2 + J_M^2).$$

This is not as good an approximation as (7), but in practice an even poorer approximation is usually used. This is

(9) <u>For projectiles for which</u> $\nu A/B$ <u>is of the order of</u> 1/50, <u>the following equation gives the yaw of repose to about</u> 5 <u>per cent</u>.

$$\lambda_r = gdU^{-2} \cos \theta \; i \nu A/J_M Bk^{-2}$$
$$= gdU^{-2} \cos \theta \; i \nu A/J_M md^2.$$

This expression indicates that the repose position of a shell is with its axis pointing slightly to the right of the trajectory. An inspection of the more precise form (6) shows that in general the shell also points above its trajectory, although to a much lesser extent. This rest position shows the reason for the drift of a projectile. A shell with normal right-handed spin will, in the absence of wind, always fall to the right of the vertical plane which contains its initial velocity vector. The equation (9) will permit us to compute the amount of this deviation.

Corresponding to the yaw of repose there is an angular velocity μ_r which, according to equations (1), is given by

$$(10) \qquad \mu_r = (\lambda_r' - a_1\lambda_r - b)/a_2.$$

In case the spin is zero (equation (7)), both λ_r' and $a_1\lambda_r$ are of the order of J times b. In the other case, (equation (9)), these terms are of the order of 1/50 of b. In either case, to an accuracy corresponding to the accuracy of the expression for λ_r, we may write

$$(11) \qquad \mu_r = gdU^{-2} \cos \theta/(\nu J_{XF} + i).$$

We now return to Section 2, equation (13), to compute the acceleration due to λ_r and μ_r. We have at once:

(12) <u>The component of acceleration perpendicular to the trajectory due to aerodynamic forces is given by</u>

$$a = [(J_D - J_N + i\,\nu J_F)\lambda + (\nu J_{XF} + iJ_S)\mu]\,U^2/d.$$

<u>A positive imaginary acceleration is measured to the left of the trajectory and a positive real downward.</u>

For a non-spinning projectile this gives an acceleration a_r, corresponding to λ_r and μ_r, of, (by equation (7)),

$$\begin{aligned} a_r &= g \cos \theta\, (J_D - J_N)(-J_D - k^{-2}J_H)/k^{-2}J_M \\ &\quad + g \cos \theta\, (J_D - J_N)(-2gdU^{-2} \sin \theta)/k^{-2}J_M \\ (13) \qquad &\quad + g \cos \theta\, J_S \\ &= g \cos \theta\, (J_L/k^{-2}J_M)(J_D + k^{-2}J_H + 2gdU^{-2} \sin \theta) \\ &\quad + g \cos \theta\, J_S. \end{aligned}$$

We have replaced $J_N - J_D$ by J_L (see (II.4.9)). Since J_M is negative for fin-stabilized projectiles, this acceleration will ordinarily be directed upward.

Making the corresponding calculation for spin-stabilized projectiles, employing (9) and (11), we have

$$a_r = (- J_L + i\nu J_F)(ig \cos \theta)(\nu A/J_M md^2)$$
$$+ (\nu J_{XF} + iJ_S)(g \cos \theta)/(\nu J_{XF} + i).$$

The second term here is of the order of J, while the first is of the order of 1/50. Thus, to an accuracy which is quite adequate, considering the accuracy of λ_r,

(14) $$a_r = ig \cos \theta \nu A(- J_L + i\nu J_F)/J_M md^2.$$

Of course the J's in this equation can be replaced by the aerodynamic coefficients K_L, K_F and K_M, since the density factor $\rho d^3/m$ is common to numerator and denominator.

The computation of the effects of the acceleration a_r is best made by the method of differential corrections. The real part of a_r, which represents an acceleration in the vertical plane of the trajectory, requires, for the computation of its effect, solutions either of the equations of variation or the associated adjoint system. The imaginary part of a_r permits a direct computation by means of quadratures. In fact, referring to (IX.3.10), we may state:

(15) The drift, which is the deflection due to the yaw of repose, is given by

$$\text{Drift} = -\frac{gA}{md^2}\int_{t_0}^{T}\left\{\frac{[x(T) - x(t)]\ \nu \cos \theta\ K_L}{\dot{x}(t)\ K_M}\right\} dt.$$

Of course, $\nu \cos \theta$ may be replaced by $\omega_1 d\ \dot{x}(t)\ U^{-2}$ if desired.

6. Initial motion of the center of mass; swerve.

It is the purpose of this section to display the equations describing the angular and spatial motion of a projectile in a form suitable for use in experiments in which the initial motion of a projectile is measured. In particular, we have in mind spark range measurements and yaw card trials, both of which are discussed at some length in Chapter XIII. In both of these cases one is concerned with the motion of the projectile for a relatively short distance — a distance of the order of 7000 calibers. The measurements which we will wish to interpret are of varying accuracy. In the case of spark range data, the angular position of a shell is measured to an accuracy corresponding to a probable error of about 0.1 degrees, while in yaw card trials the accuracy is roughly one degree. In either case the most important range is from a yaw of two degrees (although this is usually too small for successful yaw card experiments) to a yaw of ten degrees, with some importance attached to yaws up to twenty degrees.

It is convenient to measure the yaw of the projectile from the trajectory to the axis, instead of in the reverse direction. The coordinate system commonly used in the spark range at Aberdeen has one axis pointing along the trajectory, another, the H-axis, pointing to the left and horizontal, and the V-axis pointing upward. (These axes are obviously connected with the horizontal and vertical plates on which the silhouette of the projectile is photographed.) The components of yaw in the H- and V-directions, or, to be precise, the tangents of the angles from the H- and V-axes respectively, to the axis of the projectile, are denoted by ξ_H and ξ_V. Recalling the definition of λ, and the coordinate system (2.12), one sees that

$$\xi = \xi_H + i\xi_V = i\lambda. \tag{1}$$

The form of the variation of $\xi_H + i\xi_V$ with distance is thus the same as that of λ, a linear combination of the solutions (3.23) with the yaw of repose added. Two small simplifications are possible. Since in the spark range the projectile is fired almost horizontally the terms $gdU^{-2} \sin \theta$ may be neglected. Secondly, the yaw of repose may surely be approximated as in equation (5.9), and the cos θ term there may be replaced by one. These two simplifications made, one obtains

(2) <u>The yaw as a function of distance in calibers is given by</u>

$$\xi = \xi_H + i\xi_V = c_1 \exp \Phi_1 + c_2 \exp \Phi_2 - gU^{-2}A / J_M md,$$

<u>where</u> Φ_1 <u>and</u> Φ_2 <u>are given by</u>

$$\Phi_1, \Phi_2 = \tfrac{1}{2}\int_0^p \{ 2J_D - J_N - k^{-2}J_H - (J_D - J_A md^2/A)/\sigma^2$$

$$\pm [J_N - J_D - k^{-2}J_H - (2J_T - J_A)md^2/A]/\sigma$$

$$+ i\nu A(1 \pm \sigma)/B \} dp.$$

We now consider the motion of the center of mass, taking the point of view that the yaw is a known function of the arc length p. Referring to equation (2.13), the aerodynamic force on the projectile has the form

$$\text{Force} = [(J_D - J_N + i\nu J_F)\lambda + (\nu J_{XF} + iJ_S)\mu] mU^2/d.$$

This is, of course, the representation of the force in the coordinate system of λ and μ. If λ is known, according to equations (3.2) and (3.3), μ is given by

$$\mu = [\lambda' - (J_D - J_N + i\nu J_F)\lambda + \gamma dU^{-2}] /(1 + \nu J_{XF}).$$

If we now inspect the expression for force above, we see that terms within the bracket which are much smaller than $J\lambda$-terms may be neglected. In computing the force, therefore, one may surely replace the expression for μ by the simpler form $\mu = \lambda'/(1 + \nu J_{XF})$.

If the numerator and denominator of this fraction are multiplied by the conjugate of the denominator, inspection of the expression for force shows that the νJ_{XF} term may be neglected, so the aerodynamic force takes the form:

$$(3)\quad \text{Force} = [(J_D - J_N + i\nu J_F)\lambda - i(\nu J_{XF} + iJ_S)\lambda']mU^2/d.$$

We are now in a position to determine the motion of the center of mass. The spatial coordinate system used in the spark range has the x-axis horizontal and to the left, the y-axis pointing vertically upward and the z-axis horizontal and (approximately) along the initial line of fire. Because the photographic plates used are small (8 inches by 10 inches) the line of fire is always within a few mils of the direction of the z-axis. Equation (3) gives the component of aerodynamic force perpendicular to the trajectory. The component perpendicular to the z-axis, $m(\ddot{x} + i\ddot{y})$, will be the expression (3) multiplied by the cosine of a very small angle, together with a component of the aerodynamic force along the trajectory, the drag. The coordinate system in which (3) is valid has its real axis downward, the imaginary to the left. Using S as an abbreviation for x + iy, it then follows that

$$(4)\quad m\ddot{S} = m(\ddot{x} + i\ddot{y}) = -i(\text{Force}) + m\dot{U}\dot{S}/U - img,$$

where "Force" is given by (3). The drag is $m\dot{U}$, and $\dot{S}/U$ projects this force perpendicular to the z-axis. The two terms in S can be combined to give a simpler and more convenient form. Using primes, as usual, to denote derivatives to arc length, p, we compute:

$$m\ddot{S} - m\dot{U}\dot{S}/U = (\ddot{S}U - \dot{U}\dot{S})m/U = (\dot{S}/U)^{\cdot}mU$$
$$= (S'/d)'mU^2/d = S''mU^2/d^2.$$

Modifying (4) by means of this computation and replacing $i\lambda$ by ξ gives

$$(5)\quad S'' = -[(J_D - J_N + i\nu J_F)\xi - i(\nu J_{XF} + iJ_S)\xi']d - igd^2/U^2.$$

We now obtain S by repeated integration. According to (2) the yaw ξ is of the form

$$c_1 \exp \Phi_1 + c_2 \exp \Phi_2 - gU^{-2} \nu A/J\, md,$$

where Φ_1 and Φ_2 are almost linear in arc length p. In what follows the integration for S will be performed as if these were linear functions. This amounts to ignoring the variation of ν. Thus,

$$\int_0^p \exp \Phi_1 \, dp = (\exp \Phi_1 - 1)/\Phi_1',$$

$$\int_0^p \int_0^p \exp \Phi_1 \, dp\, dp = (\exp \Phi_1 - 1)/(\Phi_1')^2 - p/\Phi_1',$$

and similarly for $\exp \Phi_2$. Using this equation, and grouping terms in the order constant, linear function of p, exponential, drift and gravity drop, leads to the following equation for S:

$$\begin{aligned}
S = {} & S_0 + (-J_L + i\nu J_F) d\,[c_1/(\Phi_1')^2 + c_2/(\Phi_2')^2] \\
& - i(\nu J_{XF} + iJ_S) d(c_1/\Phi_1' + c_2/\Phi_2') \\
& + pS_0' + p(-J_L + i\nu J_F) d(c_1/\Phi_1' + c_2/\Phi_2') \\
& - i(\nu J_{XF} + iJ_S) dp(c_1 + c_2) \\
& + c_1 d \exp \Phi_1 \,[-(-J_L + i\nu J_F)/(\Phi_1')^2 \\
& \qquad + i(\nu J_{XF} + iJ_S)/\Phi_1'] \qquad (6) \\
& + c_2 d \exp \Phi_2 \,[-(-J_L + i\nu J_F)/(\Phi_2')^2 \\
& \qquad + i(\nu J_{XF} + iJ_S)/\Phi_2'] \\
& + [gA(-J_L + i\nu J_F)/J_M m] \int_0^p \int_0^p U^{-2} \nu \, dp\, dp \\
& - igd^2 \int_0^p \int_0^p U^{-2} \, dp\, dp.
\end{aligned}$$

One term, that corresponding to the rate of change of the yaw of repose, $- gU^{-2}A\nu / J_M md$, has been omitted, and $J_D - J_N$ has been replaced by $- J_L$. It is possible to obtain several useful facts from the form of (6). In particular we can discover the effects of the initial yawing motion. The exponential terms in (6) will damp out if the projectile is stable, and the constant terms involving c_1 and c_2 are too small to be of practical significance. However, there is a term linear in p which involves c_1 and c_2. This term shows that as a result of the initial yawing motion there is actually a change in direction of the mean trajectory. We state explicitly for future reference:

(7) *An initial yawing motion*, $c_1 \exp \Phi_1 + c_2 \exp \Phi_2$, *changes the direction of motion of a projectile by the angle (in radians)*

$$(- J_L + i\nu J_F)(c_1/\Phi_1' - c_2/\Phi_2') - i(\nu J_{XF} + iJ_S)(c_1 + c_2),$$

where

$$\Phi_1', \ \Phi_2' = \tfrac{1}{2} \{2J_D - J_N - k^{-2}J_H - (J_D - J_A md^2/A)/\sigma^2$$
$$\pm [J_N - J_D - k^{-2}J_H - (2J_T - J_A)md^2/A]/\sigma$$
$$+ i\nu A(1 \pm \sigma)/B \}.$$

This result will be used in the discussion of the effects of yaw in aircraft fire.

It will also be convenient to state explicitly the facts from equation (6) which will be used in the spark range reduction.

(8) *If the initial yawing motion is given by*

$$\xi = c_1 \exp \Phi_1 + c_2 \exp \Phi_2,$$

the motion of the center of mass is given by

$$S = (\text{linear function of } p) + (\text{drift}) + (\text{gravity drop}) + r_1 \exp \Phi_1 + r_2 \exp \Phi_2,$$

where

$$r_1 = c_1 d[- (- J_L + iwJ_F)/(\Phi_1')^2 + i(w_{XF} + iJ_S)/\Phi_1']$$

and r_2 is a similar expression in c_2 and Φ_2.

As far as spark range work is concerned it is possible to use approximate forms for the drift and gravity drop in the above equation which will be very simple computationally. The equation governing the velocity permits simplification since the fire is very nearly horizontal, so that U is the solution of the equation $U' = - UJ_D$. Hence $U = U_o \exp(- J_D p)$. By straightforward computation one then obtains:

$$gd^2 \int_o^p \int_o^p U^{-2}\, dp\, dp$$

$$= gd^2 U_o^{-2} \int_o^p \int_o^p \exp\,(2J_D p)\, dp\, dp$$

$$= (gd^2 U_o^{-2}/2J_D) \int_o^p [\exp\,(2J_D p) - 1]\, dp$$

$$= (gd^2 U_o^{-2}/2J_D) \{[(\exp 2J_D p - 1)/2J_D] - p \}.$$

For normal rounds fired in the range the velocity loss is at most 10 per cent, and the exponential term above can be approximated by a polynomial. Replacing $\exp 2J_D p$ by

$$1 + 2J_D p + (2J_D p)^2/2 + (2J_D p)^3/6 + \ldots,$$

we have

$$(9) \quad gd^2 \int_o^p \int_o^p U^{-2}\, dp\, dp = \tfrac{1}{2} gd^2 U_o^{-2} p^2 [1 + (2J_D p/3) + \ldots].$$

The procedure which is used in the spark range reduction is to subtract from the measured values of S the quadratic and cubic terms of (9) and the terms of the same order in the expression for drift. The revised values should then be of the form, a linear function of p added to a combination of the exponentials exp Φ_1 and exp Φ_2. Since these exponentials are obtained from the yaw reduction it is then possible to fit the revised values of S with a linear combination of four known functions, namely, a constant, the function p and the two exponentials. This can be done by a more or less routine least squares procedure, and the values of r_1 and r_2 (see equation (8)) so obtained can be used to evaluate the coefficients K_L, K_F, K_{XF} and K_S. Actually, as we shall see, only the first of these, K_L, can usually be evaluated with meaningful accuracy. In line with this program an approximate expression for the drift is also needed.

The drift is of the form

$$\int_0^p \int_0^p U^{-2} \nu \, dp \, dp$$

multiplied by a constant. In approximating this integral we shall use the fact that the spin-decelerating moment is small, so that if ν is written $\omega_1 d/U$, the variation of ω_1 can be neglected. Thus,

$$\int_0^p \int_0^p U^{-2} \nu \, dp \, dp = \omega_1 d \int_0^p \int_0^p U^{-3} \, dp \, dp$$

$$= \omega_1 d U_o^{-3} \int_0^p \int_0^p \exp(3J_D p) \, dp \, dp$$

$$= \omega_1 d U_o^{-3} \exp(3J_D p)/(3J_D)^2 + (\text{linear terms})$$

$$= \nu_o U_o^{-2} \exp(3J_D p)/(3J_D)^2 + (\text{linear terms})$$

$$= \tfrac{1}{2} \nu_o U_o^{-2} p^2 (1 + J_D p + \ldots).$$

Actually, v_o should be taken as a mean value of v. Making use of this approximate formula gives at once the desired form:

$$(10)\quad [(-J_L + i\nu J_F)/J_M]\,(gA/m)\int_o^p\int_o^p U^{-2}\,\nu dp\,dp = \tfrac{1}{2}p^2[(-J_L + i\nu J_F)/J_M](g\nu_o A/mU_o^2)(1 + J_D p + \dots).$$

7. Initial motion; evaluation of constants.

The two constants, c_1 and c_2, which appear in the expression for the yaw ξ, are determined by the initial yaw and the initial angular velocity of the projectile. We will now evaluate the constants in terms of these quantities. This evaluation will be needed in computing the effects of particular launching conditions.

In making this computation it is quite justifiable to ignore the yaw of repose, since this ordinarily has the magnitude of only a fraction of a degree. The yawing motion is then of the form

$$(1)\qquad \xi = c_1 \exp \Phi_1 + c_2 \exp \Phi_2,$$

where Φ_1 and Φ_2 are given by (6.2). For $p = 0$, the equation (1) and the equation obtained by differentiating it show that

$$(2)\qquad \begin{aligned} \xi_o &= c_1 + c_2, \\ \xi_o' &= c_1\Phi_1' + c_2\Phi_2'. \end{aligned}$$

These equations may be solved for c_1 and c_2 easily, resulting in the following values for the constants:

$$(3)\qquad \begin{aligned} c_1 &= (\xi_o' - \xi_o\Phi_2')/(\Phi_1' - \Phi_2'), \\ c_2 &= (\xi_o' - \xi_o\Phi_1')/(\Phi_2' - \Phi_1'). \end{aligned}$$

These equations are in a fairly convenient form except that it is desirable to have ξ_o' expressed in terms of the initial angular velocity. Let us denote the component of initial angular velocity perpendicular to the initial tangent to the trajectory by Ω_o — or, to be more precise, let Ω_o be the complex number representation of this vector, taken in the same coordinate system as ξ. The relation between λ and μ is given by equations (3.2) and (3.3), but for present purposes a very approximate form of these equations will be used. We shall assume $\lambda' = i\mu$. Since μ is a representation of the cross angular velocity multiplied by d/U, this equation gives very simply a relation between Ω_o and ξ_o'. In the previous section we noted that $i\lambda = \xi$, and a direct comparison of the coordinate systems (2.12) and that of the previous section shows that $-i\mu_o = \Omega_o d/U$. Thus

$$(4) \qquad \xi_o' = -i\,\Omega_o d/U.$$

This value may now be used to replace ξ_o' in the equations (3) for c_1 and c_2. Summarizing the information contained in equations (1), (3) and (4) we may state:

(5) <u>The initial yawing motion of a projectile which is launched with yaw ξ_o and cross angular velocity Ω_o is given by</u>

$$\xi = (-i\,\Omega_o d/U - \xi_o \Phi_2')\exp \Phi_1/(\Phi_1' - \Phi_2')$$
$$+ (-i\,\Omega_o d/U - \xi_o \Phi_1')\exp \Phi_2/(\Phi_2' - \Phi_1'),$$

<u>where Φ_1 and Φ_2 are given by</u> (6.2).

NOTE: We now give the omitted justification of the approximation (5.5). Let (y_1, y_2, y_3) be the unit vector pointing upward. In our coordinate system, the x_2-axis is coplanar with this and the velocity vector (u_1, u_2, u_3), so the vector (0, 1, 0) is a combination of those vectors. That is, there are numbers h and k such that $hy_1 + ku_1 = 0$, $hy_2 + ku_2 = 1$, $hy_3 + ku_3 = 0$. Then $y_3 = y_1(u_3/u_1)$, which is nearly $(u_3/u) \sin \theta$. The x_1-axis lies above the trajectory by a small angle C whose sine is u_2/u, so to a close approximation

$$y_2 = -\cos(\theta + C) = -\cos\theta \cos C + \sin\theta \sin C$$

$$= -\cos\theta + (u_2/u) \sin\theta.$$

Hence

$$y_2 + iy_3 = -\cos\theta + [(u_2 + iu_3)/u] \sin\theta.$$

On multiplying by g and transposing, this becomes (5.5).

Chapter XII

L A U N C H I N G   E F F E C T S

1. Preliminaries.

It is the purpose of this chapter to compute the magnitude and direction of the deviations of certain trajectories from normal; in particular, deviations which are due to the character of the launching. There are really two distinct sorts of effects which will be considered. One sort may be called systematic. Of primary importance among these is the effect of launching with large initial yaw, as is done in sideways fire from aircraft. Such effects are predictable, since the character of the launching is known, and they form an integral part of computed firing tables. The second sort of effect is not systematic, but simply adds to the dispersion of the weapon. Among these are the effect of yaw in the gun (although the magnitude of this yaw is predictable, its direction is not) and the effects of eccentricity of mass. The study of these is not intended as a preliminary to firing table computation but simply as a means of setting up suitable tolerances in the manufacture of projectiles and guns. It also gives an indication of the importance of these factors in dispersion.

As a part of the procedure for the computation of the systematic effects, Sections 2 and 3 are devoted to a description of the experimental procedure and the analysis of data required to evaluate these. The second of these procedures, drift firing, is almost self-explanatory. The first, yaw card firings, is essentially the same experiment as that conducted in the spark range. The angular motion of the shell is analyzed more or less completely. Because the same data, but given to a higher order of accuracy, are analyzed fully in our discussion of the spark range, and because the method has been described previously by Fowler, Gallop, Lock and Richmond, the discussion of yaw card firings here is rather sketchy.

The fourth and fifth sections are devoted to the computation of windage jump and yaw drag. These sections are definitely designed for use in construction of aircraft tables, where the range is short and the trajectory very flat. This specialization is justified since this is the only case in which projectiles are systematically launched with large yaw. Further, since this yaw occurs when the projectile has its maximum velocity, the effect is vastly greater than in the case of bombing, since for most bombs aerodynamic factors are least important at launching. However, there is, in the case of bombing, a "sheltering" effect which is of some importance.

The discussion of the effect of yaw in the gun, of eccentricity of mass and of muzzle blast is intended to give some estimate of those components of dispersion which are due to the non-particle character of the trajectory. A complete discussion of dispersion is beyond the scope of this book. The effects considered here, together with those considered in previous chapters, such as the effect of non-standard ballistic coefficient, include the principal components of dispersion which are primarily exterior ballistic in character.

The accuracy required in the computations which follow is not extreme. As has been remarked, the windage jump and yaw drag computations are done primarily for sideways fire from aircraft, where the trajectory is very flat and the range is short. On the other hand, in the computation of effects which are not systematic there is no reason to require high accuracy. An accuracy of 10 per cent is more than adequate. Accordingly, the simplest forms of the various computations made in the previous chapter can be used. It is convenient to assemble all the facts from that chapter which are needed in what follows, and to adopt a less complex notation. Accordingly, we rewrite equations (XI.3.22), in a simplified form. For convenience the term $J_N - J_D$ will be replaced by $-J_L$. The factor $\sqrt{\sigma_0/\sigma}$ will be replaced by 1. Finally, the integration indicated in (XI.3.22) will be performed as if the integrand were constant. These simplifications then yield:

(1) If the yaw of repose is neglected, the yawing motion of a projectile is given by

$$\xi = c_1 \exp (h_1 + if_1)p + c_2 \exp (h_2 + if_2)p,$$

where c_1 and c_2 are constants determined by the initial conditions, p is the arc length along the trajectory measured in calibers and h_1, h_2, f_1 and f_2 are given by the following, where the subscript 1 corresponds to the positive choice of sign:

$$h_1, h_2 = \tfrac{1}{2} \{ - J_L - k^{-2}J_H + J_A \, md^2/A$$
$$\pm [J_L - k^{-2}J_H - (2J_T - J_A)md^2/A]/\sigma \},$$

$$f_1, f_2 = A \; v \; (1 \pm \sigma)/2B.$$

The symbols in this equation are defined in Section 2 of the preceding chapter, except for $\sigma = \sqrt{1 - 1/s}$, where s is the stability factor defined in (XI.3.18). It is to be remembered that the h's are very much smaller than the f's. For normal shell, the ratio h/f would hardly be more than one or two hundredths.

For use in the discussion of drift firings we also
ote from (XI.5.15).

) A projectile drifts to the right of the vertical ane containing its initial velocity vector by the ount

$$\text{Drift} = \frac{gA}{md^2} \int_{t_0}^{T} \frac{[x(T) - x(t)]\, v \cos\theta\, K_L}{\dot{x}(t) K_M}\, dt.$$

ere x is the horizontal component of the distance om the muzzle to the shell.

The constants contained in the expression (1) for e yaw were evaluated in Section 7 of the previous hapter. It was shown in (XI.7.5) that if the initial otion of a shell was given by

$$\xi = c_1 \exp \Phi_1 + c_2 \exp \Phi_2,$$

hen the constant c_1 was

$$(-i\Omega_0 d/U - \xi_0 \Phi_2')/(\Phi_1' - \Phi_2'),$$

where Ω_0 is the initial cross angular velocity and ξ_0 is the initial yaw. Referring to (1), we see that in the present notation $\Phi_1 = (h_1 + if_1)p$. The derivative Φ_1' contains two terms, but the larger of these if if_1. Using this term as the derivative we may the state:

(3) The constants c_1 and c_2 of (1) have the values

$$c_1 = -(\Omega_0 d/U + \xi_0 f_2)/(f_1 - f_2),$$

$$c_2 = -(\Omega_0 d/U + \xi_0 f_1)/(f_2 - f_1),$$

where Ω_0 is the initial cross angular velocity and ξ_0 is the initial yaw.

There is one more result which will be needed. According to equation (XI.6.7), an effect of the initial yawing motion $\xi = c_1 \exp \Phi_1 + c_2 \exp \Phi_2$ is to change the direction of the mean trajectory by the angle

$$-(J_L - i\nu J_F)(c_1/\phi_1' - c_2/\phi_2') - i(\nu J_{XF} - iJ_S)(c_1 + c_2).$$

In the present notation, and to the accuracy used here, this states:

(4) <u>An effect of the yawing motion given by (1) is to change the direction of the mean trajectory by the angle</u>

$$i(+J_L - i\nu J_F)(c_1/f_1 + c_2/f_2).$$

2. Yaw card firings.

We now proceed to discuss a method of observation of the angular motion of a projectile and the deductions that can be made from the observation. This method was devised by Fowler, Gallop, Lock and Richmond, and even today is used on almost all firing table construction. A series of cards, called yaw cards, are placed at intervals in front of the muzzle of the gun. Upon firing, the projectile punctures each card. From the shape of the hole, an estimate can be made of the yaw of the shell at that position. (We discuss this estimate further in the next paragraph.) Thus, what amounts essentially to a continuous record of the yaw as a function of distance is obtained. This record is compared with the theoretically known form of the yawing motion and the pertinent constants are deduced. This is, in outline, the procedure. In practice there are many difficulties, some of which will be discussed below. The reduction of the data is a rather delicate operation, and the accuracy of the results depends greatly on the person performing the reduction. For optimum results, one might suggest getting H. P. Hitchcock to look at the data.

The precise relation between the perforation in the yaw card and the yaw of the projectile depends, of course, on the shape of the shell. If, for example, it is known that the bourrelet of the shell (this is the point of forward support, in the gun, and is the first point where the shell is a full caliber in width) and the band cut the top and bottom of the hole, the yaw of the shell could be deduced as follows. The upper half of the bourrelet would cut out a half ellipse having a major axis d and a minor axis d cos δ. If the distance from the bourrelet to the band is b, there would be a straight section of length b sin δ, and then a lower half ellipse of minor axis d cos δ. The maximum diameter of the perforation is then

$$d \cos \delta + b \sin \delta,$$

and the yaw is approximately

$$(\text{maximum diameter} - d)/b.$$

This is an oversimplified case, but it is clear that the geometry of the shell together with the maximum diameter of the perforation determines the magnitude of the yaw. The orientation of the yaw is determined by the line of symmetry along the direction of maximum diameter. An individual measurement of yaw is good to about one degree. The analysis is complicated by the peculiar properties of the cards. For example, it is possible for a shell to pass through a cardboard leaving a perforation of diameter smaller than that of the shell! Very careful examination is necessary in order to secure accurate measurements.

It is unfortunately true that the cards themselves have an effect on the motion of the shell. Certain corrections can be made for this effect. One can assume that the effect of each card is to impart a discrete amount of angular momentum to the shell, this amount depending linearly on the yaw. Firings can then be made with dense and with sparse distribution of cards, and the change in behavior of the shell attributed to the change in number of cards. This gives a means of evaluating the amount of angular momentum transmitted to the shell. (Another way of thinking of this is to extrapolate from a dense distribution and a sparse distribution to a vacuous distribution.) We shall not go into the methods of making these corrections. It may be well to point out that they presuppose a certain amount of uniformity in the yawing motion from round to round. This uniformity actually exists, at least to a degree. If shell are loaded in the same manner they will have similar initial conditions and hence similar initial motion. This is, however, true only in a general way. No two shell are identical, and no two are seated in precisely the same way in the bore of the gun.

Let us suppose that we have available a continuous record of the yaw ξ as a function of distance, a series of points in the plane, each corresponding to a known distance from the muzzle. What sort of pattern is to be expected? We begin the analysis with a geometric description of this yawing motion. According to (1.1), the yaw is given by

$$\xi = c_1 \exp (h_1 + if_1)p + c_2 \exp (h_2 + if_2)p.$$

Let us consider first the case where c_2 is zero. Then

$$\begin{aligned} \xi &= c_1 \exp (h_1 + if_1)p \\ &= c_1 (\exp h_1 p)(\cos f_1 p + i \sin f_1 p). \end{aligned}$$

This has a simple meaning. The point ξ is moving in the xy-plane with an angular velocity of f_1 radians per caliber of travel about the origin. The distance of the point ξ from the origin is $|c_1 \exp (h_1 + if_1)p|$ and if the shell is stable this distance decreases, although relatively slowly. If the xy-plane is thought of as being situated with its origin on the trajectory, and moving so that it is always one unit ahead of the center of mass of the shell, this circular pattern with decreasing radius is the pattern that would be traced on the plane by the axis of the shell. Or the shell may be thought of as writing this pattern on a blackboard which travels along the trajectory ahead of it. In three dimensions, one may consider the axis of the shell to be moving around on a conical surface, the vertex angle of which is constantly decreasing.

The other simple motion, $\xi = c_2 \exp (h_2 + if_2)p$, is of the same type, a damped circular motion. In general, one would expect, then, to have a combination of these two motions, with neither c_1 nor c_2 zero. This is a vector sum, and the pattern can be visualized as follows. An arm rotates about the origin with speed f_2, the arm decreasing at the logarithmic rate h_2. At the end of this arm another arm is pivoted, and this arm rotates at rate f_1 and shrinks at the logarithmic rate h_1. The pattern drawn by the end of the second arm is the pattern described by ξ. This is called an epicycle, and the motion is called damped epicyclic motion.

It might be remarked that epicyclic motion is not uncommon. Suppose that we refer the positions of all objects to a coordinate frame fixed with respect to the earth. To the ancients these were the only reasonable reference frames. We know that they are inconvenient in astronomy, because they are not inertial frames. Nevertheless we continue to use them for many purposes; for example in celestial navigation the first step is the measurement of the angular distance from the sun or a star to the horizon, which is an earth-bound reference frame. To a first approximation, in a system fixed to the earth it will be found that the sun travels in a circle centered on the earth, the angular velocity being 15 degrees per hour. To a first approximation, the motion of a planet in our reference system is then an epicycle; the first arm rotates about the earth, and the sun is its other end; the second arm starts at the sun, and the planet is its other end. Before 100 A. D. it was known that the description is not adequate. In the second century Claudius Ptolemy showed that it could be made highly accurate by using more than a mere two revolving

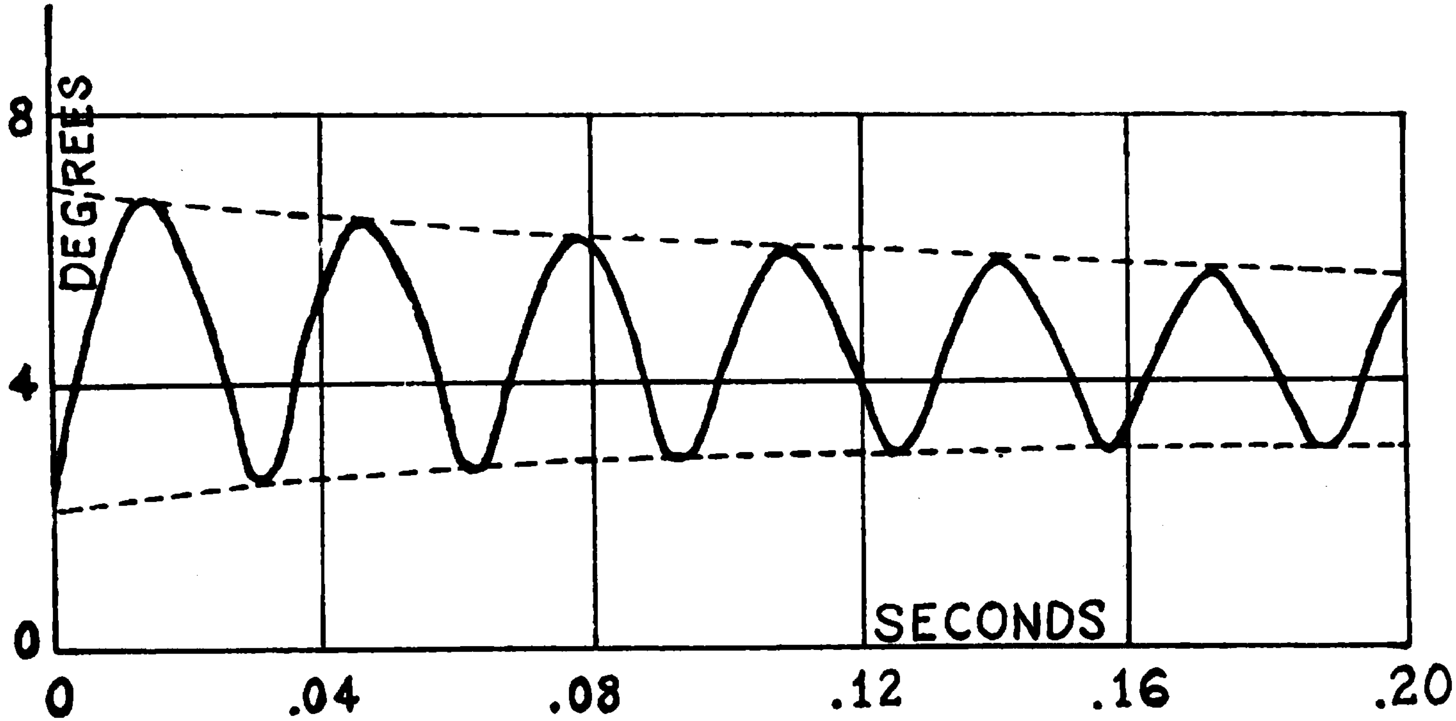

Figure XII.2.1

Yaw vs. Time, 3.3 Inch Projectile,
from Hayes, <u>Elements of Ordnance</u>

arms, forming what might be called epi-epi-epicycles (but nevertheless were called epicycles). Today we prefer inertial frames, and compute orbits as ellipses with superposed perturbations. But this preference is based on the essential simplicity and comprehensiveness of the Newtonian theory, not on any defect of accuracy of the Ptolemaic predictions.

We now proceed to consideration of the yaw card data. Suppose that the absolute value of the yaw is plotted against distance, giving a graph of which Figure 1 is typical. We may ask what the distance should be between successive maxima of yaw. If the damping rates h_1 and h_2 were both 0, so that the motion were a true undamped epicycle, maxima would occur when the two epicyclic arms pointed in the same direction, and minima when the arms pointed in opposite directions. This is not strictly accurate in the presence of damping. But h_1 and h_2 are so much smaller than f_1 and f_2 that even in the most accurate spark range measurements the maxima occur at an imperceptible distance from the places where the epicyclic arms have the same direction. The angle between the arms changes at the rate $f_1 - f_2$ radians per caliber since one rotates at the rate f_1 and the other at the rate f_2. The distance between maxima is then 2π divided by this rate. Thus, substituting the values from (1.1) in the expression $2\pi/(f_1 - f_2)$,

(1) <u>The distance between two maxima of yaw is</u> $2\pi B/A\nu\sigma$ <u>calibers of travel</u>.

Since A and B for a shell can be measured before firing, the measurement of the distance between maxima yields at once a value of $\nu\sigma$. An approximate value for ν may be obtained from the known twist of the rifling. Thus if the barrel has rifling making one turn in thirty calibers, ν must be very nearly $2\pi/30$. We say "very nearly" because there is some increase in the velocity of the shell due to muzzle blast, whereas there is no corresponding increase in spin. However, this increase in velocity is small, perhaps of the order of one per cent. Further, most of the change in v beyond the muzzle is due to change in the velocity, and this change can be computed if the drag of the shell is known. Thus, the value of v is known and, since $v\sigma$ is known, the value of σ may be deduced. Since $\sigma^2 = 1 - 1/s$, and

$$s = A^2 v^2 / 4B^2 J_M k^{-2},$$

(equations (XI.3.18)) the value of J_M and then of K_M may be computed. Thus, from the distance between two maxima of yaw and the twist of the rifling, together with certain physical constants, the value of K_M may be deduced.

So far, from the record of the yaw ξ, only the distance between successive maxima has been used. The yaw is a plane vector, and one might well inquire what can be deduced from the angular orientation of the yaw, in particular, the angle between the positions of ξ at two successive maxima. Since the maxima correspond to the geometric situation where the two arms have the same direction, this angle is precisely the angle through which the slow arm turns in the period between two maxima. This is

$$f_2 2\pi B/Av\sigma = \pi(1 - \sigma)/\sigma .$$

This then gives another possible way of finding σ. The equation (1) could then be used to find v. Actually, all of this information should be used simultaneously to determine σ. We state formally the above result.

(2) <u>The angle between the vector yaws corresponding to two successive maxima is</u> $\pi(1 - \sigma)/\sigma$.

The amplitude of the maxima and the minimum yaw can also be plotted as a function of distance (Figure 1). Recalling that the maxima correspond to positions where the two arms have the same direction and the minima to positions where the arms have opposite directions, it is at once clear that

$$(3)\quad \begin{aligned} |\xi|_{max} &= \Big| |c_1| \exp h_1 p + |c_2| \exp h_2 p \Big| , \\ |\xi|_{min} &= \Big| |c_1| \exp h_1 p - |c_2| \exp h_2 p \Big| . \end{aligned}$$

It is easy to decide whether $|c_1|$ or $|c_2|$ is larger by inspecting the pattern of yaw. On the basis of this decision either one or the other of the signs in the following can be chosen.

$$(4)\quad \begin{aligned} (|\xi|_{max} \pm |\xi|_{min}) &= 2|c_1| \exp h_1 p, \\ (|\xi|_{max} \mp |\xi|_{min}) &= 2|c_2| \exp h_2 p. \end{aligned}$$

These equations are obtained by adding and subtracting the equations (3).

Equations (4) permit the computation of $\exp h_1 p$ and $\exp h_2 p$. If the logarithms of these are taken, the slopes of the resulting functions of p are h_1 and h_2 respectively. One can thus compute h_1 and h_2, and, referring to equations (1.1), one may compute

$$(5)\quad \begin{aligned} (h_1 + h_2)m/\rho d^3 &= -K_L - k^{-2}K_H + K_A md^2/A, \\ (h_1 - h_2)\sigma m/\rho d^3 &= K_L - k^{-2}K_H - (2K_T - K_A)\, md^2/A. \end{aligned}$$

Adding and subtracting these equations we see that

$$(6)\quad \begin{aligned} &-2k^{-2}K_H - 2(K_T - K_A)md^2/A \\ &\qquad = -2k^{-2}\{K_H + (K_T - K_A)B/A\} \text{ and} \\ &2K_L - 2K_T md^2/A \end{aligned}$$

can be computed. It is not possible to evaluate the particular aerodynamic coefficients which occur here. Nevertheless, the damping rates h_1 and h_2 can be evaluated, and the combinations of coefficients given by (6). To summarize what information is available as a result of yaw card trials, one can obtain σ and ν and from these K_M; further, h_1 and h_2 can be evaluated and, from these and the values of σ, the combinations of coefficients (6) can be found.

Unfortunately, the accuracy of determination leaves something to be desired. The determination of K_M is usually sufficiently accurate, but the determination of the damping rates h_1 and h_2 is usually quite poor. This is to be expected from the nature of the measurements. The value of K_M depends on the distance between two successive maxima of yaw. Cards may be placed to locate the approximate distance between two successive maxima, and more cards may be used much farther away to locate a later maximum. Knowing the approximate period, the number of unobserved periods can be calculated and an evaluation made over a very long distance. On the other hand, the measurement of the damping rates requires differentiating the experimental data, in itself a notable way of losing accuracy. Further, the evaluation depends on the derivative of the logarithm of the experimentally determined function, so that the percentage accuracy is the determining factor. The measurement of the minimum yaw is a particularly bad source of error. Further, with normal launching the yaw is very small near the muzzle — except for effects at the muzzle itself, it would be simply the yaw in the gun. This means that the two arms of the epicyclic motion are initially nearly equal, so that the minimum yaw is initially almost zero. The rate of increase of this minimum yaw is (logarithmically) the difference of the damping rates, and it is sometimes impossible to distinguish the minimum yaw from zero at any position. At one time, due to experimental difficulties, it was common practice to assume the two damping rates equal. This amounts to assuming that the second of the expressions in (5) was always zero. This is obviously an incredibly unlikely identity — one that could be invalidated by changing the mass distribution, since this expression contains the physical constants k^{-2} and md^2/A. Naturally enough, this assumption leads to inconsistencies.

If h_1, h_2 and K_M are known for a shell there are still important effects which cannot be computed.

Computation of these effects requires in particular the value of K_L. This cannot be obtained from yaw card firings of the sort so far discussed. A possible procedure is to manufacture shell with the same external contour but with different center of mass position. By yaw card firings, K_M may be evaluated for both types. The difference in the moment coefficient values thus obtained is, referring to (II.5.7), the normal force coefficient K_N multiplied by the distance in calibers between the two centers of mass. This method is quite practical. It is the method used by Fowler et al in their classical experiments. However, it is time-consuming and expensive to modify standard shell — a dozen special shell are generally more expensive than 200 standard. Another method of finding K_L exists which does not require special shell, but does require a special gun. This is the method which is now in standard use at Aberdeen, and we now turn to the discussion of this experiment.

3. Drift firings.

We recall that the drift of a projectile is the lateral deviation due to the yaw of repose. It is necessary to list the drift in any firing table, and hence the prediction of drift is an integral part of exterior ballistic computation. A measurement of drift can be deduced from any range firing. The only causes of deflection, that is, the only causes for motion of the projectile perpendicular to the vertical plane containing its initial velocity vector, are cross wind, the effect of the rotation of the earth and the force causing drift. The first two of these can be computed, since meteorological data are taken and the drag coefficient is known — or at least independently deducible from the range firing. Any remaining systematic deflection after the observed data have been corrected for cross wind and rotation of the earth must be drift. For artillery fire it is possible to use this direct measurement of drift for a range table. It is not exactly correct, it is true, since the drift is actually a function of density and range wind structure, etc. However, the discrepancies caused by these terms are essentially of second order, and there is justification for using the measured drift directly. Nevertheless, in many cases, there are cogent reasons for arranging special firings to measure the drift. The cross wind effects may be quite large relative to the drift. The computation of the cross wind effect is subject to errors, and these in turn are reflected in the deduced value of the drift. One of the most successful ways to lose accuracy in a measurement is to obtain it as the difference of two relatively large quantities. In cases where maximum accuracy is desired, such as anti-aircraft firings, or in cases where the drift measurement is to be used for purposes other than prediction of drift, such as for guns fired sideways from aircraft, a drift firing is considered essential.

The basis for such firings lies in the fact that reversing the spin of a shell reverses the direction of the drift. Thus, if guns are available having both right- and left-handed rifling, shell may be fired alternately in a comparative firing. The wind conditions, density etc., will be the same for rounds fired from both guns. The difference in the trajectories is then only the difference in direction of the drift.

For small caliber guns a cardboard target is usually set up at a distance from each gun.* After careful boresighting, a number of rounds are fired from both guns. From the target cards the horizontal distance from the point of boresight to the center of impact of the rounds fired may be measured. The difference

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*A Mann barrel is usually used for the firing. This is simply a very heavy gun barrel which is permitted to slide freely in a V-block. This arrangement leads to extremely great accuracy in the firing — much higher than can be obtained with ordinary guns.

in these distances, for rounds fired from the right-handed and left-handed barrels respectively, is clearly twice the drift.

For anti-aircraft guns the procedure is in principle the same, although naturally more complicated. Instead of target cards the positions of burst are observed, either by cameras or by mirror position finders. (These are optical instruments designed for visual observation of space positions.) The drift is still obtained as half of the difference between the deflections observed for rounds from the right-handed and left-handed gun barrels.

In any case the result of a drift firing is a collection of values of drift corresponding to different ranges, and, in the case of anti-aircraft fire, also to different angles of elevation. We recall from equations (1.2) that

$$(1)\quad \text{Drift} = \frac{gA}{md^2}\int_{t_0}^{T} \frac{[x(T) - x(t)]\, \nu \cos\theta\, K_L}{\dot{x}(t)K_M}\, dt.$$

For small caliber guns it is justifiable to assume that K_L/K_M is constant over the rather short ranges used. These may then be taken outside the integral sign, and we see that the drift is K_L/K_M multiplied by physical constants and by the integral of known functions. This integral may be computed quite accurately — the only difficulty lies in the estimation of ν, and this question was discussed in the previous section. Thus, from drift firings one may obtain an estimate of K_L/K_M.

In the case of drift firings in the larger calibers certain other questions arise. Both K_L and K_M are functions of the Mach number, (speed)/(speed of sound). This variation can hardly be neglected over a long trajectory. Several methods have been used to allow for this variation. One of the most attractive of these is to assume a form for the variation of K_L/K_M with Mach number. For example, above sound one might well choose a form a + b/M, where M is Mach number. The constants a and b are then evaluated by fitting the observed drifts by means of a least squares procedure.

4. Windage jump.

If a gun is fired sideways from an aircraft the projectile is launched with an initial yaw, since it has, initially, the velocity of the aircraft superimposed on its muzzle velocity. There are two effects of this "abnormal" launching, one which can be interpreted as a change in the line of departure and another whcih can be interpreted as an instantaneous loss in velocity. The first of these, the windage jump, will now be evaluated.

We begin by simply computing the effect of an initial yaw ξ_o with zero initial cross angular velocity. According to (1.3), the constants c_1 and c_2 then have the values, setting $\Omega_o = 0$,

$$
(1) \qquad \begin{aligned} c_1 &= -\xi_o f_2/(f_1 - f_2), \\ c_2 &= -\xi_o f_1/(f_2 - f_1). \end{aligned}
$$

According to (1.4) the effect of the yawing motion with constants c_1 and c_2 is to change the line of departure by the angle $i(J_L - ivJ_F)(c_1/f_1 + c_2/f_2)$. Computing, using the values of the constants from (1),

$$
\begin{aligned} c_1/f_1 + c_2/f_2 &= -\xi_o\left[\frac{f_2}{f_1(f_1 - f_2)} + \frac{f_1}{f_2(f_2 - f_1)}\right] \\ &= +\xi_o(f_2 + f_1)/f_1 f_2. \end{aligned}
$$

Since $f_1, f_2 = Av(1 \pm \sigma)/2B$, $f_2 + f_1 = Av/B$ and

$$
f_1 f_2 = A^2v^2(1 - \sigma^2)/4B^2 = A^2v^2/4B^2s = k^{-2}J_M.
$$

These last two equalities come from the definitions of σ and s, equations (XI.3.18). Hence

$$(f_2 + f_1)/f_1 f_2 = A\nu/md^2 J_M.$$

Thus:

(2) The initial yaw ζ_o causes a change in the angle of departure of $\zeta_o i(J_L - i\nu J_F)A\nu/md^2 J_M$ or

$$iA\nu(K_L - iK_F)\zeta_o/md^2 K_M.$$

This is the basic result which is needed to compute the windage jump. The nose angle η is defined to be the angle from the direction of motion of the aircraft to the direction of the gun. If v_p is the speed of the aircraft and u_o the muzzle velocity then the initial speed of the projectile with respect to the air is

$$u_o = \sqrt{v_o^2 + v_p^2 + 2v_o v_p \cos\eta},$$

and the yaw at emergence is arc sin $[(v_p/u_o)\sin\eta]$. This is the magnitude of the yaw. The vector yaw lies in the plane of the tilt, that is, the plane containing the velocity vector of the plane and the initial velocity vector of the projectile. Further, the yaw vector points forward. Thus i multiplied by this vector is a vector which on the right-hand side of the aircraft points above the plane of tilt and on the left-hand side points below. We may thus state the principal result of this section:

(3) One of the effects in firing sideways from aircraft, the windage jump, changes the effective line of departure. This change is equivalent to turning the plane containing the velocity vector of the aircraft and the line of the barrel by

$$(A\nu K_L/md^2 K_M)\ \text{arc sin}\ [(v_p/u_o)\sin\eta].$$

If the projectile has right-handed spin, on the right-hand side of the aircraft the line of fire is raised and on the left-hand side it is lowered.

5. Yaw drag.

Another effect of the initial yawing motion of a projectile is to increase the drag on the shell. Although this is a second-order effect, i.e., it depends on the square of the magnitude of the yaw, it is quite appreciable. We shall compute first the yaw drag effect of an initial yawing motion

$$\xi = c_1 \exp (h_1 + if_2)p + c_2 \exp (h_2 + if_2)p.$$

This effect will be computed in the following form. We ask if, instead of the given yawing motion and initial velocity u_o, it is possible to find an equivalent trajectory with no initial yawing motion and with an initial velocity u_f — a "fictitious" initial velocity. According to (II.9.2) the variation of drag coefficient with yaw is of the form

$$(1) \qquad K_D = K_{DO}(1 + K_{D\delta}\delta^2),$$

where K_{DO} is the drag coefficient at zero yaw, $K_{D\delta}$ is the yaw drag coefficient, and δ is the yaw. The equation governing the velocity is (XI.2.9),

$$U' = - UJ_D - (gd \sin \theta)/U,$$

where the derivative is with respect to arc length measured in calibers. For the present purposes it will be adequate to ignore the gd sin θ/U, or one may consider that the Siacci approximation is to be used and solve first for velocity. The two trajectories which we wish to compare are then given, as far as velocity is concerned, by

$$(2) \qquad \begin{aligned} U' &= - UJ_{DO}(1 + K_{DS}\delta^2), \text{ initially } U = u_o, \\ U' &= UJ_{DO}, \text{ initially } U = u_f. \end{aligned}$$

Since $\delta^2 = \xi\bar{\xi}$ is known as a function of p, both of these equations are easily solvable. The solutions are

$$U = u_o \exp\left[- \int_0^p J_{DO}(1 + K_{D\delta}\xi\bar{\xi})\, dp \right],$$
(3)
$$U = u_f \exp\left[- \int_0^p J_{DO}\, dp \right].$$

The problem may now be phrased in the following way: what value of u_f will make the ratio of these two solutions approach one as p goes to infinity? This ratio is

$$(u_o/u_f) \exp\left[- \int_0^p J_{DO}K_{D\delta}\,\xi\bar{\xi}\, dp \right],$$

and in order that this ratio approach one it is necessary and sufficient that

(4) $$u_f = u_o \lim_{p\to\infty} \exp\left[- \int_0^p J_{DO}K_{D\delta}\,\xi\bar{\xi}\, dp \right].$$

It remains to compute this limit. Since

$$\xi = c_1 \exp(h_1 + if_1)p + c_2 \exp(h_2 + if_2)p,$$

$$\xi\bar{\xi} = c_1\bar{c}_1 \exp(2h_1p) + c_2\bar{c}_2 \exp(2h_2p)$$
$$+ c_1\bar{c}_2 \exp(h_1 + h_2 + if_1 - if_2)p$$
$$+ \bar{c}_1c_2 \exp(h_1 + h_2 - if_1 + if_2)\, p.$$

We recall that the f's are many times larger than the h's. Upon integrating with respect to p, the first two terms of this expression will be multiplied by factors of the order of the reciprocal of h, while the last two will be of the order of the reciprocal of f. The last two may then be neglected, and we may write

$$\int_0^p \xi\bar{\xi}\, dp = c_1\bar{c}_1 \left[\exp(2h_1p) - 1\right]/2h_1$$
$$+ c_2\bar{c}_2\left[\exp(2h_2p) - 1\right]/2h_2.$$

Since h_1 and h_2 are negative (we are not interested in making this computation for unstable shell), the limit of this integral as p approaches infinity is $-c_1\bar{c}_1/2h_1 - c_2\bar{c}_2/2h_2$. Applying this result to (4) we have

(5) Suppose that a projectile is launched so that its initial yawing motion is given by

$$\zeta = c_1 \exp (h_1 + if_1)p + c_2 \exp (h_2 + if_2)p,$$

and its initial velocity is u_o. Then, if a shell is launched with initial yaw and initial cross angular velocity zero, and if its initial velocity is given by

$$u_f = u_o \exp J_{DO}K_{D\delta} (c_1\bar{c}_1/2h_1 + c_2\bar{c}_2/2h_2),$$

the ratio of the velocities of the two shell will approach one as p approaches infinity.

Of course the trajectories of the two shell will not be precisely the same. The time it takes to reach a given distance p, and the drop below the line of fire corresponding to that distance will be slightly different for the two shell. However, if the Siacci approximation is used, both time and drop can be obtained as integrals with respect to p and in both cases the integrands contain factors which are negative powers of the velocity. Small differences in velocity in a region where the velocity is large therefore make relatively little difference in the values of time and drop obtained. Thus as an approximation, the effect of the initial yawing motion is to change the initial velocity from u_o to the u_f given by (5).

This can be immediately applied to the problem of sideways fire from aircraft. As was remarked in the previous section, the yaw at emergence is $(v_p/u_o) \sin \eta$, where v_p is the speed of the airplane, u_o the initial velocity of the projectile and η is the angle from the nose to the line of fire. The constants c_1 and c_2 are then, by (1.3),

$$(6) \qquad \begin{aligned} c_1 &= -(v_p/u_o)\sin\eta\; f_2/(f_1 - f_2), \\ c_2 &= -(v_p/u_o)\sin\eta\; f_1/(f_2 - f_1). \end{aligned}$$

We now compute the expression (5). We have:

$$(c_1\bar{c}_1/2h_1) + (c_2\bar{c}_2/2h_2)$$
$$= (v_p/u_o)^2 \sin^2\eta \;[\, f_2{}^2/2h_1(f_1 - f_2)^2 + f_1{}^2/2h_2(f_1 - f_2)^2 \,].$$

Recalling that $f_1, f_2 = A\nu(1 \pm \sigma)/2B$,

$$f_2{}^2/(f_1 - f_2)^2 = (1 - 1/\sigma)^2/4,$$

and

$$f_1{}^2/(f_1 - f_2)^2 = (1 + 1/\sigma)^2/4.$$

Thus

$$c_1\bar{c}_1/2h_1 + c_2\bar{c}_2/2h_2$$
$$= (v_p/u_o)^2 \sin^2\eta \;[\, (1 - 1/\sigma)^2/8h_1 + (1 + 1/\sigma)^2/8h_2 \,],$$

and we may summarize:

(7) The effect of the yaw drag in aircraft fire may be approximated as a change in initial air speed of the shell from u_o to

$$u_f = [\, u_o \exp J_{DO}K_{D\delta}(v_p/u_o)^2 \sin^2\eta]$$
$$\cdot[\,(1 - 1/\sigma)^2/8h_1 + (1 + 1/\sigma)^2/8h_2\,],$$

where v_p is the velocity of the aircraft, η is the nose angle, and h_1 and h_2 are the damping rates of the yaw.

6. Effect of yaw in the gun.

In seeking the causes of dispersion one of the first conjectures might well be that "undue" clearance in the gun might add greatly to the random error. That this is actually the case has been shown repeatedly by experiment. For example, some firings conducted at Aberdeen under carefully controlled conditions of

105 mm. shell showed a statistically significant difference in range for a difference in diameter at the bourrelet of 0.001 inch. This difference in range was entirely due to exterior ballistic differences, for the ranges were corrected to a standard muzzle velocity, and since the firings were conducted on a comparative basis (one round of each of several classifications of shell alternately from the same gun) no other factor enters. In this section we shall make an analysis which will indicate the reasons for this situation. We shall first derive a formula of R. H. Kent giving the relation between the yaw in the gun and the yaw in flight, and it will then be possible to compute, as has been done for sideways fire from aircraft, the effects of the yawing motion.

In making the analysis it is necessary to know something about the motion of the shell in the gun. The rotating band, at the rear of the body of the shell, serves to center the rear of the shell quite accurately, and to give essentially zero clearance at the back. The other point of support of a shell is the bourrelet. Any clearance at the bourrelet will usually be immediately reflected in yaw of the projectile. An analysis of the motion of the shell within the gun has been made by one of the authors and also by L. H. Thomas. For non-pathological shell it turns out that the bourrelet (after certain possible initial bouncing) rides smoothly along one land of the rifling. This is shown clearly by the markings on recovered shell, which, typically, show engraving marks on one side of the bourrelet only. This is intuitively very reasonable, for the shell is supported at the band behind the center of mass and, if the center of mass at any time does not lie on the axis of the gun, centrifugal force will cause the center of mass to move farther out. Further, the resultant of the force exerted by burning of the powder gas lies along the axis of the gun, so that any displacement of the center of mass of the shell results in a torque tending to increase the displacement.

This, it may be remarked, is in contrast with the motion of a rocket in a tube, where the propulsive force is along the axis of the projectile and results in a torque exerted by the tube on the bourrelet of the rocket.

Let us now suppose that the clearance at the bourrelet is of magnitude e, that is, this is the difference between the diameter of the shell and that of the bore of the gun. If the distance between the bourrelet and band is b, then the yaw in the gun should be e/2b, and this we denote by ξ_g. This will ordinarily be a rather small quantity. However, the spin of the shell, under the description given about of its motion in the gun, will be about the axis of the gun. If ω is the spin per caliber imparted by the rifling, there will be a component of angular velocity perpendicular to the axis of the shell, a cross angular velocity, of ω multiplied by the sine of the yaw. This angular velocity has the direction opposite that of the yaw, and is therefore equal to $-\xi_g\omega$. The initial conditions for the trajectory are therefore $\xi_0 = \xi_g$ and $\Omega_0 = -\xi_g\omega$, if the effect of muzzle blast is neglected. According to equations (1.3), the values of the constants c_1 and c_2 are then

$$c_1 = -(-\xi_g\omega d/U + \xi_g f_2)/(f_1 - f_2),$$
$$c_2 = -(-\xi_g\omega d/U + \xi_g f_1)/(f_2 - f_1).$$

Making the substitutions $f_1, f_2 = A\nu(1 \pm \sigma)/2B$, and writing ν for $\omega d/U$, these reduce to the statement:

(1) <u>If the yaw in the gun is ξ_g then the constants c_1 and c_2 which determine the yawing motion are given by</u>

$$c_1 = (\xi_g B/A\sigma)\ [1 - (1 - \sigma)A/2B],$$
$$c_2 = -(\xi_g B/A\sigma)\ [1 - (1 + \sigma)A/2B].$$

From this form it is easy to see that c_1 and c_2 are vectors pointing in opposite directions, i.e., initially the yaw is at a minimum. The maximum yaw can be computed easily from this form. It will have the magnitude of $c_1 - c_2$. This is Kent's formula.

(2) If the yaw in the gun is ξ_g then the maximum yaw is $(2B/A - 1)|\xi_g|/\sigma$.

This formula shows at once the importance of having the yaw in the gun small. For ordinary shell the factor B/A is of the order of ten, and σ is perhaps 0.7. The yaw in the gun ξ_g then results in a maximum yaw of approximately 28 $|\xi_g|$. Thus, if it is desired to keep the maximum yaw less than 3 degrees, it is necessary to have the yaw in the gun less than 0.1 degree, which is less than two thousandths radians. If the shell is of one caliber length from band to bourrelet, this implies that the clearance in the gun must be less than 2/1000 of the diameter of the bore.

The quantitative effects of the yaw in the gun can be computed rather easily. They are of two sorts, yaw drag and a "jump," just as the effects of initial yaw in firing from aircraft. We first compute the jump, making use of the fact that by equations (1.4) this jump has the form $i(J_L - iwJ_F)(c_1/f_1 + c_2/f_2)$. The constants c_1 and c_2 are given by (1) above, and from Section I, $f_1, f_2 = Av(1 \pm \sigma)/2B$. Thus, computing,

$$
\begin{aligned}
(c_1/f_1 + c_2/f_2) \\
&= (\xi_g B/A\sigma)\left[\frac{1 - A(1-\sigma)/2B}{Av(1+\sigma)/2B} - \frac{1 - A(1+\sigma)/2B}{Av(1-\sigma)/2B}\right] \\
&= (\xi_g B/A\sigma)\left\{[-2\sigma + 2A\sigma/B]/[Av(1-\sigma^2)/2B]\right\}.
\end{aligned}
$$

According to (XI.3.18),

$$1 - \sigma^2 = 1/s,$$

and

$$s = A^2v^2/4B^2k^{-2}J_M.$$

Thus

$$(c_1/f_1 + c_2/f_2)$$

$$= (\xi_g B/A\sigma)\{-2\sigma(1 - A/B)/(2Bk^{-2}J_M/A\nu)\}$$

$$= -\xi_g \nu(1 - A/B)/k^{-2}J_M.$$

Substituting in the form for the jump, we have

(3) <u>If the yaw in the gun is ξ_g, the direction of motion of the shell is changed by the amount</u>

$$\text{Jump} = -i\xi_g\nu(1 - A/B)(K_L - i\nu K_F)/k^{-2}K_M.$$

Since the ratio A/B is of the order of 1/10 and K_L/K_M may be expected to be of the order of unity, the jump due to yaw in the gun is of the order of magnitude of the yaw multiplied by ν. The direction of this jump is at right angles to the yaw in the gun and can be predicted only if one has some way of knowing the direction of ξ_g.

We now compute the effect of yaw drag due to yaw in the gun. To do this, we make one simplification. The velocity effect given by (5.5) depends on the squares of the magnitudes of c_1 and c_2. The values of c_1 and c_2 for the present situation may be approximated to about 5 per cent by

(4) $$c_1 = \xi_g B/A\sigma, \qquad c_2 = -\xi_g B/A\sigma.$$

This approximation will lead to an error of the order of 10 per cent in the squares of the magnitudes, but this accuracy is quite adequate for the present discussion. According to (5.5), the effect of the yawing motion can be described as the same as the effect of launching the shell with no yaw, but with a "fictitious" velocity

$$u_f = u_0 \exp J_{DO}K_{D\delta}(c_1\bar{c}_1/2h_1 + c_2\bar{c}_2/2h_2).$$

Using the values of c_1 and c_2 from (4) we have at once:

(5) <u>The yaw drag effect of the yaw ξ_g in the gun is equivalent to changing the muzzle velocity from u_0 to</u>

$$u_f = u_0 \exp J_{DO} K_{D\delta}(1/h_1 + 1/h_2) \mid \xi_g \mid^2 B^2/2A^2\sigma^2.$$

We shall make an estimate of the order of magnitude of this change in velocity in order to indicate the sort of tolerance this imposes on gun and shell design. The ratio of J_{DO} to one of the h's is about 1/10, and $K_{D\delta}$ may be expected to be of the order of 16 or 20 (per radian ^{2}). The factor $B/A\sigma$ is ordinarily of the order of 15, so that the entire exponent is of the order of $2 \mid 15\,\xi_g \mid^2$. Since $\exp x = 1 + x + \ldots$ the effect is to reduce the velocity by the factor

$$2 \mid 15\xi_g \mid^2 u_0.$$

If it is required that the yaw in the gun reduce the velocity by not more than ½ per cent, which is a reasonable requirement, this requires that

$$2 \mid 15\,\xi_g \mid^2 < .005,$$

$$\mid \xi_g \mid < .003\ldots.$$

Again, as for the jump, we see that the effects of yaw in the gun require that the clearance be kept to a minimum. There is, however, one difference between the velocity effect and the jump. The former is systematic, always resulting in a decrease in velocity, whereas the latter is essentially random, since the initial orientation of the yaw vector usually cannot be specified.

7. Effect of eccentricity of mass.

Another cause of dispersion which is of importance is possible eccentricity of mass of shell. Since shell are first forged, forming the inner cavity, and then turned on a lathe to form the outside surface, it requires rather fine machining to be certain that the axis of the cavity coincides with the axis of the outside form of the shell. Lack of coincidence results in one or both of two different kinds of unbalance. The first of these, called static unbalance, consists of the center of mass of the shell lying off the axis of form of the projectile. The second, called dynamic unbalance, consists of the axis of inertia of the shell being not parallel with the axis of form. As a measure of the static eccentricity we may use the distance e, measured from the axis of form to the center of mass, and the dynamic eccentricity is measured by the angle ε between the axis of inertia and a line parallel to the axis of form.

The static eccentricity has in the past been measured by rolling the shell on a pair of parallel bars. The angular motion of the shell is mathematically similar to that of a pendulum, since there is a torque proportional to the eccentricity e, the mass of the shell and the sine of the angle by which the line from the center of mass to the axis of the shell is displaced from vertical. Unfortunately, the rolling friction is of such magnitude that it is almost true that shell which show static unbalance under this test are not fit for use. A more direct, but an experimentally delicate, procedure is the following. The shell is placed in a V-block which is attached to a rigid framework, the framework being pivoted at one end and the other end resting on a balance system. The line from pivot to balance is perpendicular to the axis of the shell. Balance readings are then taken, the shell is turned on its axis in the V-block and the readings are again taken. If the shell is eccentric, the balance readings, plotted against angle, will show a sinusoidal variation. If a is the distance from pivot to the axis of the shell and b is the distance from the pivot to the balance, the maximum balance reading will be $(a + e)m/b$ and the minimum, $(a - e)m/b$, more than the balance reading when the framework, but not the shell, is present. The difference $2em/b$ then leads directly to an evaluation of the eccentricity e. Of course, in this form the eccentricity is obtained as the difference of two large quantities and adequate accuracy is difficult to achieve. This may, to some extent, be offset by properly counterbalancing the framework, but the experiment still requires a very exacting technique.

The measurement of dynamic unbalance is rather more complicated. It is necessary that the projectile be spun upon its axis and the resulting motion observed. This is much the same problem as the problem occurring in the balancing of crankshafts, and a machine whose basic design was for the purpose of measuring the unbalance of crankshafts has been used very satisfactorily for measurements on shell. Another machine has been constructed for use on small projectiles which spins the projectile by means of an air jet and makes a selection of projectiles with small eccentricity. Both of these machines are based on the principle that a rigid body "prefers" to rotate about a principal axis of inertia. We shall not go into the details of these measurements.

The problem of predicting the motion of an eccentric shell is purely a question of mechanics. The aerodynamic force system is known, for, of course, this system is independent of the mass distribution. However, the problem presents one of the most involved computations imaginable, and its importance scarcely warrants devoting the amount of space necessary to present it. We shall therefore be content to summarize the results of the computation, which was made by one of the authors (amid blood, sweat and tears).

The effect of static unbalance is extremely easy to describe. The shell, in the gun, rotates about the axis of the tube. Its center of mass is therefore moving about the axis of the tube at the rate $e\omega$, where ω is the spin of the shell. This velocity is in a direction perpendicular to the axis of the gun, so that angular deflection results which has the magnitude $e\,\omega/U$, or $\nu e/d$. The direction of this deflection is perpendicular to the line joining the center of mass to the axis of the shell (it is precisely the same physical principle used at least as far back as the historic engagement of David and Goliath). If the gun has rifling having one twist in thirty, ν is $2\pi/30$; and in order that the effect of static unbalance be less than one mil, it would be necessary to specify that the static eccentricity be less than (approximately) 5/1000 of one caliber.

The effects of dynamic unbalance can be described in terms almost as simple. The angular motion of the axis of form of a dynamically unbalanced shell is tricyclic. That is, superimposed on the normal epicyclic motion is a third circular motion. For shell having the normal ratio of moments of inertia, a rather accurate description is to say that the axis of minimum inertia describes the same pattern as for noneccentric shell. By far the largest effect of the eccentricity ε is one which is physically very reasonable under these circumstances. Since the shell will spin about its axis of form in the gun, its angular velocity vector will have, relative to its axis of minimum inertia, the same relation as if the shell had a yaw in the gun of ε. The effects of the eccentricity ε are, as a matter of fact, almost precisely the effects of a yaw in the gun of ε. These effects may therefore be computed by means of the formulas of the previous section, and the remarks made there about orders of magnitude are valid in this situation. That is, eccentricity ε will cause a jump of the order $\varepsilon\,\nu$, and a velocity loss of the order $5 \cdot 10^4 \varepsilon^2$ per cent. (ε is in radians.)

It should be remarked that the effects of yaw in the gun and eccentricity are probably not statistically independent. Although no theory of motion in the gun is sufficiently complete to predict with certainty, it seems highly probable that these effects are, at least to a degree, comulative. That is, whereas if the two effects are truly independent, the mean square deflection when both are present would be the sum of the mean square deflection due to each, it is probably more nearly true that the deflection due to both effects is the sum of the deflections due to each.

8. Muzzle blast.

The phenomenon of muzzle blast is responsible for part of the dispersion of shell. As soon as the driving band on the shell is free of the gun the powder gas surges past the shell, imparting momentum to it. The remarkable series of photographs in Figure 1, made by A. C. Charters, shows clearly this phenomenon. An exact analysis of the situation appears to be impossible. However, a rather crude investigation can be made which should indicate the order of magnitude of the effects.

As can be observed on the photographs, the period in which the velocity of the gas exceeds that of the projectile is very short. It therefore seems reasonable to consider the phenomenon as impulsive, that is, to presume that there is an instantaneous transfer of momentum and angular momentum to the shell. This assumption then implies that the momentum transferred does not depend on the angular velocity of the projectile, but only on its position. From the symmetry of the situation, if the shell has no yaw at the muzzle the momentum transferred must be in the direction of the axis of the shell. This momentum can be measured, at least roughly, since there is no increase in angular momentum, and measurement of the spin immediately beyond the muzzle blast then indicates a spin lower than that given by the twist of the rifling. Such measurements indicate that an increase in the linear velocity of the shell of the order of one per cent may result from the muzzle blast. We shall use this figure to get an order of magnitude for other estimates.

Suppose now that the shell does not have zero yaw at the muzzle. If ξ is a complex number representation of the vector yaw, the component M of momentum perpendicular to the axis of the shell may be seen, by the same symmetry arguments used on the aerodynamic force system, to be of the form $M = c\xi$, where the imaginary part of the complex number c vanishes if the spin is zero. If the analogy with the aerodynamic situation is carried further, one might expect $|c|$ to be of the order of ten times the increment in momentum along the axis of the shell, since K_M/K_D is usually about ten. Using the previous estimate for increase in axial momentum $|c|$ should be of the order of magnitude of mu/10. The effect of M is to change the direction of the shell by M/mu radians; thus a deviation of $|\xi|/10$ radians might be expected. Since the yaw at emergence is always small, this effect is presumably negligible in most cases. However, if the boattail is eccentric, and in small caliber projectiles this may easily occur, the effective value of $|\xi|$ may be large. For example, if the boattail is sufficiently eccentric that the effective value of $|\xi|$ is 4 degrees, a deviation of 7 mils might well result.

A similar rough analysis can be made on the angular impulse resulting from muzzle blast. If T is the angular momentum, we might expect it to be of the order of mud$|\xi|/10$. For normal shell, having a radius of gyration of about one caliber. the result would be a cross angular velocity of about u$|\xi|/10$d. Applying an analysis similar to that in Section 7, this would result in a jump of magnitude

$$|\xi|(1 - A/B)(K_L - i\nu K_F)/10k^{-2}K_M.$$

This is of the same order of magnitude as that obtained from the cross component of linear momentum, but in a direction perpendicular to the previous effect. Again, no serious dispersion should result from this cause except for projectiles with eccentric afterbodies.

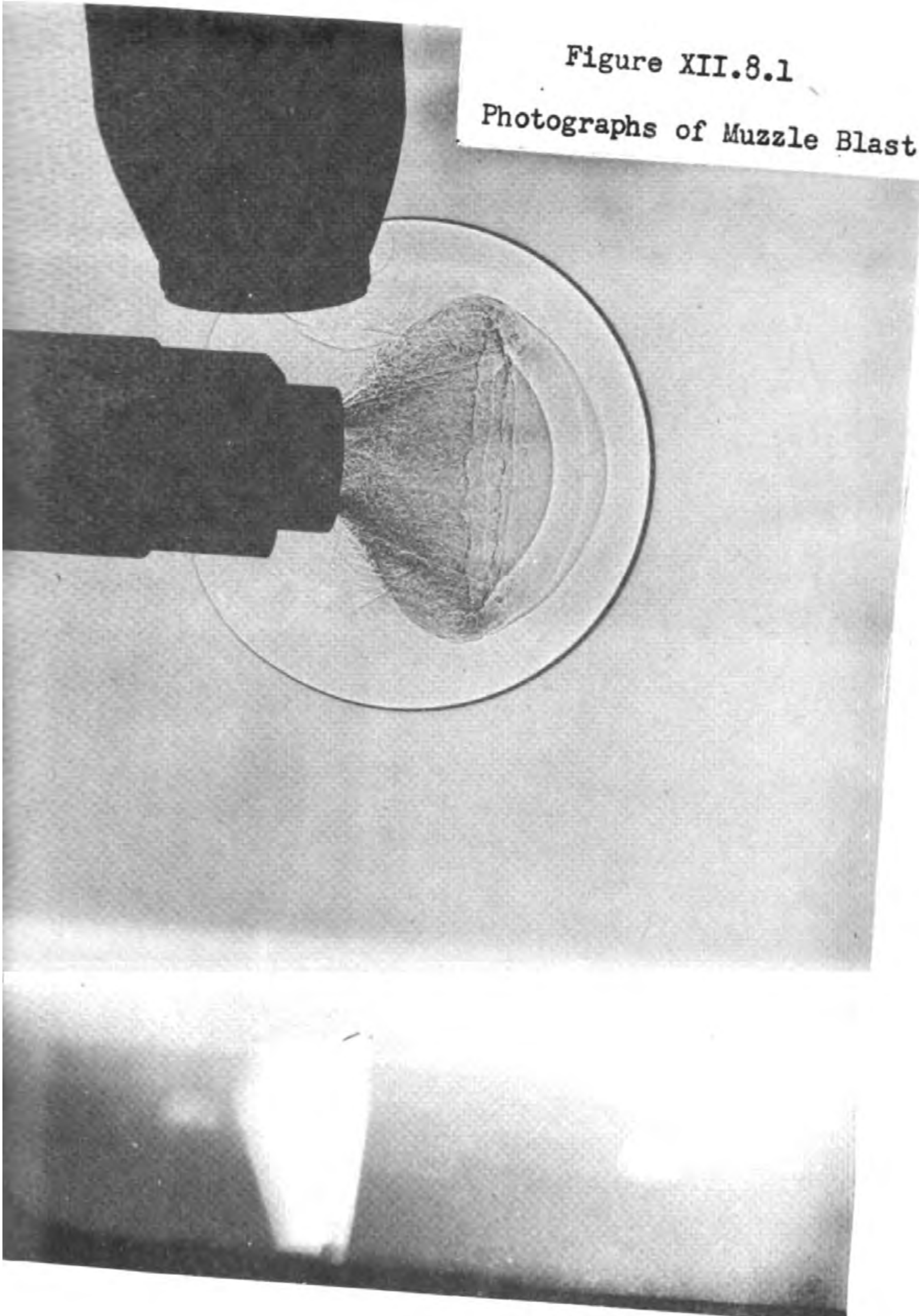
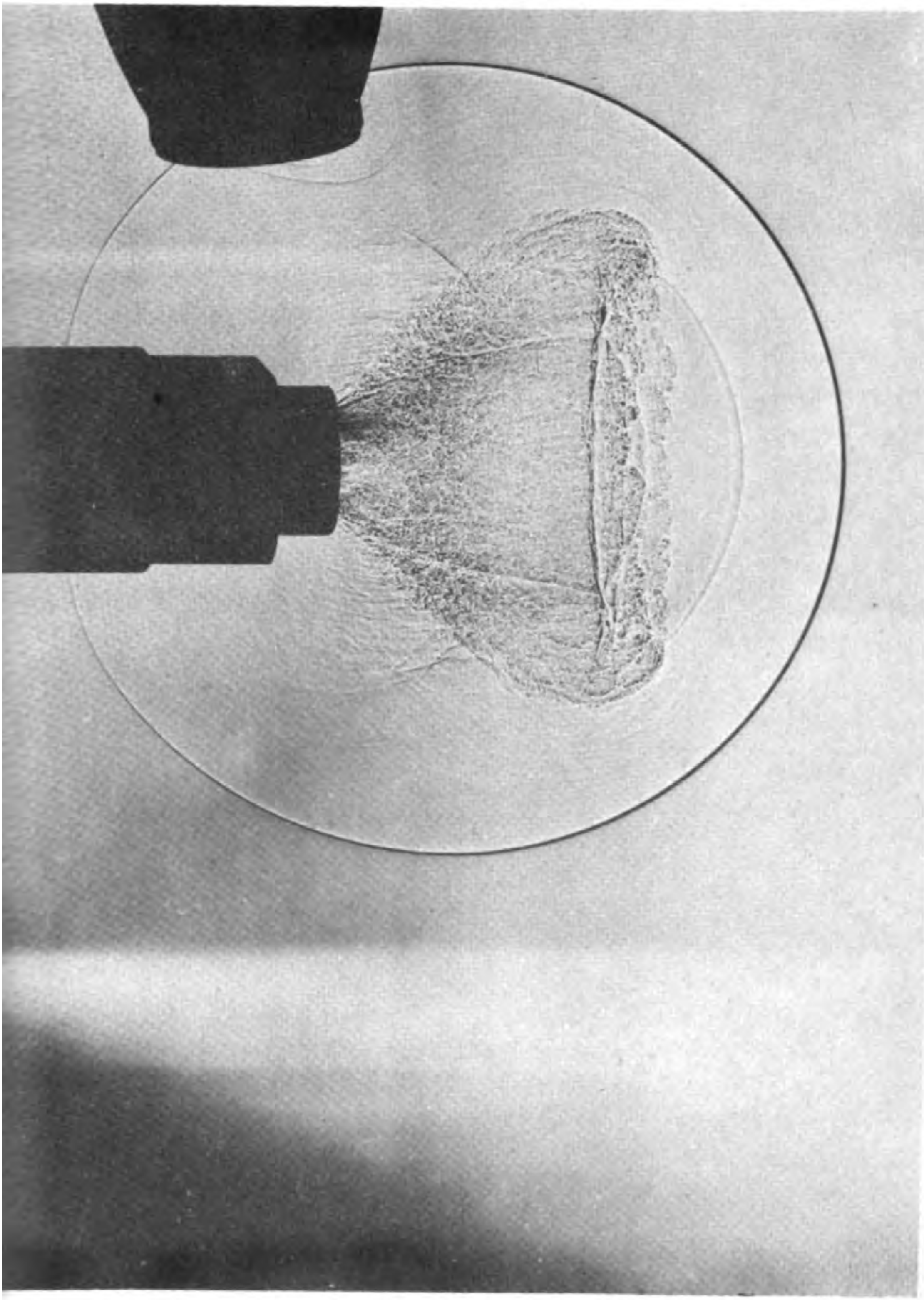
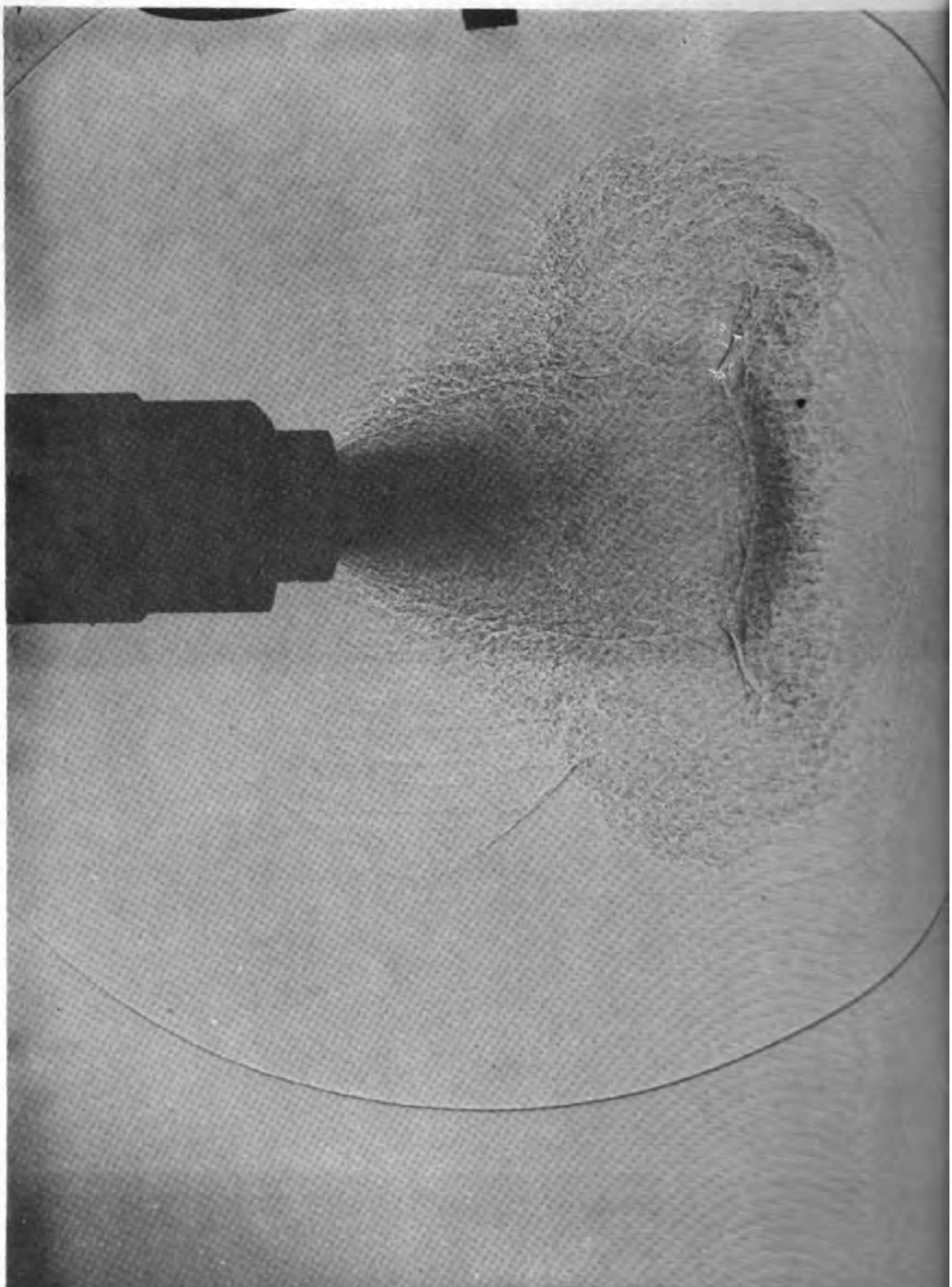
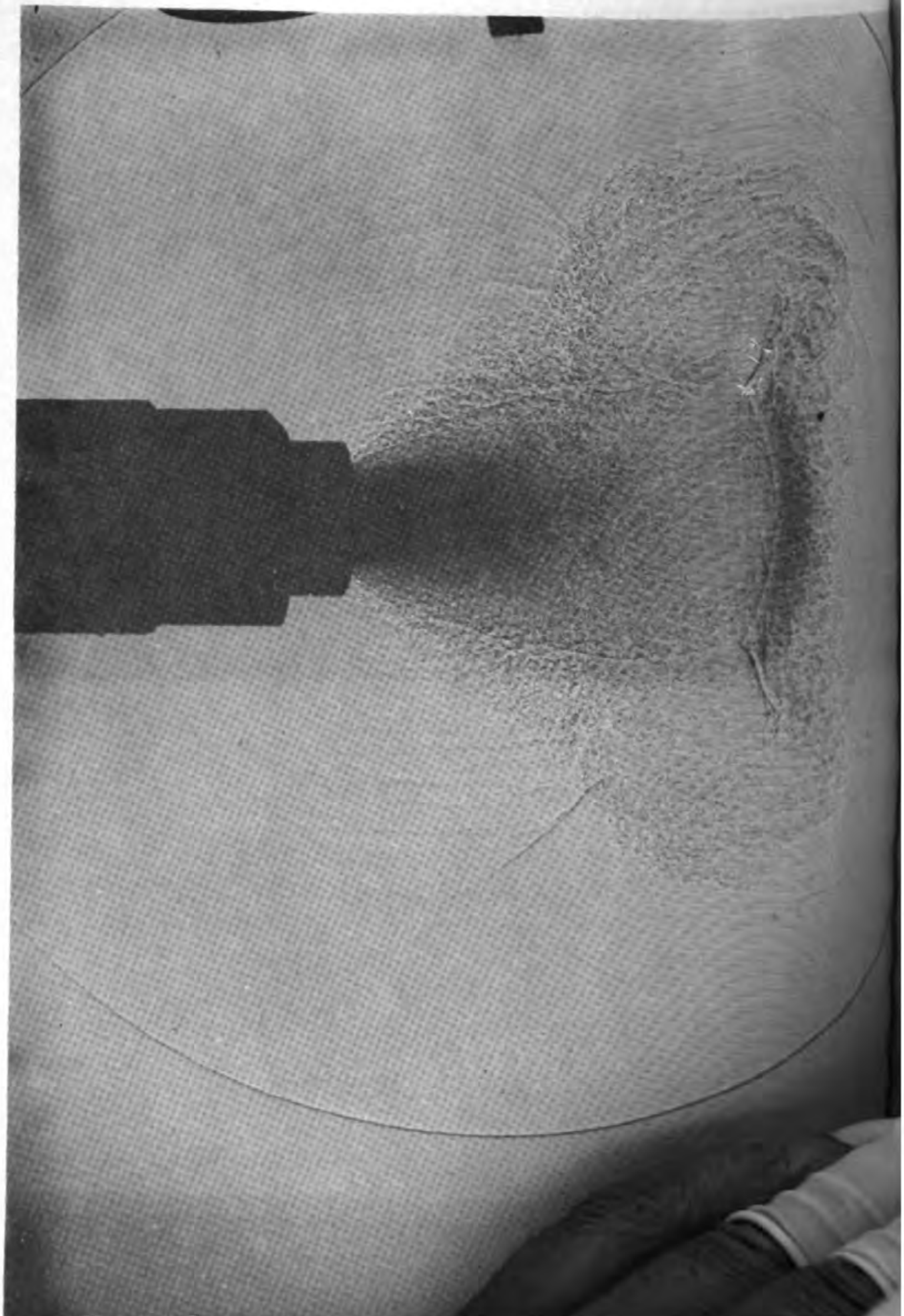


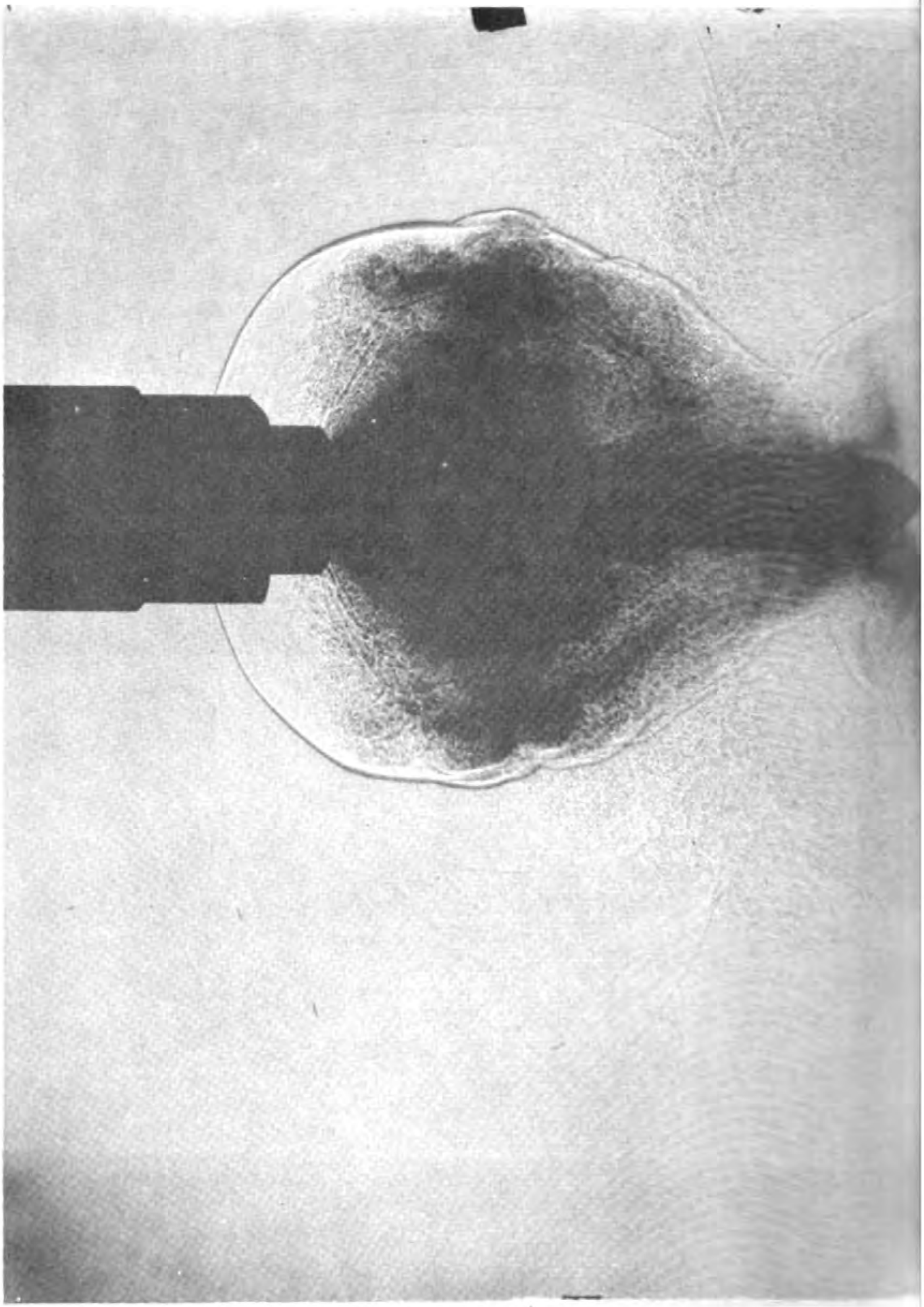
Figure XII.8.1

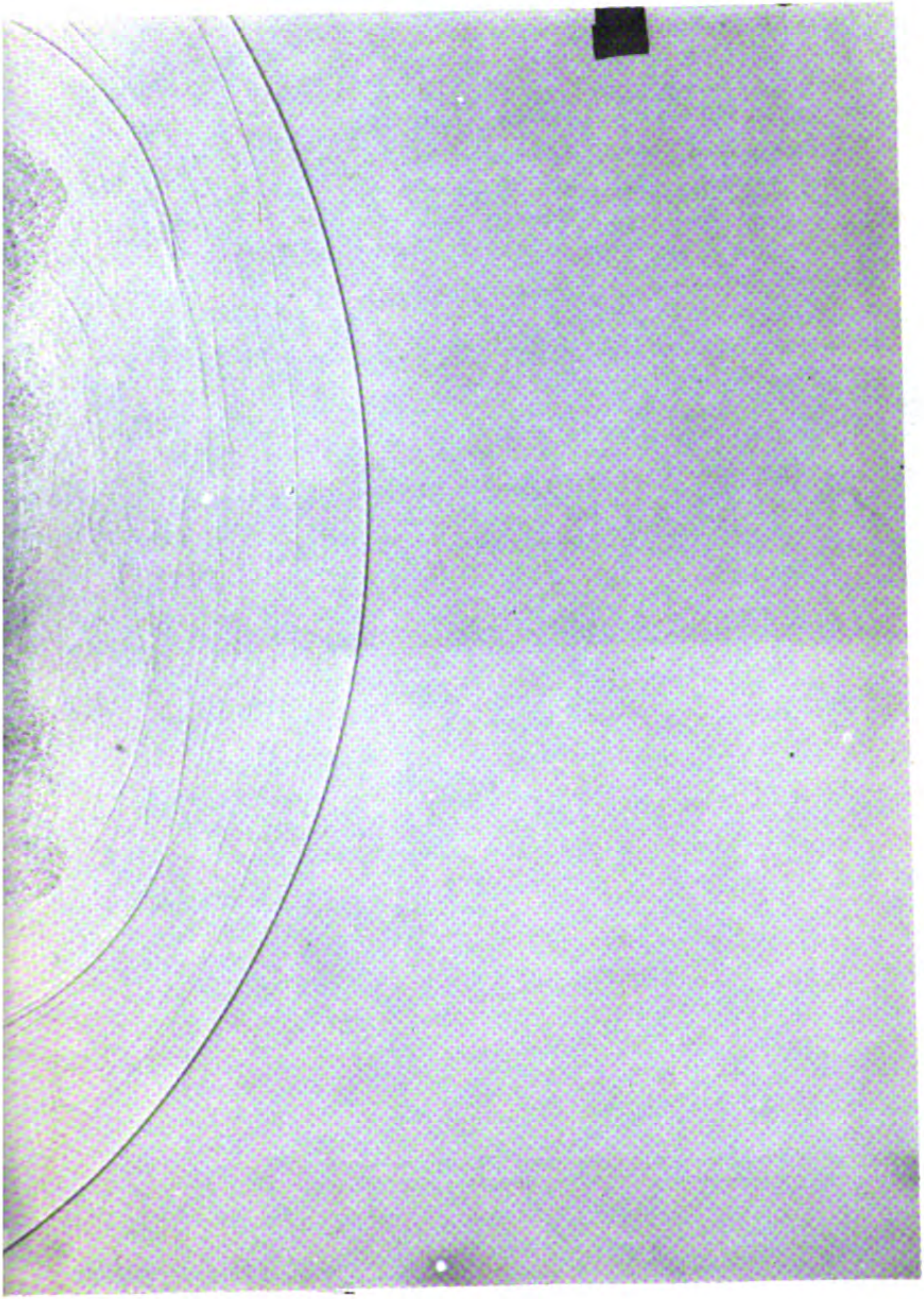
Photographs of Muzzle Blast

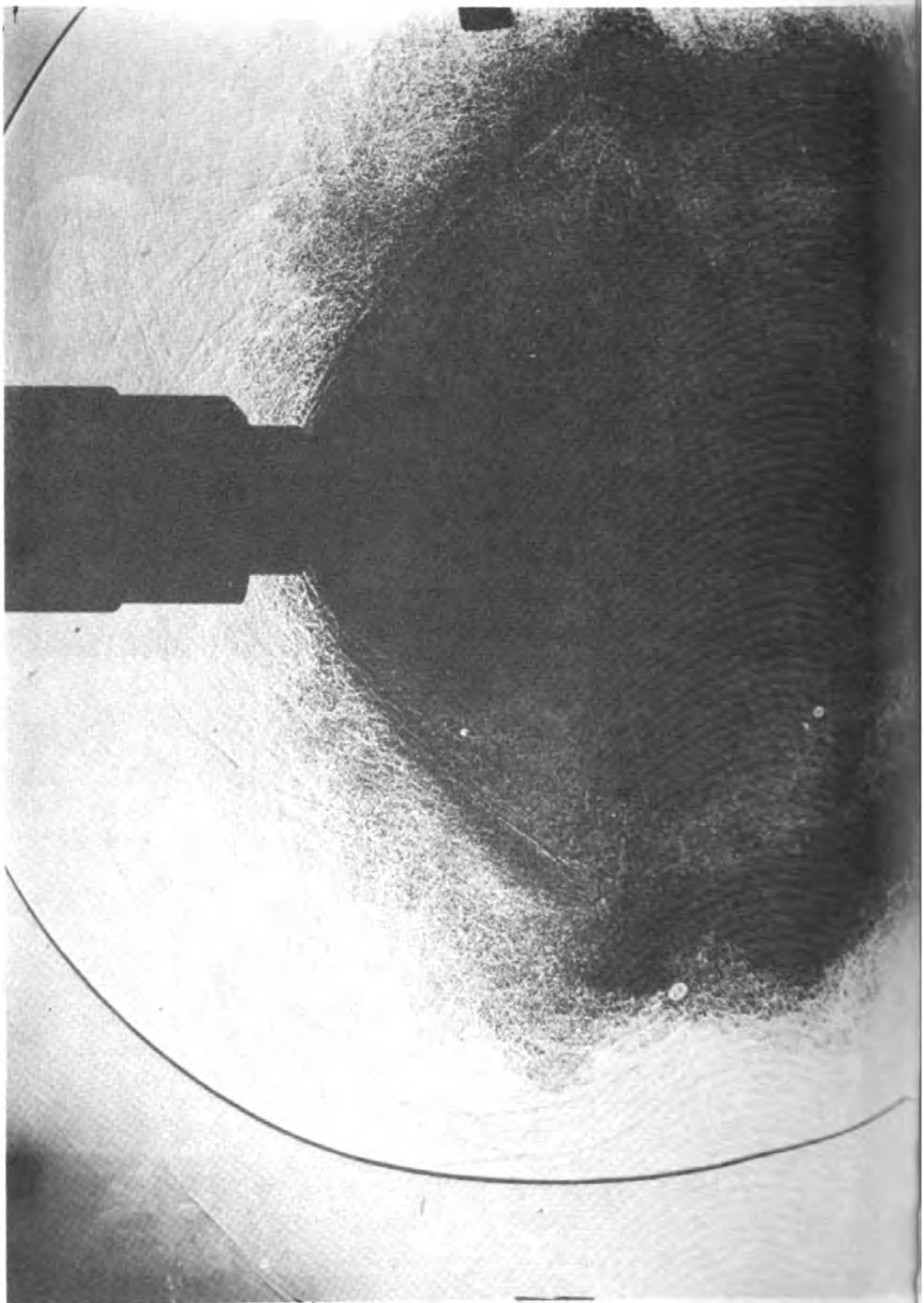


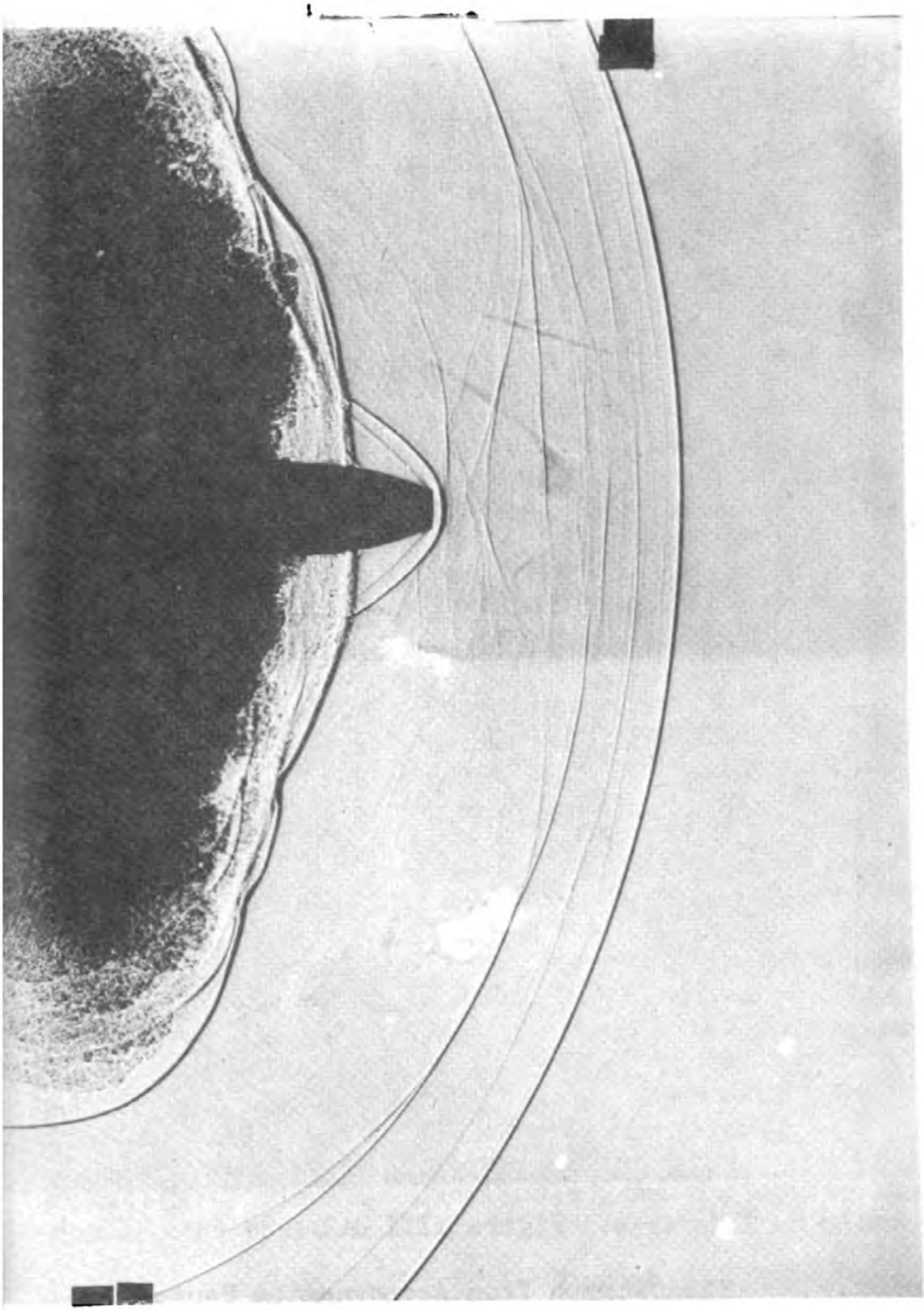












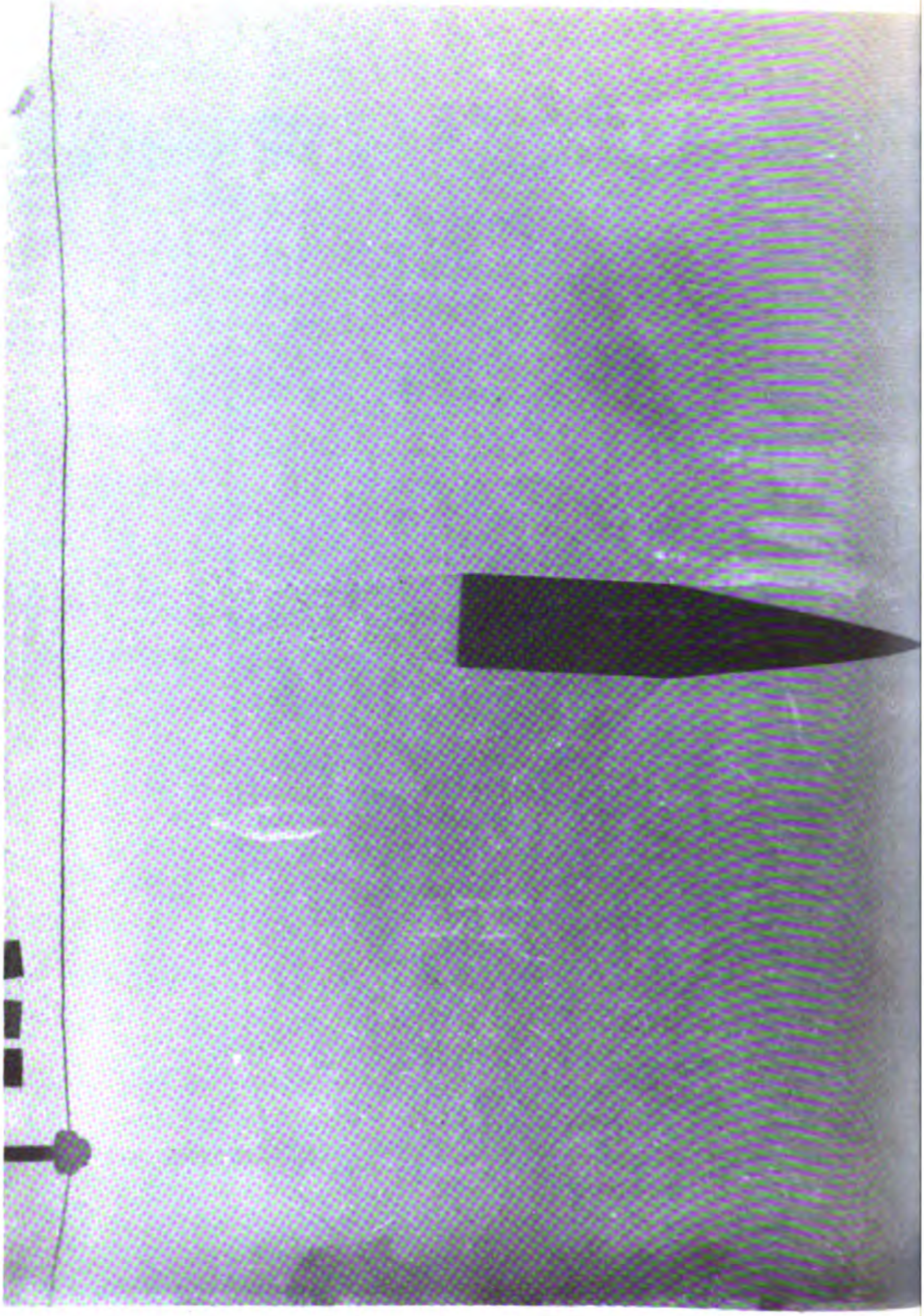


Figure XIII 1.1

Shadowgraph from Aerodynamics Range

Chapter XIII

REDUCTION OF SPARK RANGE DATA

1. Preliminaries.

The most accurate and reliable values which have so far been obtained for the various aerodynamic coefficients have been based on free flight data, obtained in a spark range. The basic equipment consists of a series of spark stations, placed at intervals along the early part of the trajectory. The Aberdeen range, which we intend to discuss in this chapter, is about 300 feet long and contains 25 spark stations. Each station consists of two photographic plates, mutually perpendicular, arranged so that the projectile passes above the horizontal plate and to the left of the vertical plate. Projectiles fired from guns normally have a static charge of electricity and this fact is used to trigger a spark. As the shell passes through the antenna, which is at the leading edge of the plates, a condenser is discharged through a spark gap, which is at a distance of about five feet from the vertical plate. The spark leaves a silhouette of the shell on the vertical plate, and a mirror, above the horizontal plate and at an angle of approximately 45 degrees, reflects the spark and gives a silhouette of the projectile on the horizontal plate. Since the position of the spark gap is known, and the positions of the silhouettes are known, it is a question of a little elementary projective geometry to find the position of the projectile in space — the position is actually overdetermined since it is obtained as the intersection of two rays. Similarly the angular position of the projectile can be determined from the angular positions of the silhouettes. We shall not give the details of this determination, since it is quite straightforward. The result of the measurements is that one knows to about 0.01 inch the position of the center of mass of the projectile at 25 positions down the range, as well as the angular position of the axis of the projectile to about 0.0015 radians. At certain stations a record is made of the time of the spark, so that the time at which the projectiles passed certain of the stations is known, usually to an accuracy of about one microsecond (0.000001 second). The performance of this experiment requires the utmost in experimental technique — it is a rather difficult proposition to measure the positions of fixed equipment, spread along a 300-foot distance, to an accuracy of 0.01 inch. The length of the range, for example, is subject to seasonal variations depending on the condition of the ground adjoining.

The results of a spark range firing deserve and receive very careful reduction. The basic data require treatment quite different from that obtained from yaw cards. The number of measurements is relatively much smaller, and the accuracy is of an entirely different order of magnitude. This chapter is devoted to a discussion of the method of reduction of these measurements. The procedure is primarily one of curve-fitting. We attempt in every case to fit the observed data with a curve of the type predicted by the theory of Chapter XI. The methods used are still in a process of evolution, and we shall point out, as we proceed, certain shortcomings of the process. The reduction as given here is primarily the work of the authors and of Professor H. Federer.

For convenience we list here certain formulas from Chapter XI which will be needed. These furnish the specification of the type of curve to which the fitting will be made. First, quoting (XI.6.2),

(1) The yaw ξ as a function of distance in calibers is given by

$$\xi = \xi_H + i\xi_V = c_1 \exp \Phi_1 + c_2 \exp \Phi_2 - gU^{-2}A\nu/mdJ_M$$

where c_1 and c_2 are constants and Φ_1 and Φ_2 are given by

$$\Phi_1, \Phi_2 = \frac{1}{2} \int_0^p \{ 2J_D - J_N - k^{-2}J_H - (J_D - md^2J_A/A)/\sigma^2 \pm [J_N - J_D - k^{-2}J_H - (2J_T - J_A)md^2/A] /\sigma + iA\nu(1 \pm \sigma)/B \} \, dp.$$

We also use the following notation, where k_1, k_2, ϕ_1 and ϕ_2 are real:

$$k_1 \exp i\phi_1 = c_1 \exp \Phi_1$$

and

$$k_2 \exp i\phi_2 = c_2 \exp \Phi_2.$$

In what follows we shall consider the real part of Φ_1' and Φ_2' to be constants, but will permit the imaginary part to be a slowly varying function of distance. The last term in the expression for ξ is the yaw of repose. It is very small in magnitude, and in certain cases is entirely negligible. It is always possible to neglect this term as a first approximation. (The notation used is that of Chapter XI, the definitions of the aerodynamic coefficients being given in Chapter II.) We further recall, from (XI.3.18) that

(2)
$$\sigma^2 = 1 - 1/s,$$
$$s = A^2\nu^2/4B^2J_Mk^{-2}.$$

The formula for the motion of the center of mass, the swerve formula, will be quoted later in the chapter when we come to the reduction of the measurements of the center of mass positions. We shall derive later the form of the time-distance relation which is used there. Both of these forms are used only after the

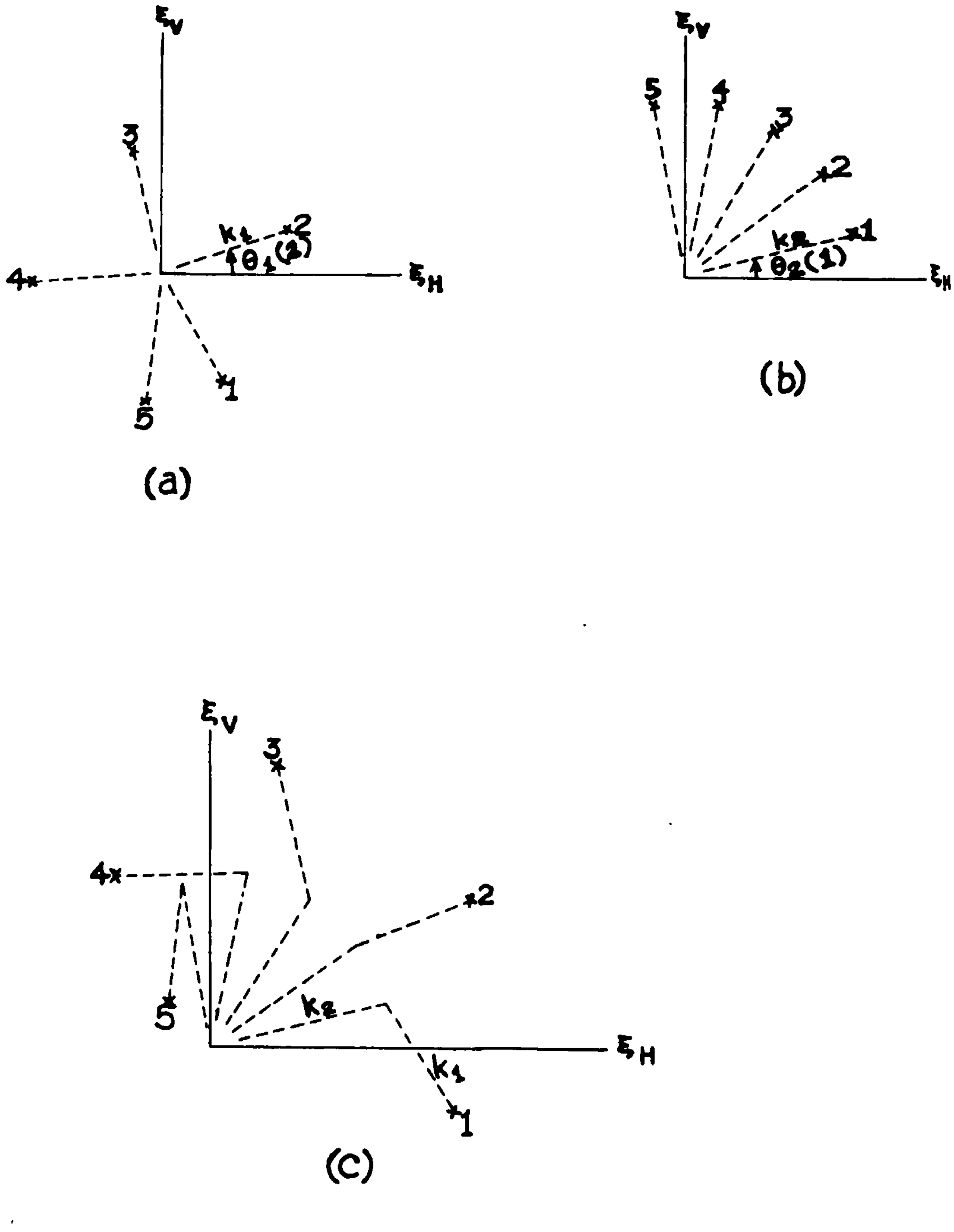


Figure XIII.2.1

Epicyclic Motion

yawing motion has been reduced, and we turn now to the detailed consideration of the angular motion.

2. Yaw reduction; first approximation.

The primary task, once the photographic plates for a round have been measured, is to reduce the measured angles and positions to space positions and direction cosines of the axis of the projectile. (We shall not give the details of this computation.) The coordinate system for the range is taken with the y-axis vertically upward, the x-axis horizontal and to the left of the line of fire, and the z-axis along the approximate line of fire. The direction cosines of the tangent to the trajectory are computed approximately, and the values of the yaw are then deduced from these and the direction cosines of the axis of the shell. The numbers ξ_H and ξ_V are respectively the horizontal and vertical components of the yaw, measured from a coordinate system with H-axis horizontal and V-axis (upward) perpendicular to the H-axis and to the tangent to the trajectory. A plot of ξ_H and ξ_V at successive stations is called a plot of the yaw.

The stations are arranged in groups down the range, five in each group. The stations within a single group are spaced at 5 or 7.5 feet, so that a very short range is covered. Within this range it is quite permissible to consider the yawing motion to be <u>pure epicyclic</u>. That is, we may write (1.1) in the form

$$(1) \qquad \xi = k_1 \exp i\theta_1 + k_2 \exp i\theta_2,$$

where k_1 and k_2 are real constants and θ_1 and θ_2 are real linear functions of distance. (See (1.1).) Such a motion is susceptible to easy geometric interpretation. (See Section 2 of Chapter XII for greater detail). The term $k_1 \exp i\theta_1$ represents a circular motion at constant rate, as shown on (a) of Figure 1. The second term is of the same type, but it has a

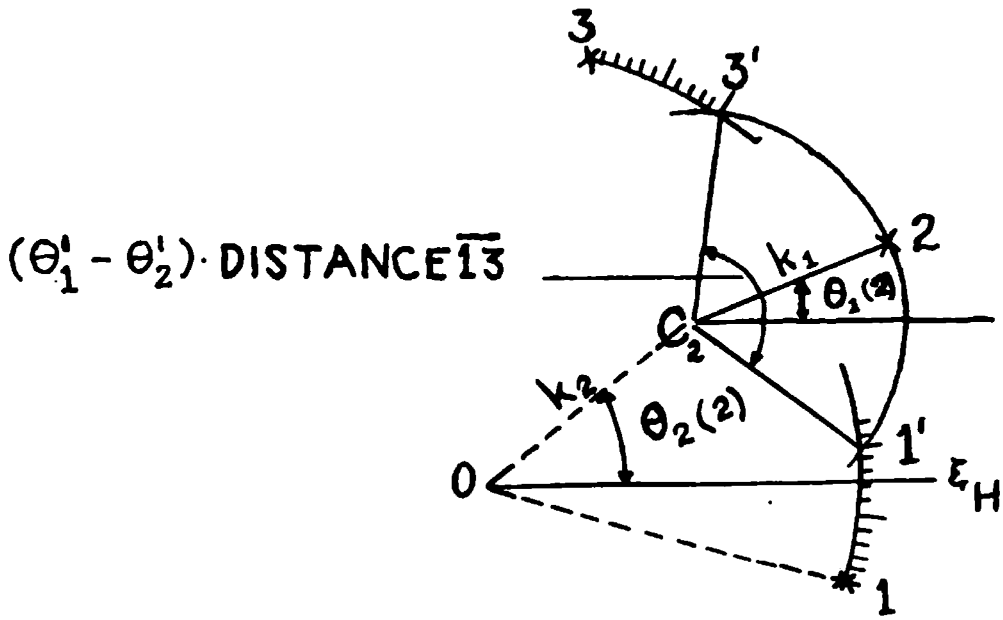


Figure XIII.2.2

Three-point Determination

different rate and a different "arm" k_2. The combination is then as shown on (c) of Figure 1. The problem to be solved in the reduction of the data from a single group of stations is to deduce from the plot of the yaw the "arms" k_1 and k_2 and the phase angles θ_1 and θ_2.

Since, within a group, θ_1 and θ_2 are supposed linear functions of distance, this amounts to determining six constants: two arms, two rates (the derivatives of θ_1 and θ_2) and two phase angles. Since two numbers, ζ_H and ζ_V are measured at each station, it is clear that three stations should be precisely enough to determine these constants. It is, of course, preferable to use more stations since one station may fail on a particular round, and the determination by means of three stations is rather sensitive to error. We give, in the following, several methods of making the determination, none of which is entirely satisfactory, but all of which have been and are being used. In the several methods the geometric picture is the same. Observing Figure 1(c), we see that if each observed yaw, 1, 2, 4 and 5, is rotated about the origin until the arms k_2 coincide with k_2 at 3 the result will be 5 points spaced on a circle with center at the end of the $k_2(3)$ arm. If, and this is for the preliminary reduction possible, we assume the yaws are measured at equal intervals of distance, the rotated points will be at equal intervals along the circle. In practice this amounts to, by some method, estimating the rate of change of θ_2, selecting a station, say 3, which is left invariant, and then rotating 1 and 2 forward, and 4 and 5 backward by angles proportional to the distance of the corresponding measurement from station 3 until the five rotated points lie equispaced on a circle. We now go into some detail on the particular methods of doing this.

a. <u>Geometric three-point determination.</u>

Three successive values, say 1, 2, 3, of the yaw, taken at equal intervals of distance, may be used to determine k_1 and k_2, θ_1 and θ_2 as follows. Pass circles with center at the origin O through 1 and through 3. Mark, clockwise from 3 and counterclockwise from 1, scales showing every 2 degrees. (See Figure 2.) With one point of a pair of dividers fixed at 2, find the smallest angle such that the other end of the dividers strikes points on the 1 and 3 scales with the same scale coordinate. Mark the points on the 1 and 3 scales respectively as 1' and 3', find the center of the circle through 1', 2 and 3' and mark it C_2. Then the epicycle through 1, 2 and 3 can be described as follows:

$k_2 = \overline{OC_2}$, $k_1 = \overline{C_2 2}$,

θ_2 at 2 is the angle from H-axis to $\overline{OC_2}$,

θ_1 at 2 is the angle from a parallel to the H-axis to $\overline{C_2 2}$,

θ_2' = derivative of θ_2 measured in degrees per foot,

= number of degrees from 1 to 1' (or 3 to 3') divided by the distance from station 1 to station 3,

$\theta_1' - \theta_2'$ = number of degrees subtended at C_2 by 1'3' divided by the distance from station 1 to station 3.

This determination is rather sensitive to experimental errors, and it fails if distance 12 = distance 23. However, the procedure is indicative of the general method, and is itself occasionally useful.

b. Geometric determination for a group of four or more equispaced stations.

If four or more stations in a group have functioned, the above procedure is modified in a trial-and-error sort of way. A reference station is first chosen, say 3. As before, draw circles with centers at the origin through each point corresponding to a station different from the reference station. For stations 2 and 4 two-degree intervals are marked counterclockwise and clockwise respectively from the plotted points. For stations 1 and 5, four-degree intervals are marked. Then, using a pair of dividers, one attempts to find a division mark so that the corresponding marks for each station are equispaced. (The first step in doing this would generally be to use 2, 3, 4 and 1, 3, 5 as in the three-point determination.) This can be done only approximately, but from this, one obtains an approximate idea of the magnitude of the change in θ_2 from station to station. The approximation is then continued by attempting to pass a circle through the division marks thus obtained. Eventually, through successive trials, one reaches the following stage: for each value 1, 2, 4, 5, there corresponds a rotated value 1', 2', 4' and 5', the rotation being through an angle proportional to the distance of the corresponding station from 3, such that the points 1', 2', 3, 4' and 5' lie approximately equispaced on a circle. This done, the center of the resulting circle is marked C_3, k_2 is the distance OC_3 and k_1 is the distance $C_3 3$. The angles from the H-axis to these two lines are θ_2 and θ_1 at 3 respectively. The value of θ_2' is the angular shift on the rotated points divided by the distance from the reference station. The total angle spanned on the circle by 1' to 5' is the change in $\theta_1 - \theta_2$ corresponding to the distance from station 1 to station 5, so that $\theta_1' - \theta_2'$ and hence θ_1' can be computed.

c. <u>Analytic determination from four equispaced observations.</u>

An epicycle in which the two arms are damped still permits a rather easy determination from four equispaced observations — the addition of the two damping rates requires observation of two more numbers. Since this method is not, however, tremendously useful, we shall give a rather sketchy treatment.

We first describe the epicyclic motion in slightly different form. At the first station the yaw is the sum of two complex numbers,

$$\xi_1 = c_1 + c_2, \tag{2}$$

corresponding to the two arms k_1 and k_2 multiplied by an exponential. At the next station the corresponding quantities are c_1 and c_2 multiplied by two complex numbers, say r_1 and r_2, which is the quotient of the exponential at 2 by the exponential at 1. The important fact is that these quotients are the same for every successive pair of stations. Thus,

$$\begin{aligned} \xi_2 &= c_1 r_1 + c_2 r_2, \\ \xi_3 &= c_1 r_1{}^2 + c_2 r_2{}^2, \\ \xi_4 &= c_1 r_1{}^3 + c_3 r_2{}^3. \end{aligned} \tag{3}$$

These equations can be used to determine c_1, c_2, r_1 and r_2 as follows. We first notice that if r_1 and r_2 can be determined the values of c_1 and c_2 are easy to deduce, using any two of the four equations. On the other hand, there are four linear equations in the three quantities 1, c_1 and c_2 so that the matrix of coefficients of these must be of rank at most two. Hence thé two following determinants must be zero.

$$\begin{vmatrix} \xi_1 & 1 & 1 \\ \xi_2 & r_1 & r_2 \\ \xi_3 & r_1{}^2 & r_2{}^2 \end{vmatrix} = 0 = \begin{vmatrix} \xi_2 & r_1 & r_2 \\ \xi_3 & r_1{}^2 & r_2{}^2 \\ \xi_4 & r_1{}^3 & r_2{}^3 \end{vmatrix} . \tag{4}$$

Straightforward expansion, using the fact that r_1 is not equal to r_2 and neither is zero then gives

$$(5)\qquad \begin{aligned} r_1r_2\xi_1 - (r_1 + r_2)\xi_2 + \xi_3 = 0, \\ r_1r_2\xi_2 - (r_1 + r_2)\xi_3 + \xi_4 = 0. \end{aligned}$$

These equations can be solved for the product r_1r_2 and the sum $r_1 + r_2$. It then follows that both r_1 and r_2 satisfy the same quadratic equation, which in determinant form is

$$(6)\qquad \begin{vmatrix} \xi_1 & \xi_2 & 1 \\ \xi_2 & \xi_3 & r \\ \xi_3 & \xi_4 & r^2 \end{vmatrix} = 0.$$

This equation is solvable by the usual elementary methods. Of course, the roots will in general be complex numbers. The logarithm of these gives at once the change in θ_1 and θ_2 between two consecutive stations, and the subsequent determination of c_1 and c_2 leads to the evaluation of k_1 and k_2 as well as the value of θ_1 and θ_2 at the first station.

3. Yaw reduction; continuation and second approximation.

We now proceed with the reduction. Suppose that $\xi = \xi_H + i\xi_V$ has been obtained for each of the twenty-five stations, and that the methods of the previous section have been used to fit, within each of the five groups of stations, the yaw by an epicycle motion. At each of the five reference stations, $R_1, \ldots, R_5$ we then have values of the amplitudes k_1 and k_2, the phase angles θ_1 and θ_2 and approximate values of the rates θ_1' and θ_2'. Using the rough rates it is possible to find the correct multiple of 360 degrees which must be added to θ_1 and θ_2 at R_2 to get a continuous phase down the range. We denote, in conformity with (1.1) the continuous phase angles thus obtained by

ϕ_1 and ϕ_2. Using k_1, k_2, ϕ_1 and ϕ_2 as given at the five reference stations, we then differentiate numerically, finding $(\log_e k_1)'$, $(\log_e k_2)'$, ϕ_1' and ϕ_2'. These are the first estimates of the quantities which we really wish to measure.

We now return to refine the reduction within each group. The motion within the group with reference station R_n is given closely by

$$
\begin{aligned}
(1)\qquad & k_1(z_{ns}) \exp i\phi_1(z_{ns}) \\
& \qquad \cdot \exp \{ z - z_{ns} \} \{(\log_e k_1)' + i\phi_1' \} \\
& + k_2(z_{ns}) \exp i\phi_2(z_{ns}) \\
& \qquad \cdot \exp \{ z - z_{ns} \} \{ (\log_e k_2)' + i\phi_2' \},
\end{aligned}
$$

where z_{ns} is the standard coordinate of R_n. Setting

$$
\begin{aligned}
(2)\qquad & c_1 = c_1(z_{ns}) = k_1(z_{ns}) \exp i\phi_1(z_{ns}), \\
& c_2 = c_2(z_{ns}) = k_2(z_{ns}) \exp i\phi_2(z_{ns}),
\end{aligned}
$$

we have the form

$$
\begin{aligned}
(3)\qquad & c_1 \exp \{ z - z_{ns} \} \{ (\log_e k_1)' + i\phi_1' \} \\
& + c_2 \exp \{ z - z_{ns} \} \{ (\log_e k_2)' + i\phi_2' \}.
\end{aligned}
$$

<u>We now determine</u> c_1 <u>and</u> c_2 <u>at the standard coordinate for</u> R_n <u>using a least squares procedure, using the multipliers of</u> $z - z_{ns}$ <u>as obtained from the first rangewise approximation</u>. To this end a subsidiary result is needed.

(4) <u>Lemma</u>. <u>Suppose that it is desired to fit observations</u> ζ_j, z_j <u>with a function of the form</u>

$$\zeta = c_1 f_1(z) + c_2 f_2(z),$$

where z is real, c_1 and c_2 are complex, and f_1 and f_2 are known complex-valued functions of z. The values of c_1 and c_2 which minimize

$$M = \Sigma_j \{ c_1 f_1(z_j) + c_2 f_2(z_j) - \xi_j \} \cdot \{ \bar{c}_1 \bar{f}_1(z_j) + \bar{c}_2 \bar{f}_2(z_j) - \bar{\xi}_j \}$$

are

$$c_1 = \frac{(\bar{f}_1 \xi)(f_2 \bar{f}_2) - (\bar{f}_2 \xi)(\bar{f}_1 f_2)}{(f_1 \bar{f}_1)(f_2 \bar{f}_2) - (\bar{f}_1 f_2)(f_1 \bar{f}_2)},$$

$$c_2 = \frac{(\bar{f}_2 \xi)(f_1 \bar{f}_1) - (\bar{f}_1 \xi)(\bar{f}_2 f_1)}{(f_1 \bar{f}_1)(f_2 \bar{f}_2) - (\bar{f}_1 f_2)(f_1 \bar{f}_2)}.$$

The value of M which is given by these constants is

$$M_{min} = (\xi \bar{\xi}) - c_1 \bar{c}_1 (f_1 \bar{f}_1) - c_2 \bar{c}_2 (f_2 \bar{f}_2) - c_1 \bar{c}_2 (f_1 \bar{f}_2) - \bar{c}_1 c_2 (\bar{f}_1 f_2).$$

In the foregoing the notation $(\bar{f}_1 \xi)$ is used for $\Sigma_j (\bar{f}_1(z_j)\, \xi_j)$ etc. The potential usefulness of the result is clear. It determines 4 parameters, the real and imaginary parts of c_1 and c_2, by means of a computation essentially as simple as the ordinary least squares procedure for two parameters. The lemma is established in perfectly straightforward fashion. Differentiating the original expression for M with respect to the real and imaginary part of c_1 gives

$$\begin{aligned} &\bar{c}_1(f_1 \bar{f}_1) + \bar{c}_2(\bar{f}_2 f_1) - (f_1 \bar{\xi}) \\ &\qquad + c_1(f_1 \bar{f}_1) + c_2(\bar{f}_1 f_2) - (\bar{f}_1 \xi) = 0, \end{aligned}$$

(5)

$$\begin{aligned} &\bar{c}_1(f_1 \bar{f}_1) + \bar{c}_2(\bar{f}_2 f_1) - (f_1 \bar{\xi}) \\ &\qquad - c_1(f_1 \bar{f}_1) - c_2(\bar{f}_1 f_2) + (\bar{f}_1 \xi) = 0. \end{aligned}$$

These two equations are equivalent to the real and imaginary parts of the following:

$$(6) \qquad c_1(\bar{f}_1 \, f_1) + c_2(\bar{f}_1 \, f_2) - (\bar{f}_1 \xi) = 0.$$

Similarly for c_2,

$$(7) \qquad c_1(\bar{f}_2 \, f_1) + c_2(f_2 \, \bar{f}_2) - (\bar{f}_2 \xi) = 0.$$

Solution of these equations gives the values of c_1 and c_2 as stated. To evaluate M for these values of c_1 and c_2 we first multiply and sum the factors in the definition of M, obtaining

$$(8) \qquad \begin{aligned} M = {} & c_1\bar{c}_1(f_1 \, \bar{f}_1) + c_2\bar{c}_2(f_2 \, \bar{f}_2) + (\xi \, \bar{\xi}) \\ & + c_1\bar{c}_2(f_1 \, \bar{f}_2) + c_2\bar{c}_1(f_2 \, \bar{f}_1) \\ & - c_1(f_1 \, \bar{\xi}) - \bar{c}_1(\bar{f}_1 \, \xi) \\ & - c_2(f_2 \, \bar{\xi}) - \bar{c}_2(\bar{f}_2 \, \xi). \end{aligned}$$

If the terms of the type $(f\xi)$ are replaced by their equals from (6) and (7) the given form for M_{min} results.

The lemma can be applied directly to the problem at hand, setting

$$(9) \qquad \begin{aligned} f_1(z) &= \exp \{ z - z_{ns} \} \{ (\log_e k_1)' + i\phi_1' \}, \\ f_2(z) &= \exp \{ z - z_{ns} \} \{ (\log_e k_2)' + i\phi_2' \}. \end{aligned}$$

For computational purposes, (9) can be replaced by

$$(10) \qquad \begin{aligned} f_1(z) &= \left(1 + \{ z - z_{ns} \} (\log_e k_1)' \right) \\ & \qquad \cdot \exp i\phi_1' \{ z - z(R_1) \}, \\ f_2(z) &= \left(1 + \{ z - z_{ns} \} (\log_e k_2)' \right) \\ & \qquad \cdot \exp i\phi_2' \{ z - z(R_1) \}, \end{aligned}$$

and the computation of c_1 and c_2 is quite straightforward. The value of M_{min} can be used to estimate the probable error in each measurement of ξ_H or ξ_V. We have determined four constants from 2m observations, if m is the number of stations within the group, so that

$$\sqrt{M_{min}/(2m - 5)}$$

is an estimate of the standard deviation of the individual measurement. Since c_1 and c_2 are obtained as linear combinations of observed values, their accuracy may also be estimated.

It is to be noted that, while the errors in the first approximation to the motion are quite large, the second approximation usually gives small errors.

Having found c_1 and c_2, simply changing these complex numbers to polar form gives, according to (2), $k_1(z_{ns})$, $k_2(z_{ns})$, $\phi_1(z_{ns})$ and $\phi_2(z_{ns})$. These values are, very conveniently, at the standard coordinates of the reference station. The derivatives of ϕ_1 and ϕ_2 are now computed on the basis of a least squares fit by a quadratic in z. One fact should be noticed here. Since

ϕ_1 = arc tan (imaginary part of c_1/real part of c_1),

the accuracy of the determination is directly proportional to k_1. The least squares procedure is therefore done with weights k_1. Once the fitting has been done, referring to (1.1), we have

$$(\phi_1 + \phi_2)' = A\nu/B,$$

(11)
$$(\phi_1 - \phi_2)' = A\nu\sigma/B,$$

$$(\phi_1 + \phi_2)'' = A\nu'/B.$$

(To be precise, (11) holds if the "units" are radians per caliber.) From the quantities (11) the following result:

$$\phi_1' \cdot \phi_2' = k^{-2} J_M,$$

$$(12) \quad (\phi_1 + \phi_2)''/(\phi_1 + \phi_2)' = \nu'/\nu$$
$$= J_D - J_A md^2/A.$$

The last equation results from (XI.2.10). Thus, from the phases ϕ_1 and ϕ_2 can be obtained the spin ν, the moment coefficient K_M, and after the drag data have been reduced, the spin-decelerating moment coefficient K_A.

Determination of $(\log_e k_1)'$ and $(\log_e k_2)'$ requires slightly more care. We know that each k should be of the form a exp bz, and from the first approximation an estimate b_o for b and an estimate a_o for a exists. Writing

$$b = b_o + \Delta b, \; a = a_o + \Delta a,$$

the form of k, to first-order terms, is

$$(13) \quad k = a_o \exp b_o z + \Delta a \exp b_o z + a_o \Delta b\, z \exp b_o z.$$

In this form, the solution consists of fitting k with a linear combination of known functions, and the procedure is straightforward.

The values of $(\log_e k_1)'$ and $(\log_e k_2)'$ having been determined, from (1.1) we have at once

$$(14) \quad \begin{aligned} &(\log_e k_1)' + (\log_e k_1)' \\ &\quad = 2J_D - J_N - k^{-2} J_H - (J_D - md^2 J_A/A)/\sigma^2, \\ &(\log_e k_1)' - (\log_e k_2)' \\ &\quad = \{ J_N - J_D - k^{-2} J_H - (2J_T - J_A) md^2/A \}/\sigma. \end{aligned}$$

The combinations of coefficients (12) and (14) represent all the information that can be gained from the yawing motion.

4. Drag reduction: general remarks.

We shall see that the reduction of the experimental data on drag can be reduced to the following simplified case. Given measurements $t_1, t_2, \ldots, t_m$ of the time at which the projectile passed the stations which are at distances $z_1, z_2, \ldots, z_m$, it is required to find that linear combination of three known functions which best fits the experimental data — "best fits" in the least squares sense. That is, if the time is written

$$t = a_o + a_1 f^1 + a_2 f^2 + a_3 f^3, \tag{1}$$

where f^1, f^2, f^3 are known functions of z, we wish to choose a_o, a_1, a_2 and a_3 in such a way as to minimize the sum of the squares of the errors. This amounts to the problem of choosing the a's so that they satisfy

$$\Sigma_j (\Sigma_i a_i f_j^{\,i} - t_j)^2 = \text{minimum}, \tag{2}$$

$$i = 0, 1, 2, 3; \; j = 1, \ldots, m.$$

where $f_j^{\,i}$ is the notation for $f^i(z_j)$. Differentiating with respect to a_i, we must have

$$\Sigma_j (\Sigma_k a_k f_j^{\,k} - t_j) f_j^{\,i} = 0, \text{ for each } i = 0, 1, 2, 3. \tag{3}$$

Rearranging these equations, we have four equations for the four unknowns a_i as follows:

$$\Sigma_k a_k (\Sigma_j f_j^{\,i} f_j^{\,k}) = \Sigma_j t_j f_j^{\,i}, \text{ for } i = 0, 1, 2, 3. \tag{4}$$

For convenience, let us denote

$$\Sigma_j f_j^{\,i} f_j^{\,k} = b_{ik}, \; i, k = 0, 1, 2, 3; \; j = 1, \ldots, m. \tag{5}$$

Then equations (4) take the form

$$\Sigma_k b_{ik} a_k = \Sigma_j t_j f_j^{\,i}, \text{ for } i = 0, 1, 2, 3. \tag{6}$$

Now let (B_{pi}) be the matrix inverse to (b_{ik}). Each element B_{pi} is the signed minor of (b_{ik}) corresponding to the i-th row and the p-th column divided by the determinant $|b_{ik}|$. Thus

$$(7) \qquad \Sigma_i \, B_{pi} \, b_{ik} = \begin{cases} 0, & \text{if } p \neq k, \\ 1, & \text{if } p = k. \end{cases}$$

Multiplying the i-th equation of (6) by B_{pi} and summing,

$$\begin{aligned} a_p &= \Sigma_i \;\; \Sigma_j \;\; t_j \, f_j{}^i B_{pi} \text{ for } p = 0, 1, 2, 3 \\ (8) \qquad &= \Sigma_j \;\; t_j (\Sigma_i \, f_j{}^i B_{pi}) \\ &= \Sigma_j \;\; t_j \; c_{pj}, \end{aligned}$$

where

$$c_{pj} = \Sigma_i f_j{}^i B_{pi}.$$

These coefficients, together with the experimental values t_j, thus determine the required values of the a's. The computation has been put in the above form for the following reason. Values of the time will be measured corresponding to the same values of distance for all rounds. The c_{pj} do <u>not</u> depend on the values of t. The sums b_{ik} of (5) must be computed, and the inverse matrix found, either by determinants or otherwise. (The equations (7) may be solved by Gaussian elimination for the inverse.) Once the inverse matrix has been obtained, the bracket in (8) which defines the coefficients c_{pj} can be computed. If the yaw is negligible these coefficients may be used for all rounds.

There are certain objections to the above procedure. First of all, there is <u>a priori</u> no reason why one should use a least squares technique. What is really desired for drag measurement is the best possible estimate of the second derivative of t with respect to z. Fortunately, as we shall show below, the least squares fit does furnish the best estimate of the second derivative. This result will justify use of the above formulas. We shall also make an investigation of the optimum spacing of the timing stations. That is, given enough equipment to make a certain number of time measurements, what spacing should be used in the range in order to assure a best estimate of the drag. Since neither of these results is needed in the actual reduction of a round, readers not interested in optimum spacing and willing to believe the authors honest on the result concerning estimation of the second derivative may omit the rest of this section.

We now consider the problem of fitting the data from the point of view of making a best estimate of the individual coefficients. The result which we shall prove was first communicated to us by A. P. Morse in a somewhat more general form. As before we assume that the true functional relationship is of the type (1) and we wish to estimate the a's from a collection of observed data $t_1, \ldots, t_m$ corresponding to fixed distances $z_1, \ldots, z_m$. The estimate of a certain coefficient, say a_p, will be taken to be a linear function of the observed values of the form

(9) $$a_p = \Sigma_j \, d_{pj} t_j.$$

The coefficients d_{pj} are now to be evaluated. First we require that the estimate of a_p be unbiased, in the sense that if the observed data actually fit (1) exactly then the computation (9) is to give the coefficients of this combination. That is, if

$$t_j = \Sigma_k \, a_k f_j{}^k, k = 0, 1, 2, 3,$$

then

$$a_p = \Sigma_k \, d_{pj} t_j = \Sigma_j \, d_{pj} \; \Sigma_k \, a_k f_j{}^k$$

must be an identity for all values of $a_0, \ldots, a_3$. Hence the d_{pj} are subject to the conditions

$$(10) \quad \Sigma_j \, d_{pj} f_j^{\,k} = \begin{cases} 0 \text{ for } k \neq p, \\ 1 \text{ for } k = p \end{cases} \qquad k = 0, 1, 2, 3.$$

Suppose now that the measured value of t_j is in error by e_j, so that the value given by the measurement is $t_j + e_j$. The error in the estimate of a_0 is then $\Sigma_j d_{pj} e_j$ and it is our task to minimize its square, $(\Sigma_j d_{pj} e_j)(\Sigma_j d_{pj} e_j)$. Now the error at one distance, say z, should be statistically independent of the error at another distance, say z_p, and the expected value of the error at each distance should be zero. The stations being alike, the variance of the errors e_j should be the same for all j, say equal to $(S.D.)^2$. Then the expected value of the above product is

$$\Sigma_j \, (d_{pj})^2 \, (S.D.)^2.$$

The problem now reduces to the minimizing of

$$(11) \qquad \Sigma_j \, (d_{pj})^2$$

subject to conditions (10). We now use the Lagrange multiplier rule, that is, we seek values of d_{pj} and values of multipliers m_{pk}, k = 0, 1, 2, 3 such that

$$(12) \qquad \Sigma_j (d_{pj})^2 + \Sigma_k \, m_{pk} (\Sigma_j d_{pj} f_j^k - \delta_k^p) = \text{minimum}$$

where $\delta_k^p = 0$ if $k \neq p$ and 1 if $k = p$. Taking the derivative of this function with respect to d_{pq},

$$(13) \qquad 2d_{pq} + \Sigma_k \, m_{pk} \, f_q^k = 0.$$

Substituting this value of d_{pq} in the relation (10), and changing the index of summation gives

$$\Sigma_j \, \Sigma_s \, m_{ps} f_j^s \, f_j^k = -\, 2\delta_p^k,$$

or

$$\Sigma_s \, m_{ps} \, (\Sigma_j f_j^s \, f_j^k) = -\, 2\delta_p^k.$$

In the notation used in the early part of this section, in particular (5), this is

$$\Sigma_s\ m_{ps}\ b_{sk} = -\ 2\delta_p^k.$$

Using the inverse matrix, precisely as before, we have

$$m_{ps} = -\ 2B_{ps}, \tag{14}$$

and hence, referring again to (13),

$$d_{pq} = \Sigma_k\ B_{pk} f_q^k. \tag{15}$$

This is precisely the result given in (8), so that we may state: <u>the least squares fit to the data gives the best estimate of the values of the particular coefficients</u> a_p.

We now consider the question of optimum spacing of time measurements. In doing this, two simplifications are made. It is assumed that the distance z_j are symmetrically placed about $z = 0$, so that $\Sigma_j\ za_j^r$, for r odd, is zero. Second, the time measurements are fitted by a quadratic in z. Although neither of these simplifications is entirely warranted, the result of the calculation should give some help on the question of spacing.

If a quadratic is fitted the coefficients are determined by

$$\Sigma_j\ (a_o + a_1 z_j + a_2 z_j^2 - t_j)^2 = \text{minimum}, \quad j = 1, \ldots, m. \tag{16}$$

Using precisely the method of the first part of this section, this gives at once

$$(17)\quad a_2 = \frac{\begin{vmatrix} m & 0 & \Sigma t \\ 0 & \Sigma z^2 & \Sigma tz \\ \Sigma z^2 & 0 & \Sigma tz^2 \end{vmatrix}}{\begin{vmatrix} m & 0 & \Sigma z^2 \\ 0 & \Sigma z^2 & 0 \\ \Sigma z^2 & 0 & \Sigma z^4 \end{vmatrix}} = \frac{m(\Sigma tz^2) - (\Sigma t)(\Sigma z^2)}{m(\Sigma z^4) - (\Sigma z^2)^2}.$$

Here use has been made of the fact that the sum of odd powers of z_j vanishes, and the subscripts have been left off the summed quantities for convenience. This equation may at once be rewritten

$$(18)\qquad a_2 = \Sigma t_j \left\{ \frac{mz_j^2 - (\Sigma z^2)}{m(\Sigma z^4) - (\Sigma z^2)^2} \right\}.$$

If all the values t_j have the same probable error, then to minimize the mean square error in a_2 we must, by (I.21.10), have the sum of the squares of the coefficients of the t_j as small as possible. Thus the set $\{z_j\}$ should be selected so that

$$(19)\quad \Sigma_j \left\{ \frac{mz_j^2 - (\Sigma z^2)}{m(\Sigma z^4) - (\Sigma z^2)^2} \right\}^2 = \text{minimum.}$$

This reduces, on squaring and summing, to

$$(20)\qquad \frac{(m^2 - m)(\Sigma z^2)^2}{[m(\Sigma z^4) - (\Sigma z^2)^2]^2} = \text{minimum.}$$

The factor $m^2 - m$ is always positive, for unless $m \geq 3$ no determination of a_2 can be made. For further convenience write

$$(21) \qquad r_j = z_j^2/z_1^2.$$

This amounts to choosing units of length so that the range is two units long. Then, inverting, we see that (20) becomes

$$(22) \quad \left| (\Sigma\, r^2/m\,)/(\Sigma\, r/m\,) - \Sigma\, r/m\, \right| = \text{maximum.}$$

Although further work can be done to find optimum spacing, (22) gives a very convenient criterion which is easy to use. For example, suppose 4 timing stations are available and we wish to compare the two spacings

$$(a) \qquad -1, \quad 0, \quad 0, \quad 1$$

$$(b) \qquad -1, \; -\tfrac{1}{3}, \; \tfrac{1}{3}, \; 1.$$

Computing, for the two cases

(a) $\Sigma\, r/m = 2/4,$ $\qquad\qquad \Sigma r^2/m = 1/2,$

$$\left| (\Sigma^2 r/m)/(\Sigma\, r/m) - (\Sigma\, r/m) \right| = 1/2,$$

(b) $\Sigma\, r/m = 5/9$ $\qquad\qquad \Sigma r^2/m = 41/81,$

$$\left| (\Sigma\, r^2/m)/(\Sigma\, r/m) - (\Sigma\, r/m) \right| = .365\ldots.$$

Thus the arrangement (a) is much preferable to (b) for drag measurement. A quantitative idea of how much better is easy to see. The probable error of the determination of a_2 is proportional to the square root of the expression (19). This is, in turn, for a fixed number of stations, inversely proportional to the expression (22). Thus, the ratio of the probable errors of a_2 for two different arrangements of timing stations is the inverse of the ratio of the corresponding numbers computed from (22). In the example, choice of the (a) arrangement in preference to the (b) reduces the probable error of a_2 by a factor 0.73....

5. Drag determination from spark range data.

We now turn to the reduction of the time-distance measurements made in the range. For the very flat fire occurring there it is quite permissible to assume that the t-z relation is determined entirely by the drag, and that the velocity u is the same as $\dot{z}$. The equation satisfied by $\dot{z}$ is then

(1) $$m\ddot{z} = -\rho d^2\dot{z}^2K_D.$$

It will be necessary to take K_D in a more accurate form than that usually used. The coefficient K_D depends on both Mach number and the square of the yaw, and we shall take account of the first-order terms in this dependence. The velocity of sound, u_s, is fixed, depending on the temperature in the range. Thus, instead of expanding in Mach number, it is equally correct to expand K_D as a function of velocity about the velocity u_o at $z = 0$. We therefore define α, α' and β by the equation

(2) $$\rho d^2K_D/m = \alpha + (\dot{z} - u_o)\alpha' + \beta\delta^2,$$

where δ is the yaw. The relation of these numbers to quantities defined earlier in the following:

$$\alpha = \rho d^2K_{DO}/m,$$

(3) $$\alpha' = (\rho d^2/mu_s)(\text{partial derivative of } K_D \text{ with respect to Mach Number}),$$

$$\beta = \rho d\ K_{DO}K_{D\delta}/m,$$

where K_{DO} is the drag coefficient at zero yaw and $K_{D\delta}$ is the "yaw drag" coefficient of Chapter II. Another form of the second of these equations can be stated, since α'/α is simply the logarithmic derivative of the drag coefficient with respect to Mach number M, divided by the velocity of sound u_s. Symbolically,

(4) $$\alpha'/\alpha = M\varepsilon/u_o,$$

where

$$\varepsilon = \partial(\log K_D)/\partial M.$$

This form will presently be useful. For the present form (2) will be used.

Since the time measurements are made at constant distances, it is preferable to write the differential equation (1) with z as independent variable. Using primes to denote derivatives with respect to z, the necessary computation follows.

$$\ddot{z} = - \{ \alpha + (\dot{z} - u_o)\alpha' + \beta\delta^2 \} \dot{z}^2,$$

$$t' = 1/\dot{z},$$

$$t'' = (1/\dot{z})' = - \ddot{z}/\dot{z}^3 = - \ddot{z}t'^3.$$

Hence, substituting in the last of these from the first,

(5) $$t'' = (\alpha - u_o\alpha')t' + \alpha' + \beta\delta^2 t'.$$

We shall first consider the special case in which α' happens to be zero; the solution is then simpler, and by an artifice we shall later reduce the general case to this special case. In this case, (5) reduces to

(6) $$t'' = \alpha t' + \beta\delta^2 t'.$$

This can be solved by dividing by t'; the result is

(7) $$\log t' = \alpha z + \int_0^z \beta\delta^2 \, dz + \log t_o'.$$

If we introduce the notation

$$I'(z) = \int_0^z \delta^2 \, dz$$

this yields

(8) $$t' = t_o' \exp (\alpha z + \beta I'(z)).$$

Hence

(9) $$t - t_o = t_o' \int_0^z \exp (\alpha z + \beta I'(z)) \, dz.$$

We expand the exponential in a Taylor's series and integrate. If we introduce the symbol

$$(10) \qquad I(z) = \int_0^z I'(z)\, dz,$$

the result can be written in the form

$$(11) \qquad \begin{aligned} t = {} & t_0 + t_0' z + \alpha (t_0'/2) z^2 + \beta t_0' I(z) \\ & + (t_0'/2) \int_0^z (\alpha z + \beta I'(z))^2 dz \\ & + (t_0'/6) \int_0^z (\alpha z + \beta I'(z))^3 dz + \ldots . \end{aligned}$$

But by (7), together with Taylor's theorem,

$$(12) \qquad \begin{aligned} & \alpha z + \beta I'(z) \\ & = \log (t'/t_0') \\ & = (t_0''/t_0') z \\ & \quad + \tfrac{1}{2} \{ (t_0'''/t_0') - (t_0''/t_0')^2 \} z^2 + \ldots . \end{aligned}$$

Substituting this in (11) and integrating yields

$$(13) \qquad \begin{aligned} t = {} & t_0 + t_0' z + \alpha (t_0'/2) z^2 + \beta t_0' I(z) \\ & + (t_0'/6)(t_0''/t_0')^2 z^3 \\ & + (t_0'/24) \{ (3 t_0'' t_0'''/t_0'^2) \\ & \qquad - 2 (t_0''/t_0'^3) \} z^4 + \ldots . \end{aligned}$$

The last term is included only to permit verification that it is negligible, as it seems to be in every case considered. The coefficient of z^3 could be estimated by performing a least squares fit of 1, z, z^2, z^3 and $I(z)$ to the data. But this determination is not very accurate, and a much more precise estimate can be made by virtue of the fact that we have the specific formula in (13) for the coefficient. For by a preliminary rough reduction we can find t_0' with high accuracy, and t_0'' to within not worse than about two per cent when equipment of the present time (1948) is used. We ignore the last term in (13) and transpose the z^3-term to the left. That is, from each recorded t we subtract

$$(t_0'/6)(t_0''/t_0')^2 z^3.$$

The remainder is then a linear combination of 1, z, z^2 and I(z), with an error, due to ignoring small terms, which is well below the experimental errors. This is all subject to the assumption that $\alpha' = 0$.

Suppose next that the effect of the α'-terms is not negligible. They will nevertheless be small, and we can compute them with adequate accuracy by an expansion to terms linear in α'. Consider the family of all solutions of (5) with given initial values t_0, t_0' and a given value of α. The solutions will depend on the parameter α', and will satisfy (5) identically in z and α'. From (5),

(14) $$\alpha + \beta\delta^2 - u_0\alpha' = (t'' - \alpha')/t'.$$

If we differentiate both members of (5) with respect to α', substitute from (14) and then set $\alpha' = 0$, we find that for $\alpha' = 0$,

(15) $$(\partial t/\partial\alpha')'' = (t''/t')(\partial t/\partial\alpha')' + (1 - t'u_0),$$

where all the primes except those in the symbol α' denote derivatives with respect to z. For this equation, $1/t'$ is an integrating factor. The solution is

(16) $$(\partial t/\partial\alpha')'/t' = \int_0^z \{(1/t') - u_0\}\, dz,$$

since at z = 0 the value of $\partial t'/\partial\alpha'$ is 0. Integrating again,

$$\partial t/\partial\alpha' = \int_0^z t'(z)\left[\int_0^z \{(1/t'(z)) - u_0\}\,dz\right]dz \tag{17}$$
$$= -(t_0''/6t_0')z^3 + \ldots.$$

If rounds have been fired at two or three different velocities, rough reductions will furnish estimates of K_D at the corresponding Mach numbers. From these, with (4), we can estimate α'; with care, an estimate accurate to within about two or three per cent can be made. For each z, the effect of the α'-terms is $\alpha'(\partial t/\partial\alpha')$, the second factor being evaluated somewhere between 0 and α'. We make only a small error if we replace the factor α' by our estimate and the other factor by the value of $\partial t/\partial\alpha'$ for $\alpha' = 0$, as estimated in (17). Thus we estimate the effect of the α'-terms to be

$$\alpha'(\partial t/\partial\alpha) = -\alpha'(t_0''/6t_0')z^3. \tag{18}$$

If this is subtracted from the times t(z) corresponding to the solution of (5) with the α'-terms present, the remainder will be the solution of (6) with the same initial values. Thus by subtracting the quantity (18) from t we return to the special case discussed in (6) to (13). Applying the result there found, we discover that

$$t(z) + (1/6)(t_0''/t_0')(\alpha' - t_0'')z^3$$
$$= t_0 + t_0'z + \alpha(t_0'/2)z^2 + \beta t_0' I(z), \tag{19}$$

with negligible error.

Accordingly, we "adjust" the time t_i experimentally measured at coordinate z_i, computing the "adjusted time"

$$t_i^* = t_i + (1/6)(t_0''/t_0')(\alpha' - t_0'')z_i^3. \tag{20}$$

It remains to compute the least squares fit of these "adjusted times" by a linear combination of 1, z, z^2 and I(z). The coefficient of z will be an accurate determination of t_0' (although in the next step the

t_o' of the preliminary reduction would be accurate enough). The coefficients of z^2 and $I(z)$ then determine α and β, and from (3) we can compute K_{DO} and $K_{D\delta}$. However, we shall make two remarks of some importance in connection with the least squares fitting.

The first remark concerns the computation of $I(z)$. Referring to (2.1), the square of the magnitude of the yaw is

$$(21) \qquad \delta^2 = k_1^2 + k_2^2 + 2k_1k_2\cos(\phi_1 + \phi_2).$$

Here, as in Section 2, it is permissible to consider ϕ_1 and ϕ_2 as linear functions of z, and the arms k_1 and k_2 as exponentially decreasing. Then

$$\begin{aligned}\delta^2 = {} & k_{10}^2 \exp(-2h_1 z) + k_{20}^2 \exp(-2h_2 z) \\ & + 2Rk_{10}k_{20} \exp i\{(\phi_1' - \phi_2') z + c_o\}.\end{aligned}$$

Computing $I(z)$ from this is not difficult. The result is the sum of three integrals with comparable numerators and with respective denominators

$$(2h_1)^2, \ (2h_2)^2, \ (h_1 + h_2 - i[\phi_1' - \phi_2'])^2.$$

The last is far larger than the others in absolute value, so that the last term in (21) contributes negligibly little to $I(z)$. Thus we may assume

$$(22) \qquad I(z) = \int_0^z \left[\int_0^z (k_1^2 + k_2^2)\, dz \right] dz.$$

From the experimentally determined k_1 and k_2 this is easily computed by the methods of Section 3 of Chapter VI.

The second remark is concerned with the process of fitting a combination of 1, z, z^2 and $I(z)$ to the adjusted times t_1^*. This of course can be done by the methods of Section 4. However, even when a single round is being reduced some simplifications are possible. Besides this, it is usual to fire several similar rounds at nearly the same velocity; some will have considerable yaw, some will have small yaw. It is reasonable to assume that β is the same for all rounds of the group; the accuracy of its determination will not be good enough for us to hope to be able to detect any change between nearby velocities. On the other hand, α will change from round to round by a small but perceptible amount. If we have a fairly good estimate of K_{DO} at two or three Mach numbers, we will be able to estimate this small change in α with satisfactory accuracy, and to allow for its effects. However, t_o and t_o' will vary from round to round. Given N rounds, the best determination of α and β would be obtained by using all of them at once. This would amount to determining α, β, N values of t_o, and N values of t_o'. A fitting with 2N + 2 parameters is a complicated matter even if N = 2 or 3. Fortunately, in the present case notable simplifications can be made.

Given rounds j = 1, ..., j = N, to round j correspond values of air density ρ_j, velocity u_{oj} at z = 0, Mach number M_j, mass m_j, etc. We let ρ (without subscript) be a number exactly or nearly equal to the mean density $(\rho_1 + \dots + \rho_N)/N$, and likewise for m, d, M. If α is the value defined in (3) and corresponding to the mean values ρ, m, d, u_o, M, then by (3) we have

$$\alpha_j = \kappa_j \alpha, \tag{23}$$

where

$$\kappa_j = \rho_j d_j^2 K_{DO}(M_j) m / \rho d^2 K_{DO}(M) m_j. \tag{24}$$

This quantity can be estimated with considerably smaller percentage error than α_j itself, even if a rough first estimate of u_{oj} is used. Now by (19) and (20), the adjusted time t_{ji}^* for round j at station i is

$$(25) \quad t_{ji}^* = t_{oj} + t_{oj}'z_{ji} + \alpha(\kappa_j z_{ji}^2 t_{oj}'/2) + \beta t_{oj}' I_j(z_i),$$

where I_j is the function $I(z)$ computed for the yaw of round j.

Our program for determining α and β has the following basis. For each α and β, let the t_{oj} and t_{oj}' be determined so as to minimize the sum of squares of residues in (25). The corresponding sum of squares of residues is then a function of α and β. If we choose α and β so as to minimize this function, we have minimized the sum of squares of residues for all $\alpha, \beta, t_{o1}, \ldots, t_{oN}'$.

For given α, β, by Section 4 we see that the coefficients of the least squares fit are linear in α and β. We therefore fit the t_{ji}^*, the $\kappa_j z_{ji}^2 t_{oj}'/2$ and the $t_{oj}' I_j(z_i)$ by least squares linear combinations of 1 and z , and find the residues. This is not difficult. In the fitting of the t_{ji}^*, the data for the various rounds can be treated separately, since t_{oj} and t_{oj}' affect only round j. For each j, let the least squares fit of $t_{j1}^*, t_{j2}^*, \ldots$ be

$$T_j + T_j' z_{j1}, \ T_j + T_j' z_{j2}, \ \ldots,$$

with residues $\tau_{j1}, \tau_{j2}, \ldots$. Let the least squares fit of $\kappa_j z_{ji}^2 t_{oj}'/2$ be

$$Q_j + Q_j' z_{j1}, \ Q_j + Q_j' z_{j2}, \ \ldots,$$

with residues $q_{j1}, q_{j2}, \ldots$; and let the least squares fit of $t_{oj}' I_j(z_i)$ be

$$B_j + B_j' z_{j1}, \ B_j + B_j' z_{j2}, \ \ldots,$$

with residues $b_{j1}, b_{j2}, \ldots$. Then the least squares fit of

$$(26) \quad t_{ji}^* - \alpha(\kappa_j z_{ji}^2 t_{oj}'/2) - \beta t_{oj}' I_j(z_i)$$

is

(27) $$(T_j - \alpha Q_j - \beta B_j) + (T_j' - \alpha Q_j' - \beta B_j')z_{ji},$$

where $j = 1, 2, \ldots, N$ and i runs over the stations functioning for round j. The sum of squares of the residues from the fit (27) is

(28) $$\Sigma_{i,j} (\tau_{ji} - \alpha q_{ji} - \beta b_{ji})^2.$$

To minimize this, we find its derivatives with respect to α and β and set them equal to zero. The result is

(29) $$\begin{aligned} \alpha \Sigma_{i,j}(q_{ji})^2 + \beta \Sigma_{i,j} q_{ji}b_{ji} &= \Sigma_{i,j} q_{ji}\tau_{ji}, \\ \alpha \Sigma_{i,j} q_{ji}b_{ji} + \beta \Sigma_{i,j}(b_{ji})^2 &= \Sigma_{i,j} b_{ji}\tau_{ji}. \end{aligned}$$

These equations can be solved readily for α and β, which are the quantities desired. So instead of one least squares fit with $2N + 2$ parameters, we need to effect $3N$ linear fits (three for each round, the rounds being treated independently), find the residues, and then set up and solve equations (29).

6. Swerve reduction.

Of all the measurements made in the spark range only the utilization of the x- and y-coordinates of the center of mass remains to be discussed. The procedure we discuss here for utilizing these coordinates was first established in a systematic way by H. Federer, although preliminary work had been done by A. K. Goldstine and the authors. The procedure is at present just beyond the experimental stage. It is too laborious for routine hand computation, and has not yet been set up for the more sophisticated computing machines, although no serious difficulties prevent this set up. Our discussion will be rather sketchy, covering the mathematical background, but ignoring the rather formidable computing problems.

The swerve reduction relies heavily on the earlier reduction of the angular motion. We take as known the representation of ξ,

(1) $$\xi = c_1 \exp \Phi_1 + c_2 \exp \Phi_2,$$

where

$$c_1 = k_1(0), \quad c_2 = k_2(0),$$
$$\Phi_1 = z(\log_e k_1)' + i\phi_1,$$
$$\Phi_2 = z(\log k_2)' + i\phi_2.$$

Let S be x + iy. Then S is known for 25 different values of z. According to (XI.6.8), S should have the form

$$S = (\text{linear function of } p)$$

(2) $$+ (\text{drift}) + (\text{gravity drop})$$

$$+ r_1 \exp \Phi_1 + r_2 \exp \Phi_2$$

where

$$r_1 = c_1 d\{(- J_L + ivJ_F)/(\Phi_1')^2 - i(vJ_{XF} + iJ_S)/\Phi_1' \},$$

$$r_2 = c_2 d\{(- J_L + ivJ_F)/(\Phi_2')^2 - i(vJ_{XF} + iJ_S)/\Phi_2' \}.$$

Here p is the distance along the trajectory, measured in calibers. For our purposes p = z/d. The derivatives in (2) are with respect to p. Referring to (XI.6.9) the gravity drop, in the (x, y)-system, has the form

(3) $$G = (igd^2p^2/2U_o^2) \{ 1 + (2J_Dp/3) + \ldots \}.$$

The drift, by (XI.6.10), has the form

(4) $$D = (- K_L + ivK_F)(gAv/K_M mU_o^2) \cdot (1 + J_Dp + \ldots)$$

The value of the quadratic and cubic parts of the gravity drop can be computed corresponding to each station, since the mean drag coefficient K_D has already been computed. The drift, which is quite small, can be estimated to a reasonable accuracy. The coefficient K_M has been computed in the yaw reduction, and K_L is of the order of one. Using 1 for K_L and .2 for K_F has not, on the few rounds for which a swerve reduction has been done, made a second, finer approximation necessary. Having estimated the drift and computed the quadratic and cubic parts of the gravity drop, the quantity

$$(5) \quad S_c = S - D - (\text{quadratic and cubic parts of } G)$$

is computed for each station. Referring to (2) we see that S_c should be of the form

$$(6) \quad S_c = a + bz + r_1 \exp \Phi_1 + r_2 \exp \Phi_2,$$

where a and b are complex constants. The functions $\exp \Phi_1$ and $\exp \Phi_2$ are known, and we have the familiar problem of determining which combination of known functions, in this case 1, z, $\exp \Phi_1$ and $\exp \Phi_2$, best fits the data. In precisely the same fashion as in Section 3 this fitting leads to a system of four linear equations for the four unknowns a, b, r_1 and r_2. Having r_1 and r_2, the equations (2) permit solution for the quantities

$$(7) \quad \begin{aligned} &- J_L + i\nu J_F, \\ &\nu J_{XF} - iJ_S. \end{aligned}$$

Mathematically, these determinations, together with those of (3.15) and that of the drag, give all aerodynamic coefficients except K_{XT}. Practically, this is not the case for data so far obtained. The probable errors of the coefficients from (7) have been of the order of 5 per cent for K_L, 30 per cent for K_F and over 50 per cent for K_{XF} and K_S. Effectively, this means that until measurements and reduction procedures are further refined the swerve reduction must be considered a method of determining K_L and the order of magnitude of K_F.

Chapter XIV

R O C K E T S

1. Introduction.

Historically, the use of rockets as military weapons antedates the use of guns by centuries; but the earliest efforts to use rockets were abandoned, and these weapons more or less forgotten until they were revived by Congreve in the early nineteenth century. His rockets were used in the Napoleonic Wars, and rocket batteries remained in the British army until 1885. From then until the Second World War they were again ignored, because of their great inaccuracy as compared with guns. In this last war, several tactical situations arose in which the rocket was the suitable weapon, and strenuous and fruitful efforts were made to improve its accuracy. We shall now briefly consider some of these situations.

Early in the war it was found that a small light shell could be made which even at low velocities was capable of destroying a tank. It was desirable to make this into an infantry anti-tank weapon, for direct fire by infantrymen at short ranges. The shell, though small, was still over two inches in diameter, and a conventional gun of such diameter is clearly unsuitable as a shoulder weapon. The difficulty was resolved by delivering the shell by means of a rocket, well known to many as the "bazooka." (The nickname was originally applied to the tube and later transferred to the rocket itself.) The tube was quite light, since its only purpose was to support the rocket during the aiming process and to protect the gunner from the burning gases emitted from the rocket. A similar weapon was also made by the Germans, and known as the Panzerfaust.

This was one instance in which the rocket was the suitable weapon because the support of the weapon (in this case an infantryman) was not capable of withstanding great forces or carrying heavy weights. Another such instance is in firing from airplanes. Obviously a fighter plane is not a suitable gun platform for heavy artillery. But it is possible for such a plane to carry several rockets under the wing, of striking power great enough to be highly significant as a weapon. As another example, small naval craft can carry enough rockets to bombard a beach on which a landing is to be made and to clear it for a landing. Without trying to make any accurate comparison, it requires little imagination to see that the volume of fire which a small craft can deliver in the form of rockets is greater than that of which it is capable in the form of gunfire.

An entirely different application of the rocket principle was made in the design of very long-range rockets such as the German rocket popularly (or should we say extensively?) known as the V-2. Such rockets require all the complicated ballistic analysis required by long-range guns, together with several additional factors. Although the ballistics of such a weapon requires numerical integration of a more or less conventional sort, we shall not attempt to give the details of the long-range rocket ballistics, but shall confine ourselves to the analysis of rockets of the sort used by this country in the Second World War. The same sort of analysis will apply to the early part of the trajectory of a long-range rocket also.

The fundamental physical principle of the propulsion of a rocket is quite simple and well known. A rocket consists of two parts, a head and a motor. The head is the business end. It is designed as a shell for fragmentation, high explosive or any other use. The motor consists of a combustion chamber, analogous to the chamber of a gun, in which the propellant is burned. By means of a nozzle or nozzles the products of combustion are ejected at high velocity from the rear of the motor. Since, in the absence of external forces, the total momentum of rocket and gases is constant, the rocket must be given forward velocity. Of course, the mechanism by which velocity is imparted is the varying pressure over the inner surface of the motor, but the total effect can be computed if the velocity of the gas relative to the projectile is known. This velocity is known as the exit velocity, and for most of the rockets of the types under discussion it was of the order of four to six thousand feet per second. Suppose that the velocity of the rocket and unburned propellant is v at the time when the mass is m, and is $v + \Delta v$ at the time when the mass is $m + \Delta m$. If the latter corresponds to the later time, Δm is negative and Δv is positive; a mass $|\Delta m|$ of gas has been expelled with velocity $v - v_e$, where v_e is the exit velocity. Hence, by the law of conservation of momentum,

$$mv = (m + \Delta m)(v + \Delta v) + (v - v_e)|\Delta m|,$$

or

$$m \Delta v + v_e \Delta m + \Delta v \Delta m = 0.$$

This implies

$$(1) \qquad \frac{dv}{dm} = -\frac{v_e}{m}.$$

If the mass of the rocket without propellant is m_r and the mass of the propellant is m_p, and v_e is regarded as constant, integration of the last equation yields

$$(2) \qquad v = v_0 + v_e \log \{ (m_r + m_p)/m_r \}.$$

If the mass of the propellant is considerably less than that of the rocket, this is approximately the same as

$$v = v_0 + v_e m_p/(m_r + \tfrac{1}{2}m_p). \tag{3}$$

Thus, for example, if the empty rocket weighed 30 lb. and the propellant weighed 4 lb., and the exit velocity were 5,000 feet per second, the rocket would be accelerated from rest to about 625 feet per second, according to the approximation (3). The more accurate formula (2) gives 625.8 feet per second instead.

In the present chapter we shall analyze the motion of a rocket during burning. Since after propulsion has ceased the motion of a rocket can be computed on the same basis as that of a shell or bomb, we need consider only the motion during burning. Our principal concern will be with fin-stabilized rockets, since these were by far the most tactically useful in the Second World War. However, the analysis can be extended to cover the spin-stabilized rockets also. This has in fact been done, by the methods of Chapter XI. But we shall not consider this extension in the present chapter.

The aerodynamic force system considered is a very restricted one. For rockets of the type under consideration the burning time is of the order of 0.1 to 1.5 seconds, and the mean acceleration is from 1,000 to 5,000 feet per second per second. In view of the magnitude of the propulsive force it would at first glance appear possible to ignore all aerodynamic effects, but this is not so. Although the drag is of minor importance the angular position of the projectile is of primary concern, since the thrust of the motor is directed along the axis. It is therefore necessary that we be able to predict the position of the axis throughout burning. We consider only the largest of the aerodynamic torques, and assume that the rocket's motion is determined by the acceleration given by the motor, by the restoring moment, and by gravity.

Finally, we wish to point out that the application of the theory of ballistics of rocket fire is an excellent example of the way in which the results of a somewhat idealized theory can serve as an interpolating device for complicated physical experiments. We have already encountered one such example in Section 4 of Chapter IV. In the theoretical investigation of the motion of a rocket, it will be assumed that at any given temperature the propulsive acceleration is constant. This acceleration enters as a constant in the final equations. Its chief service is to determine the velocity at the end of burning. Consequently we do not measure the acceleration by direct measurements of thrust. Instead, from the data of the range firings, we find the velocity at end of burning, and then choose the constant called "propulsive acceleration" in such a way that when the velocity at end of burning is computed from the equations, it agrees with the experimental determination. We may thus feel confident that when launching conditions (other than temperature) are changed, with this same constant acceleration the theory will continue to furnish the correct velocity at end of burning. Likewise, the righting moment should not be deduced from wind tunnel experiments or from the angular motion of the rocket. Its chief service in the equations of motion is to determine the direction of motion of the rocket at the end of burning. Hence from the range-firing data, we find the direction of motion of the rocket at end of burning, and then select the value of righting moment which, being substituted in the theory, provides the same direction of motion as was experimentally observed. Then we may expect that under other launching conditions, the theory will continue to provide accurate predictions of the direction of motion at end of burning.

2. Equations of motion for a fin-stabilized rocket.

In this section we derive the equations of motion for a rocket and put them in a form suitable for computation. As usual, the X-axis and the Y-axis are respectively horizontal and vertical, the rocket being supposed launched at the origin in a direction lying in the XY-plane. If the X-axis points to the right, a positive angle is one which is measured counterclockwise. The angle from the X-axis to the tangent to the trajectory will be denoted θ, the angle from the X-axis to the axis of the rocket is ϕ. The yaw, δ, is the difference $\phi - \theta$, and u is the total velocity. Let c be the acceleration due to the motor, which is assumed constant in magnitude and directed along the axis of the rocket. The components of acceleration along and perpendicular to the tangent to the trajectory are then respectively $c \cos \delta - g \sin \theta$ and $c \sin \delta - g \cos \theta$. Hence we may write

$$
(1) \qquad \begin{aligned} \dot{u} &= c \cos \delta - g \sin \theta, \quad \delta = \phi - \theta, \\ u\dot{\theta} &= c \sin \delta - g \cos \theta. \end{aligned}
$$

According to the notation previously introduced, the restoring moment is of the form $-\rho d^3 u^2 K_M \sin \delta$, where ρ is the density of the air, d is the diameter of the projectile and K_M is the moment coefficient. However, for the purposes of the present investigation it is convenient to amend this somewhat. We shall write the restoring moment in the form $-\rho d^3 u^2 K_M \delta$. This amounts to changing the meaning of K_M by multiplying the original meaning by a factor $\delta/\sin \delta$. The difference between this factor and unity is about $\delta^2/6$, so for small yaws the redefinition is quite immaterial. For large yaws the original and the new definitions differ; but in the following mathematical study we shall replace the (new) K_M by a constant which is the same as the common value of old and new K_M at zero yaw. Making this substitution for the restoring moment gives us for the equation governing the angular motion the following:

$$
(2) \qquad B\ddot{\phi} = -\rho d^3 u^2 K_M \delta,
$$

where B is the moment of inertia of the rocket about an axis through the center of mass and perpendicular to the axis of the rocket. We now modify the form of (1) and (2) somewhat.

Referring to Section 5 of Chapter III, it is recalled that for a finned projectile in free flight the distance between two successive maxima of yaw, called the wave-length of yaw, has the form

$$\lambda = 2\pi \sqrt{B/\rho d^3 K_M}. \tag{3}$$

This quantity has the dimensions of a length and is a very convenient parameter for the present discussion. Accordingly, (2) may be written

$$\ddot{\phi} = -4\pi^2 u^2 \delta/\lambda^2, \tag{4}$$

and we shall assume that λ is a constant. We now change the form of (1). Computing,

$$(u \sin \delta)^{\cdot} = \dot{u} \sin \delta + u \dot{\delta} \cos \delta = u\dot{\phi} \cos \delta + g \cos \phi.$$

We now make further approximations, assuming that $\sin \delta$ may be replaced by δ and $\cos \delta$ by one, that $g \cos \phi$ also may be replaced by $g \cos \theta_0$, and finally that $c - g \sin \theta$ may be replaced by a constant a. The last equation and the first equation of (1) then become

$$\dot{u} = a, \qquad (5)$$
$$(u\delta)^{\cdot} = u\dot{\phi} + g \cos \theta_0.$$

The set of equations, (4) and (5), form a complete system on the variables u, ϕ, δ, involving the constants λ, a and $g \cos \theta_0$. We now select a new independent variable which will be dimensionless. This variable will be simply u multiplied by a suitable constant. Namely, we define

$$w = u \sqrt{2/a\lambda}. \tag{6}$$

The reason for the choice of the factors a and λ is fairly obvious, for these depend on the particular rocket and it will be convenient to have equations which, without even change of constants, are simultaneously valid for all rockets. The factor 2 is simply for future convenience. If we use primes to denote differentiation with respect to w, the equations (4) and (5) become

$$\phi'' = -\pi^2\delta w^2,$$

(7)

$$(w\delta)' = w\phi' + (g\cos\theta_0)/a.$$

These equations are now in the form in which we shall solve them. They determine completely the angular motion of the rocket. It is now a rather easy task to set up the equations which determine the motion of the center of mass. In terms of the original XY-coordinates, we have at once

(8)
$$\dot{X} = u\cos\theta,$$
$$\dot{Y} = u\sin\theta.$$

If we transform to slant coordinates, ξ, η, as in the Siacci method, where ξ is distance measured along the initial tangent to the trajectory and η is drop from this line, the equations simplify. Defining

(9) $\xi = X\sec\theta_0, \qquad \eta = X\tan\theta_0 - Y,$

we compute directly from (8)

$$\dot{\xi} = u\cos\theta\sec\theta_0,$$
$$\dot{\eta} = u(\cos\theta\tan\theta_0 - \sin\theta)$$
$$= u\sec\theta_0\sin(\theta_0 - \theta).$$

Under the assumptions we have made, it is quite proper to use, instead of these, the approximate equations

(10) $\dot{\xi} = u, \quad \dot{\eta} = u(\theta_0 - \theta)\sec\theta_0.$

The first of these leads to the equation

$$\xi = (u^2 - u_0^2)/2a,$$

while the second, upon changing to the independent variable w, becomes

$$\eta' = \lambda w(\theta_0 - \theta)(\sec \theta_0)/2,$$

which is the form we shall use.

For convenience in future reference we now collect together the results of this section.

The angular motion of a rocket is determined by the equations

$$\text{(11)} \quad \phi'' = -\pi^2\delta w^2, \quad \delta = \phi - \theta,$$
$$(w\delta)' = w\phi' + (g \cos \theta_0)/a,$$

where $w = u\sqrt{2/a\lambda}$, a is the mean acceleration during burning minus $g \sin \theta_0$, and λ is the wave-length of yaw. The equations determining the space position and the time are:

$$\text{(12)} \quad \begin{aligned} \xi' &= (u^2 - u_0^2)/2a, \\ \eta' &= \lambda w(\theta_0 - \theta)(\sec \theta_0)/2, \\ t - t_0 &= (u - u_0)/a. \end{aligned}$$

3. Solution of the equations of motion.

This section is devoted to the explicit solution of the system of equations (11) and (12) above. We first solve the first equation for $w\delta$, and by substituting in the equation involving $(w\delta)'$ the following is obtained.

$$\text{(1)} \quad \frac{1}{w}\left[\frac{1}{w}\phi''\right]' = -\pi^2\phi' - \frac{\pi^2 g \cos \theta_0}{a\,w}.$$

This can now be recognized as a familiar form if the independent variable w is replaced by the independent variable $p = w^2$, for the derivative of a function with respect to p is then its derivative with respect to w multiplied by $1/2w$. Equation (1) then becomes

(2) $$\frac{d^2\phi'}{dp^2} + \frac{\pi^2\phi'}{4} = -\frac{\pi^2 g \cos\theta_0}{4a\sqrt{p}}.$$

The general solution of the equation (2) for ϕ' is a linear combination of sin $(\frac{1}{2}\pi p)$ and cos $(\frac{1}{2}\pi p)$ added to a particular solution. It is, in fact, possible to obtain a particular solution from the known solution cos $(\frac{1}{2}\pi p)$ of the homogeneous equation by the method of variation of parameters. If we assume a solution of (2) of the form A cos $(\frac{1}{2}\pi p)$ where A is a function of p, substitution in (2) leads to the equation

$$A'' \cos \tfrac{1}{2}\pi p - \pi A' \sin \tfrac{1}{2}\pi p = -\tfrac{1}{4}(g/a)(\pi^2 \cos\theta_0)/\sqrt{p}\,.$$

By multiplying the equation by cos $(\frac{1}{2}\pi p)$ the left-hand side becomes the derivative of $(A' \cos^2(\frac{1}{2}\pi p))$, and a solution for A is

$$A = -\tfrac{1}{4}(g/a)(\pi^2 \cos\theta_0)\int_{p_0}^{p} \sec^2\tfrac{1}{2}\pi s\left[\int_{p_0}^{s} r^{-1/2} \cos\tfrac{1}{2}\pi r\, dr\right]ds.$$

This expression may be integrated by parts, integrating the $\sec^2(\frac{1}{2}\pi s)$. The result, after a little simplification, can be written in the form

$$A = \tfrac{1}{2}(g/a)(\pi \cos\theta_0 \sec\tfrac{1}{2}\pi p)\int_{p_0}^{p} s^{-1/2}\sin\tfrac{1}{2}\pi(s-p)\,ds,$$

and a particular solution is this value of A multiplied by $\cos^2(\frac{1}{2}\pi p)$. The general solution for ϕ' is therefore, replacing p by w^2 and s by r^2,

(3) $$\phi' = (g/a)(\pi\cos\theta_0)\int_{w_0}^{w}\sin\tfrac{1}{2}\pi(r^2 - w^2)\,dr + C\sin\tfrac{1}{2}\pi(w^2 - w_0^2) + D\cos\tfrac{1}{2}\pi(w^2 - w_0^2).$$

We now evaluate C and D by using the initial conditions. Let δ_0 and ω_0 denote the initial yaw and the initial angular velocity respectively. Since

$$\phi'' = -\pi^2\delta w^2,$$

(see equation (2.11)),

$$\phi_0'' = -\pi^2\delta_0 w_0^2 .$$

Further, ϕ' is $(d\phi/dt)/(dw/dt)$, and hence

$$\phi_0' = \omega_0\sqrt{\lambda/2a}.$$

(See the definition of w in (2.11).) Setting $w = w_0$ in equation (3) then shows that

(4) $$D = \phi_0' = \omega_0\sqrt{\lambda/2a} .$$

Differentiating the equation (3) with respect to w and setting $w = w_0$ leads to the evaluation of the other constant.

(5) $$C = \phi_0''/\pi w_0 = -\pi\delta_0 w_0.$$

These values of the constants, substituted in (3), then give the desired solution. Given this solution for ϕ' it is easy to see that solutions for all the other variables can be obtained as quadratures. Actually what is required for ballistic computation is the initial conditions for the non-burning part of the trajectory. We are therefore interested in solving for θ, ξ and η, t and u at the end of burning. There is no particular reason to find δ or ϕ at the end of burning, for a simple calculation will show that under the assumptions we have made, the yaw at the end of burning will be negligible. (This is physically very reasonable since the propulsion increases the axial velocity, which, in the absence of other factors, decreases the yaw.) We now consider the determination of θ and η. Since, (equation (11) of the preceding section), $\phi'' = -\pi^2\delta w$, and

$$\theta - \theta_0 = (\phi - \phi_0) - (\delta - \delta_0),$$

it follows that

(6) $$\theta - \theta_0 = \delta_0 + \int_{w_0}^{w} \phi'(s, w_0)\, ds + \phi''/\pi^2 w^2.$$

Differentiating (3) leads to

(7)

$$\phi'' = -(g/a)(\pi\cos\theta_o)\int_{w_o}^{w} \pi w \cos\tfrac{1}{2}\pi(r^2 - w^2)\,dr$$
$$-(\pi\,\delta_o w_o)[\pi w\cos\tfrac{1}{2}\pi(w^2 - w_o^2)]$$
$$-(\omega_o\sqrt{2a})[\pi w\sin\tfrac{1}{2}\pi(w^2 - w_o^2)].$$

Hence (6) may be written, collecting like terms,

$$\theta - \theta_o$$
$$= \delta_o + (g/a)\cos\theta_o\left[\pi\int_{w_o}^{w}\int_{w_o}^{s}\sin\tfrac{1}{2}\pi(r^2 - s^2)\,dr\,ds - (1/w)\int_{w_o}^{w}\cos\tfrac{1}{2}\pi(r^2 - w^2)\,dr\right]$$

(8) $$-\delta_o\left[\pi w_o\int_{w_o}^{w}\sin\tfrac{1}{2}\pi(r^2 - w_o^2)\,dr + (w_o/w)\cos\tfrac{1}{2}\pi(w^2 - w_o^2)\right]$$

$$+(\omega_o u_o/a w_o)\left[\int_{w_o}^{w}\cos\tfrac{1}{2}(r^2 - w_o^2)\,dr - (1/\pi w)\sin\tfrac{1}{2}\pi(w^2 - w_o^2)\right].$$

The functions of w and w_o which lie within the three sets of square brackets can be computed, and are actually combinations of Fresnel integrals

$$S = \int_0^w \sin\tfrac{1}{2}\pi r^2\,dr$$

and

$$C = \int_0^w \cos\tfrac{1}{2}\pi r^2\,dr,$$

and the related integrals

$$\int_{o}^{w} S\ dC$$

and

$$\int_{o}^{w} C\ dS.$$

These quadratures have been done at Aberdeen and from them the three functions within the brackets were evaluated. These are denoted by $F_1(w_o, w)$, $F_2(w_o, w)$ and $F_3(w_o, w)$ respectively. In terms of these functions, the value of $\theta - \theta_o$ may be written

$$\begin{aligned}\theta - \theta_o = (g/a) \cos \theta_o\, F_1(w_o, w) \\ + \delta_o [\, 1 - F_2(w_o, w)\,] \\ + (\omega_o u_o / a w_o)\, F_3(w_o, w).\end{aligned} \tag{9}$$

It should be remarked that this is an unexpectedly pleasant result. The functions F_1, F_2 and F_3 do not depend in any way on the rocket being considered, and may be (and have been) tabulated once and for all. Further, $\theta - \theta_o$ depends linearly on the yaw and the initial angular velocity, so that if the angle has been computed on the basis of zero initial yaw and zero angular velocity the corrections for these may be made by simply adding the appropriate factors.

The drop η may also be computed in an equally convenient form. Referring to equation (2.12),

$$\eta' = \tfrac{1}{2}\lambda w(\theta_o - \theta) \sec \theta_o,$$

so that

$$\eta \cos \theta_o = \tfrac{1}{2} \lambda \int_{w_o}^{w} r\ [\theta_o - \theta(w_o, r)\,]dr. \tag{10}$$

In view of (9) we may therefore define* functions G_1, G_2 and G_3 such that

$$\eta \cos\theta_o = \tfrac{1}{2}\lambda\Big[(g/a)\cos\theta_o\, G_1(w_o, w) + \delta_o G_2(w_o, w) + (\omega_o u_o/aw_o)\, G_3(w_o, w)\Big]. \tag{11}$$

Equations (9) and (11) embody the results which will be required for the computation of a firing table for rockets. In practice the mean acceleration a will be evaluated before the wave-length of yaw λ, so that instead of the factor $\lambda/2$ the equivalent number u_o^2/aw_o^2 will be used in formula (11).

The basic formulas now being available, it would seem reasonable to consider as a next step their application to the problem of analyzing experimental data and constructing a firing table. This material has, however, not been cleared for publication, so here we abruptly close.

---

*The functions which have been tabulated at Aberdeen are not actually G_1, G_2 and G_3 as defined here, but certain related functions F_4, F_5 and F_6. In terms of these the drop is defined by the equation

$$\eta\cos\theta_o = \tfrac{1}{4}\lambda\Big\{-w^2(\theta-\theta_o) + \tfrac{1}{2}(g/a)\cos\theta_o\big[F_4(w_o, w) - (w^2 - w_o^2)\big] + \delta_o F_5(w_o, w) + (\omega_o u/aw_o)\, F_6(w_o, w)\Big\}.$$

The relation between the G's and these is easily computed.

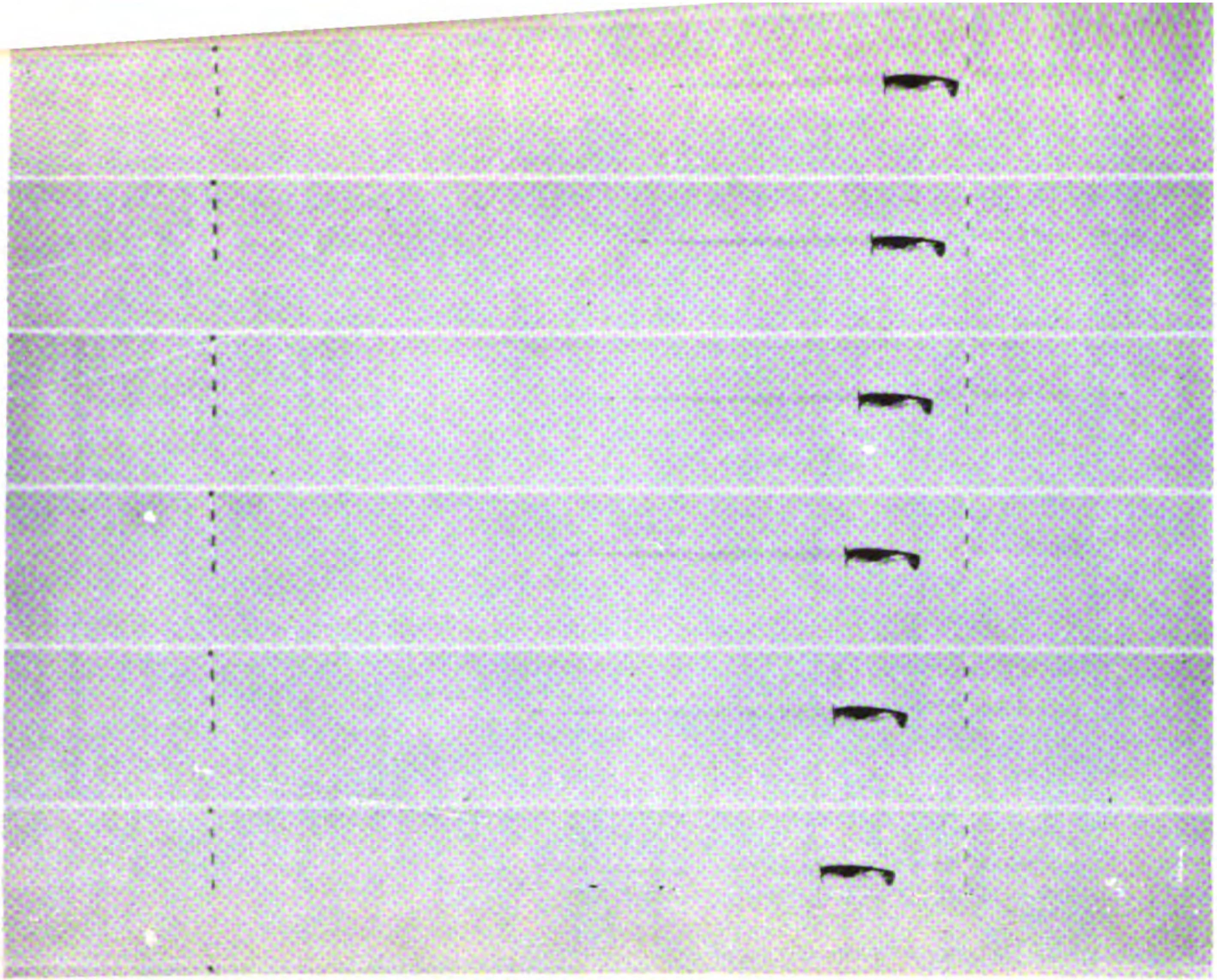
NOTE: A more complete treatment of the motion of a rocket, including the effects of jet malalignment, etc., may be found in Mathematical Theory of Rocket Flight by J. B. Rosser, R. R. Newton and G. L. Gross (New York: McGraw-Hill Book Company, Inc., 1947) and in "The Mathematical Theory of the Motion of Rotated and Unrotated Rockets," by R. A. Rankin, Philosophical Transactions of the Royal Society of London, Series A, Vol. 241 (1949), pp. 457-485.

Figure XIV.3.1

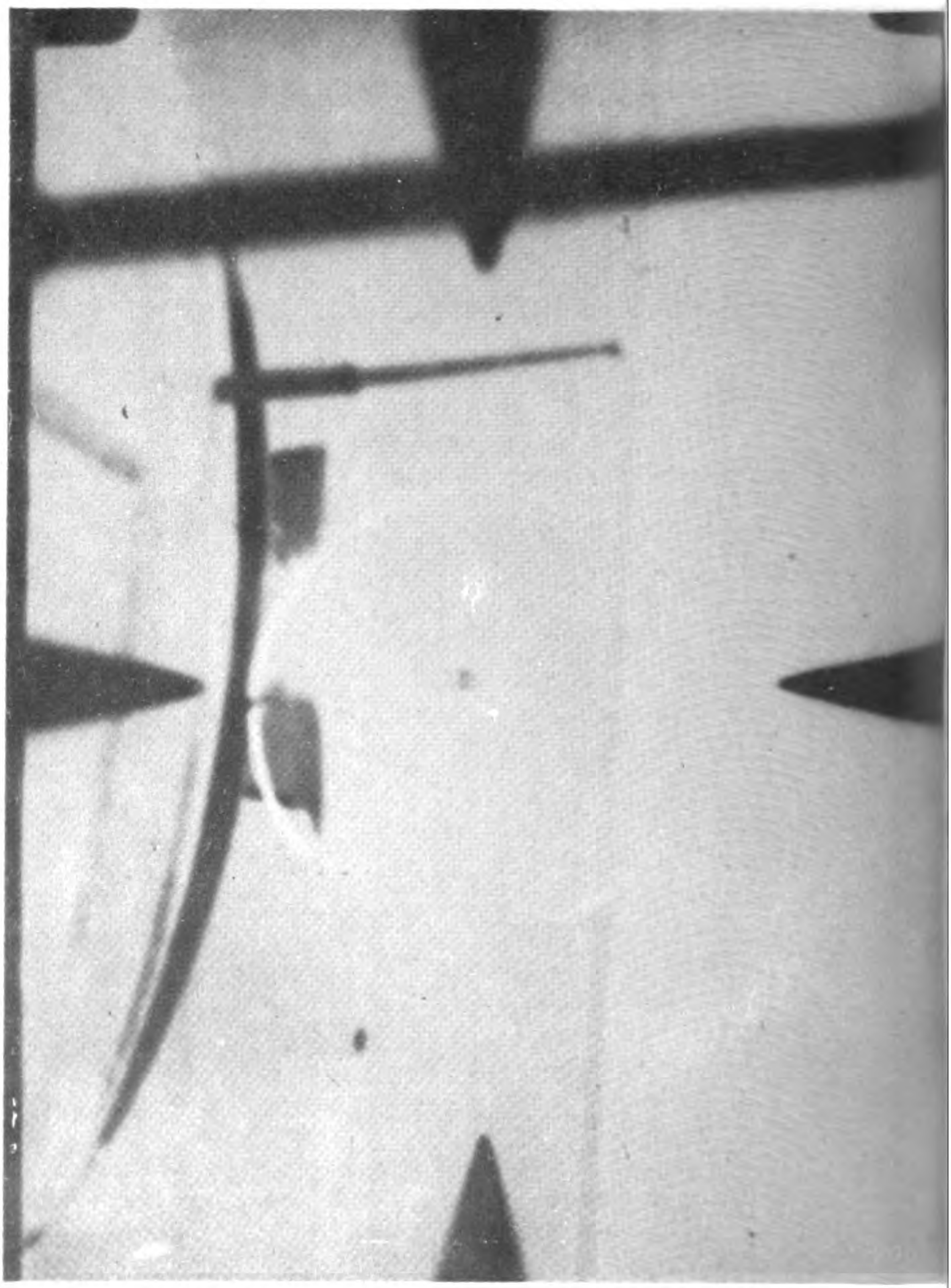
Range Firing of Five-inch High Velocity Aircraft Rocket

(a) Preliminary firings, photographed with Bowen-Knapp camera
(b) Range firing, gunsight camera record
(c) and (d) Firing, photographed from rear, Eyemo camera
(e) Firing, from side, Mitchell camera
(f) Rocket, near end of burning, Mitchell camera
(g) Impact, Eyemo camera

Photographic Measurements Section
Ballistic Research Laboratories
Aberdeen Proving Ground.



(c)



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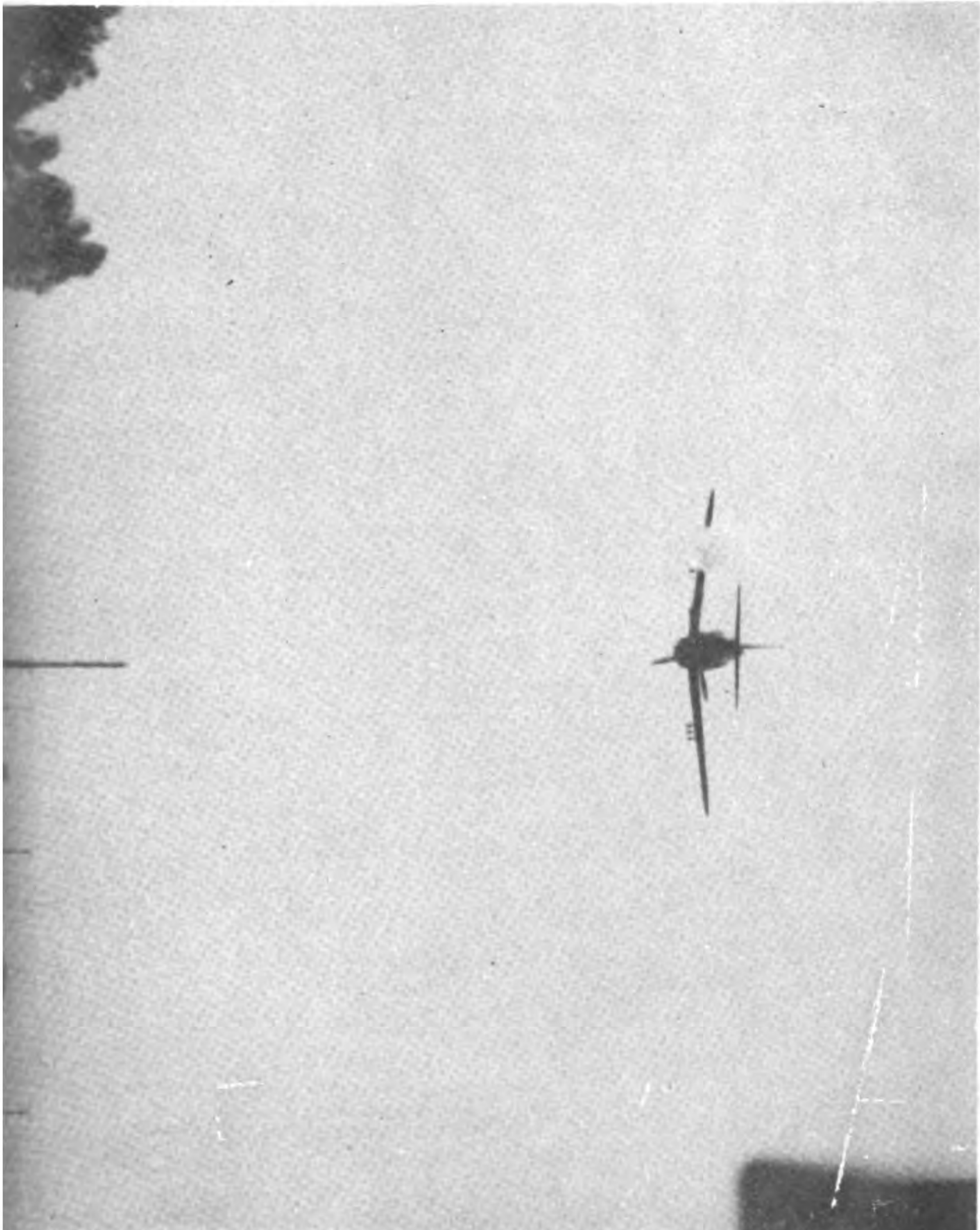
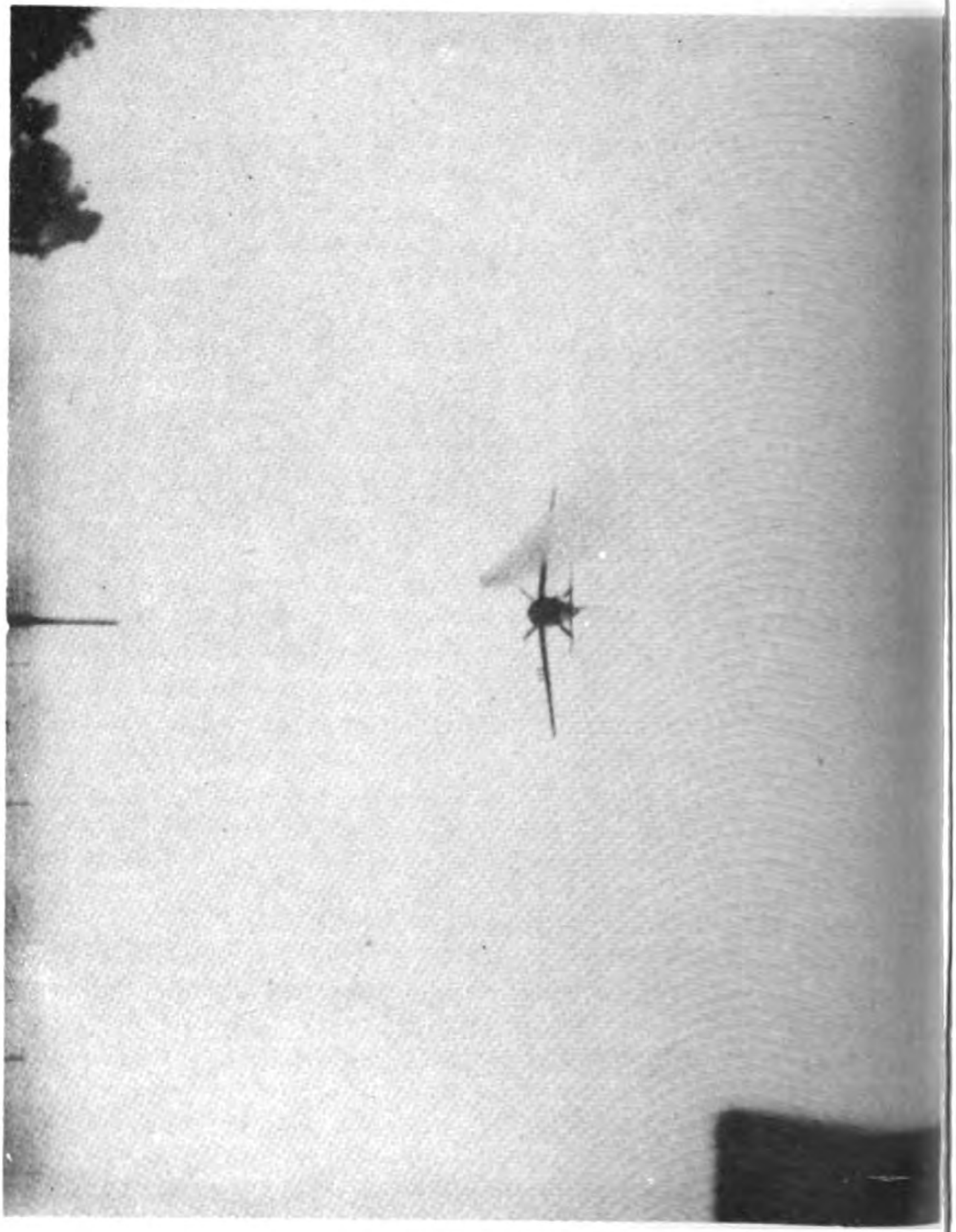
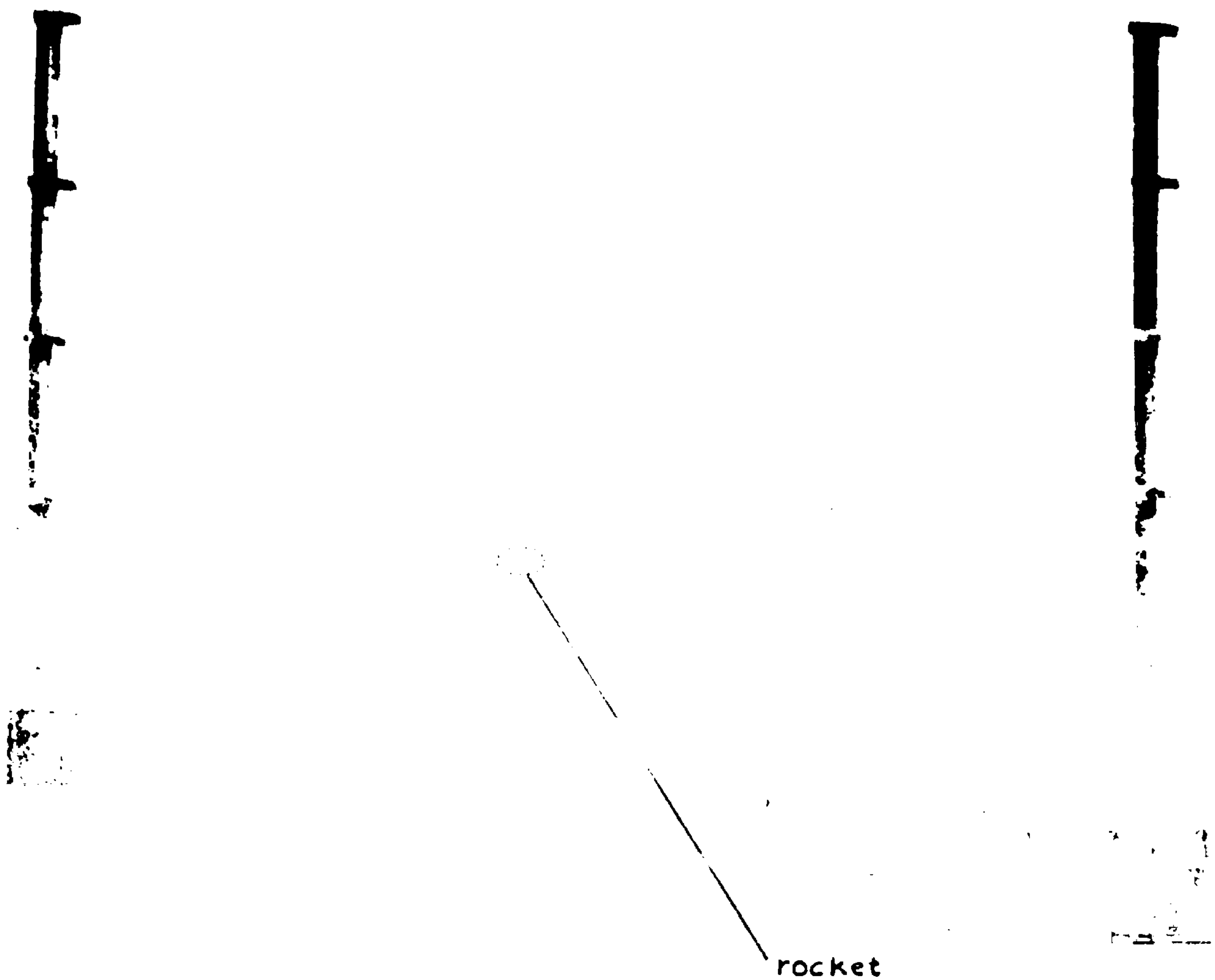
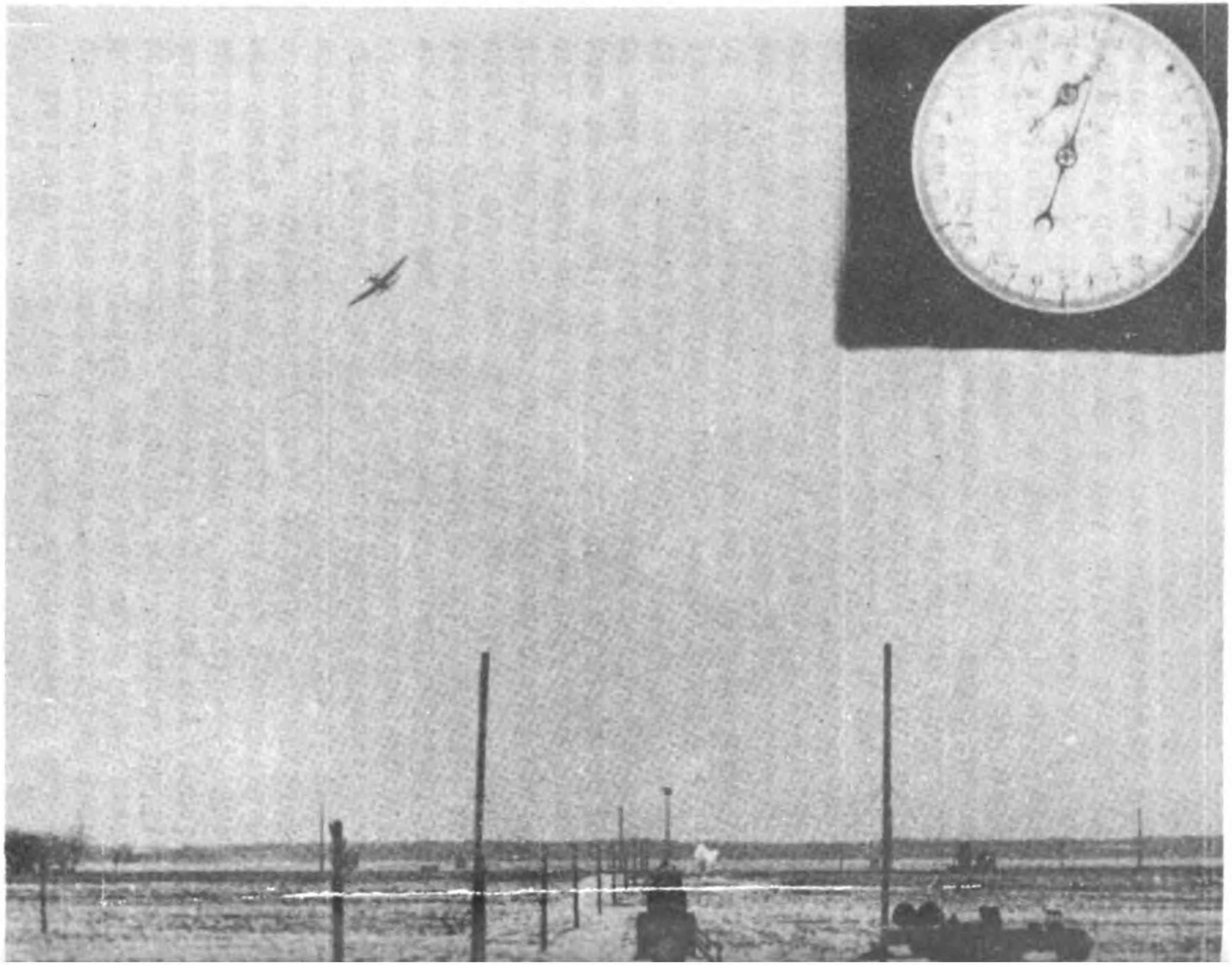




(3)

(e)


rocket



(a)

HISTORICAL APPENDIX

Ballistics is the science which treats the action of projectiles and their associated hurling machines. Modern writers divide the subject into three parts described as interior, exterior and terminal ballistics which are concerned, respectively, with the propulsion, motion and destructive action of projectiles. The modern theory of exterior ballistics has been developed as a branch of the analytical dynamics of rigid bodies moving under the influence of gravitational and aerodynamic forces. In this exact sense, exterior ballistics was founded about three hundred years ago with the work of Galileo. Interior and terminal ballistics were founded as sciences more recently than exterior ballistics, the three branches having been distinguished first during the latter part of the eighteenth century. Medievil and even ancient writers, however, used terms cognate to the English word ballistics to describe the art of making and using missive armament. The word ballistics was derived from the Latin ballista for an ancient javelin-hurling engine actuated by the torsion of skeins of sinew. The term ballista was perhaps derived, in turn, from a Greek term used by the Syracusan and Alexandrian mechanicians who developed this engine into efficient forms. Extant books entitled βελοποιϊκα, or throwing devices, were written by Philon of Byzantium, Rhodes and Alexandria during the third century B. C. and Heron of Alexandria at an uncertain date no later than the third century A. D. Ballistics at the time of Philon and Heron was the theory of the design of hurling engines such as the ballista and the similar stone-casting engine which medieval Latin writers sometimes described as a petraria. In the general sense of the cognate term used by classical writers, ballistics was an advanced art in Hellenistic times. Thus Philon described still earlier writers as "the ancients." The development of missive armament as a skilled craft began, however, millennia before writing; ballistics originated as a technical art with the inventions of the earliest weapons specialized for throwing purposes. Specialized missiles and hurling machines have been prominent everywhere among the artifacts recovered from the campsite debris left by hunting men of the Upper Old Stone Age.

The materials employed in making the most important instruments, especially weapons, have been used by anthropologists to distinguish the stages of civilization attained by the carriers of different cultures. Early Lower Old Stone Age men knew the flint fist-hatchet, primitive language and the use of fire but no weapon especially designed to be thrown. The most important advance in weapon technique during the Lower Old Stone Age was the chipped stone point provided with notches for attachment to a wooden handle. These notched stone points indicate knowledge of the spear which was probably used both for thrusting and for hurling from the time of its invention. The spear was known to Mousterian man during the last period of the Lower Old Stone Age in southwestern Europe. The oldest hurling machine was the atlatl or spear-thrower. The invention and diffusion of the atlatl has divided the cultures described as Lower and Upper Old Stone Ages over the whole earth; this instrument was known to Aurignacian man in Spain about twenty-five thousand years ago and to Chiricahua man in New Mexico about ten thousand years ago. The oldest engine, or machine for doing work by means of stored energy, was the bow. The wooden long bow was invented more than fifteen thousand years ago by an Upper Old Stone Age people in Africa or Asia. The oldest known user of the bow was Magdalenian man who showed it in his polychrome drawings on the walls of a cave near Alpera. The wooden long bow was the primary weapon of the Middle Stone Age Capsians who spread over western Europe from Africa, but it first acquired universal use among peoples with New Stone Age cultures. The reflex bow was devised in the arsenals of the Bronze Age Sumerian and Egyptian armies which conquered the oldest empires on earth. The compound reflex bow of the Iron Age Assyrian Empire was made of alternate layers of wood, horn and the neck-sinews of the stag. The cross-bow was apparently devised somewhere in the Orient shortly after 1000 B. C. by adding a stock-and-trigger to the long bow. The arbalista was developed by equipping the cross-bow with a windlass or lever used to retract the bow-stave mechanically. The catapult was probably devised from the mechanically retracted battering ram shown in Assyrian bas-reliefs: in any case, dart- and stone-casting engines precursive of arbalistas and catapults were reportedly included in the armament assembled by King Uzziah (circa 750 B. C.) for the defense of the city of Jerusalem. Double-armed skein-actuated engines such as the ballista were used in both siege and field warfare by the engineers of the army of Alexander of Macedon during the decade 333-323 B. C. The techniques employed in making the powerful mechanically retracted engines of classical times were largely unknown in western Europe during the Middle Ages, but were re-introduced in France and Italy during the eleventh and twelfth centuries. King John of England was explicitly prohibited from employing foreign "balistarios," specialists in the use of the arbalista, by a clause of the Magna Carta, which he signed in 1215. The ancient elastically actuated engines and the medieval gravitationally actuated trebuchet were replaced during the fourteenth century by the first true firearms, devices which hurled solid projectiles by the force resulting from the expansion of gases from the burning of gunpowder.

The inventions of the cannon and the rocket were made possible by the medieval Chinese invention of gunpowder. The Chinese discovery of the propellant properties of gunpowder was the ultimate consequence of Asian developments in fire arrow compositions which were probably initiated with the appearance of iron metallurgy in the countries bordering the Mediterranean Sea. The fire arrows of the Mesopotamian peoples of the late Bronze Age were covered with viscous, persistent-burning mixtures of pitch, petroleum and sulphur. Early iron craftsmen experimented with various inflammable materials which furnished large heats of combustion because iron requires a higher temperature than copper for its reduction and forging. Some of these men also observed that the colors of metals became lighter as they were heated in a forge. They recognized that metals were heated by the transfer of some mysterious entity from the flame and decided that a yellow flame was very hot. Common salt was believed to contain a large amount of this heat-producing entity because flames into which it was thrown turned yellow. Since it was desired to produce the hottest possible flame from the materials transported by fire arrows, some ancient peoples added common salt to their incendiary mixtures. Many old recipes for making fire arrow compositions included resin, petroleum and sulphur: some also employed charcoal and common salt. These recipes were apparently transmitted through the Near East to India and China, perhaps at the time of Alexander or his successors. Saltpetre may possibly have been identified first as a material distinct from common salt or soda by medieval Syrian or Arabian chemists. However, it seems more likely that saltpetre was first employed in incendiary compositions in the interior of China where common salt was scarce and saltpetre was quite widely distributed. At some time before 1200 A. D., the Chinese apparently substituted saltpetre for common salt as an ingredient of incendiary mixtures. Since charcoal was one constituent of these mixtures and sulphur another, the addition of saltpetre led to the preparation of gunpowder. Saltpetre was the critical component of gunpowder because it furnished the oxygen which enabled the combustion of the mixture to proceed rapidly without utilizing atmospheric oxygen. The Chinese presently employed saltpetre, charcoal and sulphur compositions which, if the saltpetre content was high, constituted explosive mixtures, if somewhat lower, propellant mixtures and, if still lower, simple incendiary mixtures. Explosive mixtures were enclosed in pottery cases, propellant compositions in separable paper containers attached to fire arrows and wild fire materials in flame-throwing tubes made of bamboo. The pottery-encased gunpowder charge was the earliest type of explosive shell, the self-propelled fire arrow was the prototype of the incendiary rocket and the fire-projecting bamboo tube was one of the precursors of the flame thrower. The Chinese described their explosive shell as the "heaven-shaking thunder," their incendiary rockets as the "fire that flies," and their flame throwers as "fire tubes." Both the "fire that flies" and the "heaven-shaking thunder" were employed in 1232 A. D. by Tartar defenders of the city of Kaifeng against the Mongol army commanded by Ogotai, son of Genghis Khan. Gunpowder with the concentration of saltpetre required for either an explosive or a propellant was known to the Chinese siege engineers of the Mongol "horde" which advanced into Europe as far as Hungary in the campaign of 1238-1242 A. D. However, gunpowder probably became known in the Near East through the Moslems who had been engaged in warfare with the Mongols during the khanship of Genghis. An accurate description of saltpetre as a material distinct from common salt was given in a book written in 1240 A. D. by an Arabian pharmacist sometimes called ibn al-Baithar, the "son of the horse-doctor." Ibn al-Baithar reported that saltpetre was known by the Egyptians as "snow from China." Ibn al-Baithar did not describe the deflagrating property of saltpetre but this property was well known to pyrotechnicists of the Near East a few years later. In 1279 or 1280, an Arabian military engineer named Hassan al-Rammah wrote a book on cavalry warfare in which he gave pyrotechnic and rocket propellant compositions containing saltpetre, sulphur and charcoal in approximately the proportions since used in the manufacture of gunpowder. Roger Bacon of England and Albert the Great of Germany have sometimes been credited with the invention of gunpowder but both men probably knew recipes given in a Latin manuscript with the title _Liber Ignium ad Comburendum Hostes_. This book was widely circulated in Europe during the latter part of the thirteenth century. Reputedly the work of a Byzantine called Marcus Graecus, the _Liber Ignium_ contained many incendiary compositions, directions for making a rocket and a description of a petard or cracker. Bacon and Albert were proficient in Arabic literature and they may also have read contemporary Arabic recipes for gunpowder. Although the propellant properties of gunpowder were almost certainly learned in the Near East from the Chinese, the Arabs were probably the first to use it as a propellant in firearms. The earliest known mention of a firearm was made in an Arabic document dated 1304. The first European firearm was a dart-throwing engine called a pot-de-fer. The invention of the pot-de-fer has been ascribed by a well-known German legend to a monk named Berthold Schwarz, or Bertholdus Niger, whose surname "the black," was probably a reference to his interest in alchemy. The possibly apocryphal Berthold has been variously reported as having invented the first firearm in Freiburg im Breisgau, Dortmund, Venice and Flanders. The earliest European evidence for the existence of firearms was given in documents of the city of Ghent which were dated 1313 and 1314. The bombard, a smoothbore cannon hurling stone or metal round shot, was developed simultaneously in several countries of Europe and the Near East during the fourteenth and fifteenth centuries. An old description of the battle of Crécy (1346) states that the English employed bombards "which with fire throw little balls to frighten and destroy horses." Firearms were probably ineffective at Crécy but the destruction of the Byzantine Empire in 1453 was in some degree a consequence of Turkish use of siege artillery. Constantinople, the Byzantine capital, was bombarded by huge cannon for twelve days before succumbing to the Ottoman ruler, Mahomet II. These great bombards, which hurled stone balls thirty inches in diameter, were cast in bronze by a Wallachian gun founder named Urban. After the Turkish conquest of Constantinople many Byzantine scientists and technicians fled to Italy.

The development of the modern theory of the analytical dynamics of rigid bodies was initiated during the fifteenth and sixteenth centuries by mathematicians and physicists many of whom were employed as ballisticians and military engineers in the arsenals of the Italian principalities. This development began with a critical revision of some Aristotelian theories which had dominated dynamical ideas for eighteen hundred years. Aristotle (384-322 B. C.), the founder of formal logic, was interested in the problem of projectile motion. He was skillful in applying geometry to kinematics, but he lacked both the conceptions and the methods required for an accurate development of the dynamics basic to the modern theory of exterior ballistics. He constructed an elaborate theory of the dynamics of particles which was based partly on principles derived from the hypotheses of Empedocles on the nature of matter. Empedocles of Acragas, Sicily (circa 440 B. C.) had performed experiments which established the corporeality, or material nature, of the air. This discovery led Aristotle and some of his contemporaries to argue that since air was present where matter had been supposed absent, a vacuum was impossible. Aristotle noted that in a vacuum all bodies would fall at the same speed and asserted that, since this was incredible, a "void" was a physical impossibility. He presumed that all matter had such properties as wetness or dryness and warmth or coldness and considered that these properties were the causes of the tendencies of bodies to rise or fall. He was somewhat indefinite or, possibly, inconsistent, in his statements on the rate of fall of bodies in fluid media: he overemphasized the influence of the medium on the motion. Some Aristotelian dynamical errors were corrected by his ancient successors, especially the followers of the atomistic school of philosophers whose theory of the nature of matter had been critized by Aristotle. The atomic theory of the constitution of matter was enunciated by Leucippus of Thrace (circa 440 B. C.) who had been a student of Zeno of Elia (495-435 B. C.). Leucippus was succeeded as director of the Thracian atomistic school by Democritus of Abdera. Democritus justified the atomic theory of the constitution of matter by several observed phenomena including solid friction and the drag of solids forcibly moved through liquids. He argued that a "void" could be created by exhausting an air-filled container of its content of atoms. Democritus' idea on the creation of a vacuum interested Straton of Lampsacus, Alexandria and Athens (circa 300 B. C.). Straton was the last important director of the "lyceum," the research laboratory in Athens which had been founded by Aristotle. He had earlier established experimental physics in the "museum," the famous university and research laboratory in Alexandria. He showed that a near vacuum could be made by exhausting the air from a closed vessel with siphons. He also demonstrated the compressibility of the air. Straton's researches led some ancient mechanicians to recognition that the properties of fluids could be explained by the atomic theory of the constitution of matter. Archimedes of Alexandria and Syracuse (287-212 B. C.) stated in a prefatory letter to one of his extant books that he had discovered a theorem in hydraulics by considering the probable behavior of a fluid composed of atoms as suggested by the theory of Democritus. After the time of Archimedes, almost all ancient mechanicians argued correctly that a body rose or fell according to whether its specific gravity relative to the surrounding medium was less or greater than unity. A few ancient writers apparently believed that bodies fell at rates proportional to their specific gravities relative to surrounding media. These mechanicians probably argued from observations on the vertical fall of very light bodies in dense fluids. Such observations would have agreed with the results of modern experiments on the motion in dense resisting media of light particles near "limiting" or "terminal" velocities. Stokes' law of drag is applicable to this type of motion if the particles employed have very low terminal velocities. Such experiments, of course, yield very different results from those obtained by testing the rates of fall of very dense bodies in tenuous media.

The writings of the Alexandrian mechanicians showed the development of ideas on the nature of both gravitational and aerodynamic forces which were more nearly similar to the conceptions of Galileo than to those of Aristotle. Philon and Heron always computed the magnitudes of forces by comparison with assigned weights. In the preface of his Pneumatica, Heron defended the atomic theory of the constitution of matter by arguments derived from experiment: he apparently recognized that the air exerted a resistance to the motion of solid bodies. Joannes Philiponos of Alexandria, also known as John the Grammarian, who wrote during the sixth century A. D., explicitly contradicted Aristotle's erroneous opinion that the continued motion of a projected body was a consequence of the movement of air which had been started in motion simultaneously with the body. "Why," asked Philiponos, "must the moving hand touch the stone at all if the air manages everything?" He denied the Aristotelian doctrine that geometrical position in itself exerted force on a body. He had some conception related to Galileo's modern idea of the inertia of massive bodies and Newton's first law of motion: he attributed to bodies the effort to preserve their order. He stated that Aristotle was in error in supposing that bodies of great weight fell much more rapidly than bodies of small weight: his argument that Aristotle was wrong in this supposition was based upon observations of the times of fall of dense bodies of different weights which were dropped simultaneously. Philiponos' criticism of the Aristotelian dynamics probably influenced some of the predecessors of Galileo.

The development of ballistics from a technical art to a branch of science began about the time when firearms were first introduced into warfare in western

Europe. Several conceptions basic to modern dynamics originated with writers of the fourteenth and fifteenth centuries. Jean Buridan (circa 1320) had a conception of mass which was similar to that of Newton. He also believed that celestial bodies moved according to the same laws as terrestrial bodies, thereby foreshadowing Newton's idea that the moon was acted upon by the same forces as those which determined the motion of projectiles. Several writers of the fourteenth century proposed theories of the earth's gravitation in order to explain the correspondence between the moon's period and that of the tides. Albert of Saxony (circa 1380) believed that the motion of heavy falling bodies was uniformly accelerated. Nicholas of Cusa (1401-1464) proposed timing the fall of bodies in air and thereby determining the effects on the times of fall which he thought to be caused by the resistance of the air.

The early modern development of ordnance engineering in Italy began with the work of versatile Leonardo da Vinci (1452-1519). The character of da Vinci's earliest inventions was shown by his letter of application for an appointment as engineer to Ludovico Sforza, the usurper of the principality of Milan. He wrote:

> "Again I have kinds of mortars, most convenient and easy to carry, and with these can fling small stones almost resembling a storm, and with the smoke of these causing great terror to the enemy, to his great detriment and confusion...
>
> "In case of need I will make big guns, mortars, and light ordnance of fine and useful forms, out of the common type...
>
> "And in short, according to the variety of cases, I can contrive endless means of offense and defense..."

He sketched many new weapons including rifled firearms, wheel-lock pistols, breech-loading cannon, fuzed explosive shell and even primitive machines precursive of the tank and the submarine. He attempted to formulate general laws governing the operation of the machines which he had studied or proposed. This work was described in the five thousand pages of his notebooks. These notes included discussions of researches on the mechanics of rigid and deformable bodies, aeromechanics and hydraulics. Although he retained some Aristotelian ideas, his mechanics was not primarily Aristotelian. He attempted to deduce a theoretical statics and dynamics from the motion of colliding bodies before and after impact, the motion of bodies on an inclined plane and the motion of pulleys. He knew the ideas of Albert of Saxony on the theory of impulse and examined the rebounding of spheres from plane surfaces, probably in an attempt to determine the action of cannon balls in bounding from the walls of fortresses. He concluded that

> "The blow will be less powerful than its impulse, according as the angle of percussion is nearer the right angle"

and he believed that the angles of incidence and rebound were equal. He attained some conception of the parallelogram of forces from the motion of bodies on an inclined plane. He said:

> "The weight of a body divides its gravity into two aspects, that is, according to the line along the inclined plane and according to the line perpendicular to the inclined plane."

He knew the book of Albert of Saxony on gravity. He concluded from experiments that the rapidity of movement of a sphere sliding down an inclined plane to that of a body falling freely was as the height of the vertical fall to the length of the inclined plane. He observed the free fall of heavy bodies and asserted that:

"A weight which has no support falls by the shortest route to the lowest point which is the center of the world."

His investigations of falling bodies were displayed with the coordinates suggested by Nicholas Oresmus of Lisieux (1323-1382). He plotted the time of fall of a body on a vertical scale and the velocity on a horizontal scale. He found that:

"In air of uniform density, the heavy body which falls, at each stage of time acquires a degree of movement more than the degree of the preceding time."

He enunciated axioms, or laws, of motion which appeared to conform to the results of his observations. His study of the flight of birds led him to conclude that:

"All moving tends to maintenance or rather all moved bodies continue to move as long as the compression of the force of the motors remains in them."

He remarked that

"Nothing can be moved by itself but its motion is effected through another."

He argued from observations on the motion of a parachute that

"an object offers as much resistance to the air as the air does to the object."

Da Vinci's dynamical axioms were precursive of Newton's first and third laws of motion, but he suggested no relation involving force and rate of change of momentum comparable to Newton's important second law of motion. He was, however, the first writer who attempted to find a theoretical basis for the phenomena of aerodynamics.

He knew Roger Bacon's ideas on the possibility of flight and wrote a treatise on flight which contained sketches of a parachute. He said:

> "If a man have a tent roofing of calked linen twelve bracci broad and twelve bracci high, he will be able to let himself fall from any great height without danger to himself."

He made thin wax figures which floated in the atmosphere when filled with hot air and he may have made models of a primitive helicopter which would soar. He used models to find the centers of gravity of birds and noted that occasionally the center of gravity of a bird lay outside its body. He conceived a center of pressure for a bird and interpreted its movements through the different positions for the bird's center of gravity and center of pressure. He said:

> "When a bird which is in equilibrium throws the center of resistance of the wings behind the center of gravity, then it will descend with its head downward."

This is essentially the first principle employed in stability considerations on the design of modern bombs.

Several early modern writers on the theory of gunnery knew how to prepare rudimentary forms of firing data from range observations. Johannes Müller of Germany (1436-1476) invented instruments for positional observation by military surveyors and developed exact methods in cartography for use by artillerists and navigators. Müller, better known as Regiomontanus, probably had an insufficient knowledge of the computation of gunnery data even for the requirements of his own times, but he prepared extensive and accurate tables of trigonometric functions by methods somewhat precursive of those used in making modern firing tables. Santbach of Germany wrote a book on ballistics in 1561. He attempted to apply mechanics to problems in gunnery. His mechanics was, however, Aristotelian: he considered that a cannon ball proceeded along the line of departure until its velocity was exhausted, thereafter dropping nearly vertically. The early modern writers on ballistics often divided the trajectory of the projectile into three parts which they called the motus violentus, the motus mixtus and the motus naturalis. It was supposed that during the motus violentus, the projectile moved in a straight line directed along the vector of the initial impulse. The motus naturalis was the presumably normal descent of a heavy body from an elevated position. The motus mixtus was the regime in which the motus violentus was passing into the motus naturalis. This early modern description of the trajectory was somewhat similar to the conception of some of the ancients who supposed that the path of the projectile was composed of two straight lines connected by a circular arc at the summit. This classical conception was not as inaccurate a description of the normal trajectory as would be indicated by comparison with the Galilean parabola traversed by a particle in vacuo under the influence of a uniform, vertically directed field of gravity. The descending branch of the trajectory of a particle in air was steeper than the ascending branch and, in the case of a particle with very small ballistic coefficient, the descending branch of the trajectory rapidly became nearly vertical. George Greenhill remarked in his Notes on Dynamics that this old conception of the trajectory has survived to some extent in terminology employed even in recent times. The moderately curved first part of the trajectory is often supposed to be replaceable by a straight line at "point-blank range." After the point-blank range has been attained, the trajectory is presumed to fall off rapidly under the acceleration of gravity as in the motus mixtus, and, finally, to attain a regime of nearly vertical fall. There is another way in which the motus violentus and motus naturalis have been preserved in modern ideas although they are not explicitly mentioned. Newton, investigating the motion of a particle in air of constant density with drag proportional to the square of the velocity and subject to a uniformly directed gravitational field, discovered that the trajectory had two asymptotes. The ascending branch of the trajectory, if extended indefinitely, asymptotically approaches a line with slope greater than the quadrant elevation. The descending branch of the trajectory, if extended indefinitely, asymptotically approaches a vertical line.

The first writer on theoretical ballistics in modern times was the Italian, Niccolo Fontana (1500-1557). Fontana, better known as Tartaglia, originally obtained a geometrical form of the solution of the cubic equation sometimes described as Cardan's formula. Tartaglia and Girolamo Cardano (1501-1576) were scientific consultants to arsenals of various Italian principalities for some years. The master of ordnance at the castle of Verona proposed that Tartaglia consider the problem of finding the angle of elevation of a gun which would yield maximum range for a shot. Tartaglia discovered that forty-five degrees was the angle of elevation which resulted in the maximum range in gun-fire. He then undertook a treatment of ballistics as an exact science in order "to bring it to a degree of perfection capable of directing fire in all circumstances assisted only by a few particular experiments." Although Tartaglia's mechanics was partially based on Aristotelian conceptions, he obtained an accurate description of the shape of the trajectory by empirical methods. He attempted no considerable analysis of the forces acting on the projectile in flight, confining his investigation largely to inductions from observations. He made, however, several perceptive general observations on the "way of the pellet," as he called the trajectory. He was the first known writer to assert that the trajectory was curved throughout, but that the curve at the beginning and end of the path departed but little from straight lines. His theoretical treatment of ballistics was largely described in his Nuova Scienzia, which was published in 1537, although some of his work on gunnery appeared in his Quesiti et Invenzioni Diverse, published in 1546. He had considerable facility in experimental procedure: he suggested in the Quesiti, which was translated into English in 1588, methods of causing "any great piece of artilleries to make in his discharge an exceeding great noyse and marvellous rore." A loud sound on firing was regarded as a valuable attribute for a cannon of the sixteenth century. More significant for modern ballistics than Tartaglia's methods for inducing loud noises in cannon fire was his description of a gunner's quadrant. He said:

> "This instrument will help us to judge of all the variable positions or elevations that may happen in any piece of artillerie whatsoever."

Tartaglia's quadrant was also employed for determining angles of site and plotting the positions of targets.

Simon Stevin of Holland (1543-1620) performed a crucial "experiment against Aristotle" which preceded the similar test of Galileo at the leaning tower of Pisa. In his Statics and Hydrostatics, published in 1586, Stevin wrote:

> "Let us take (as I have done ...) two leaden balls, one ten times greater in weight than the other, which allow to fall together from a height of thirty feet upon a board or something from which a sound is clearly given out, and it shall appear that the lightest does not take ten times longer to fall than the heaviest, but that they fall so equally upon the board that both noises appear as a single sensation of sound."

Galileo employed shots weighing one pound and one hundred pounds in his comparable experiments a few years later.

The founder of modern theoretical dynamics was Galileo Galilei of Italy (1564-1642). Galileo was a consultant to the Venetian Arsenal for several years: he performed experiments with cannon balls on inclined planes in establishing the basis of his theory of motion of rigid bodies. His work on dynamics was largely described in the Dialogues Concerning Two New Sciences which was completed in 1636. He treated the motion of bodies in three parts, the first dealing with uniform motion, the second with naturally accelerated motion and the third with application of the theory of these two types of motion to the analysis of the flight of projectiles. He deduced Newton's first law of motion by considering the character of motion of "hard, smooth and very round balls" on a pair of inclined planes. The acceleration of these bodies decreased as the slopes of the planes decreased and vanished when the slopes became equal to zero. Thus, in the ideal case of no friction, a body projected along a horizontal plane would move uniformly forever at the initial or launching speed. He also measured the times of descent of cannon balls along declining planes by means of a water clock which would measure to a tenth of a pulse beat. He used a water spout at the base of a pail, weighed the water discharged during the descents of the bodies and assumed that the times of descent were proportional to the ascertained weights of water discharged. By comparing the times of descent for a quarter, a third, a half and other fractions of the plane's length with the times for descent of the whole length, he found that the "spaces traversed were to each other as the squares of the times, and this was true for all inclinations of the planes." He deduced from this space-time relation the further theorem that the speed at the end of measurement was proportional to the time of fall. This result led to the conclusion that the gravitational acceleration of any one body was a constant. Comparison of the times of fall of bodies of different masses showed that the gravitational acceleration of a body was independent of its mass. Having examined the properties of uniform horizontal motion and of naturally accelerated vertical motion, Galileo compounded the two types of motion in order to determine the form of the trajectory of a projectile. He said:

> "If the (horizontal) plane is limited... then the moving particle, which we imagine to be a heavy one, will, on passing over the edge of the plane, acquire, in addition to its previous uniform and perpetual motion, a downward propensity due to its own weight, so that the resulting motion, which I call projection, is compounded of one which is uniform and horizontal and of another which is vertical and naturally accelerated."

He showed that the trajectory was a parabola and said:

> "We now proceed to demonstrate some of its properties ... From accounts given by gunners I was already aware of the fact that in the use of cannon and mortars, the maximum range, that is the one in which the shot goes farthest, is obtained when the elevation is forty-five degrees."

He also deduced a result which has "perhaps never been observed in experience namely, that of other shots, those which exceed or fall short of forty-five degrees by equal amounts have equal ranges." His fourth dialogue included several tables of trajectory elements as functions of the angle of elevation using the maximum tabular values of the elements as units.

Galileo knew sufficient properties of the trajectory that he could have computed the range of a projectile in vacuo if he had been given observed values of the local apparent gravitational acceleration, the quadrant angle of elevation and the muzzle velocity. He knew only rough methods of measuring the local apparent gravitational acceleration, but more accurate procedures were devised by his student, Evangelista Torricelli (1608-1647). Torricelli determined the value of the superficial gravitational acceleration from observations of the motion of two weights attached to opposite ends of a string which passed over a fixed pulley. The quadrant angles of elevation of cannon had been accurately measured from Tartaglia's time, but rough methods of measuring muzzle velocity were first found in the century after Galileo's death. It was then learned that the ranges actually attained by projectiles were very much smaller than the ranges predicted from Galilean parabolas initiated with the observed muzzle velocities and quadrant angles of elevation. These decrements of observed ranges from the values expected for trajectories in vacuo were usually attributed correctly to the effects of drag of the air. Galileo had known that the atmosphere resisted the motion of projectiles: he argued merely that the parabola was an accurate approximation to the trajectories of dense projectiles shot with low initial velocities. Thus, he stated that his demonstrations were accurate "in the case of no resistance" and furnished practically useful results for dense projectiles shot from bows or arbalistas or, in general, hurling engines "other than firearms." He compared the times of fall of oak and lead balls dropped from altitudes of "150 or 200 cubits" and found small but definite differences in these times of descent. He argued that the retardation or acceleration due to the drag of the air on a moving body was a function "of weight, of velocity and also of form." He stated that this resistance decreased with the projectile's density, increased with its speed and varied greatly with its shape. He may have considered that drag was proportional to the square of the velocity for he remarked that falling bodies should be "displaced" by amounts proportional to the durations of the times of descent. He developed the conception of a "terminata velocità," or "terminal velocity," of a particle in vertical fall from rest in an atmosphere of constant density. The "terminal" or "limiting" velocity has been defined as the velocity at which the drag acting on a projectile becomes equal to its weight in vertical fall from rest. The limiting velocity was the asymptote of a velocity-time or velocity-space curve under Newtonian conditions on the drag for a vertical trajectory. Galileo stated that

> "No matter how heavy the body, if it falls from a very considerable height, the resistance of the air will be such as to prevent any increase in speed and will render the motion uniform, and in proportion as the moving body is less dense (men grave) this uniformity will be so much the more quickly attained and after a shorter fall. Even horizontal motion, which if no impediment were offered would be uniform and constant, is altered by the resistance of the air and finally ceases, and here again the less dense the body, the quicker the process."

He indicated a rough, but correct, experimental procedure which would show that the limiting velocity of a small round shot was much smaller than the muzzle velocity of a typical smoothbore hand-gun of his time.

Galileo gave the modern definition of momentum which was later used by Newton. He knew that applying a force to a body would change its momentum although he did not give a relation equivalent to Newton's second law of motion. He considered aerodynamic drag as a force and he investigated some properties of the atmosphere which caused this force. He invented a primitive type of air thermometer called a thermoscope and he knew that the pressure of the atmosphere was less than that of a column of water thirty feet in height. Although accurate methods of measuring drag were first introduced by Newton, Robins and their successors, many experiments on the nature of the air as a resisting medium were performed during the seventeenth century. Otto von Guericke of Magdeburg (1602-1686) constructed an air pump with which he exhausted the air from various metal containers He showed that feathers would fall as rapidly in a near vacuum as leaden balls. He noticed that gusts of air would blow metal objects along a tube which was being exhausted and that a powerful gust could be created by compressing air in a small space. Aristotle and other ancient writers had known that the absolute weight of an air-filled leather bladder was greater than that of an uninflated container when compared on a scale balance, but Guericke was the earliest physicist to determine the density of the air. He used the thermoscope invented by Galileo and the barometer invented by Torricelli to measure the temperature and pressure of the air in various researches. He knew that the density of the air was dependent upon its pressure and temperature. Torricelli, who wrote on fluid motion and also on ballistics, apparently tried to deduce the properties of drag from the experimentally determined properties of fluids. "Torricelli's tube," the barometer, was used in the researches on fluid phenomena of Edme Mariotte of France (1620-1684). Mariotte gave an independently discovered statement of Boyle's law of gases in his Sur la nature de l'air published in 1676. He knew that the drag of the air was proportional to its density. He attempted to determine the magnitude of air resistance by experiments performed at the Paris Observatory in 1670. He timed the descents of falling bodies and found that the air resistance appeared to be proportional to the squares of the time. The work of Guericke and Mariotte on the nature of the air was extended by the experiments of the English physicists Robert Boyle (1627-1691) and Robert Hooke (1635-1691). Boyle considered that the expansion of the air might furnish evidence supporting the molecular theory of the constitution of the air. Newton presently devised a direct rational argument to deduce Boyle's law from the molecular theory of the constitution of the air.

Most seventeenth century physicists believed that aerodynamic drag was dependent upon a change in the momentum of the air mass through which the projectile moved and some of these men thought that this change in the momentum of the air mass was a consequence of the collision of the projectile with the molecules of the air. Newton's conception of the drag of fluids was based on the molecular theory of the constitution of matter and the theory of the interaction of colliding bodies. These theories had been used by some physicists to explain the resistance of the air from the time of Francis Bacon of England (1561-1626). Bacon was interested in the behavior of projectiles. He wrote:

> "The condition of weapons and their improvements are, first, the fetching a far off, for that outruns the danger, as it is seen in ordnance and musketry. Secondly, the strength of the percussion.... "

He had no clear ideas on the nature of drag, but he argued that the properties of matter could be explained by the molecular theory. He advanced the important idea that heat was a form of mechanical energy derived from molecular motion. Stevin had understood that a dynamic pressure was exerted in changing the momentum of a fluid and his ideas became known to Rene Des Cartes of France, Holland and Sweden (1596-1650). Des Cartes enunciated ten laws of nature, of which the first two were equivalent to Newton's first and second laws of motion, but the latter eight were inexact or incorrect. He had a conception of quantity of motion which he used inconsistently. This ill-defined quantity was used by some later Cartesian writers for either, or both, momentum and doubled kinetic energy. Leibnitz confused momentum and doubled kinetic energy and referred to both quantities as forces; he described momentum as "vis mortua," or "dead force" and doubled kinetic energy as "vis viva" or "living force." He wrote incorrect equations of motion on the basis of Cartesian ideas. However, several useful conceptions in theoretical ballistics were originally derived from the erroneous Cartesian hypothesis on dynamics. Leibnitz' teacher, Christian Huyghens of Holland (1629-1695) corrected Des Cartes' errors on the theory of colliding bodies. He argued from experiment that the momentum of two perfectly elastic bodies in a certain direction before collision was equal to the momentum in the same direction after collision. He may or may not have regarded the air as a continuous medium, but he determined some properties of the air resistance from experiments on the motion of pendulums in fluid media. He apparently formulated the relation between the resistance coefficient, the limiting velocity and the weight of a falling body. Conclusions on the laws of colliding bodies which were similar to those of Huyghens were announced in the same year by Christopher Wren of England (1632-1723). Wren's work on the resistance of fluids was known to Newton. John Wallis of England (1616-1703) obtained Huyghens' and Wren's results on the theory of motion of colliding bodies simultaneously and also considered the collision of imperfectly elastic bodies. Wallis' later work contained the earliest systematic use of symbolic formulas in dynamics.

Isaac Newton of England (1642-1727) was the greatest of modern founders of ballistics. Newton's work on geometrical dynamics appeared in the Philosophiae Naturalis Principia Mathematica. The Principia contained two volumes: the motion of rigid bodies was described in the first book and the motion of fluids was discussed in the second book. The definitions given in the first volume indicated that Newton undertook his treatment of mechanics with conceptions which were either original with him or which had been developed during the preceding century. Thus mass was a primary characteristic of a physical body and weight was the force on a physical body which resulted from a gravitational field. Newton stated his law of gravitation in a general form applicable to all bodies in the universe, but he began his argument supporting that famous law by considering the motion of a projectile fired horizontally from a mountain top in vacuo. He showed that, by continually increasing the initial speed of the projectile, a speed would presently be obtained which would result in the projectile moving completely around the earth and returning to pass through the firing position. He then showed that the moon moved about the earth as if it had been started in motion in a similar fashion, thereafter repeating its orbit indefinitely. In the latter part of the first book, Newton examined the motion of projectiles in uniform media resisting as the first and second powers of the speed. He discovered the asymptotes to the trajectory of a particle projected in a resisting medium of constant density. He obtained a general solution for the problem of motion of a particle in a uniform medium resisting as the first power of the speed. He also found a solution for the case of purely vertical projection in the problem of motion of a particle in a uniform medium resisting as the second power of the speed.

Newton's interest in the theory of motion of a particle in media resisting as the second power of the speed was explained by the fact that he believed the drag of actual projectiles in air and water to be in accord with this law. The second book of the Principia discussed his experimental and theoretical work on the resistance of fluids to the motion of projectiles. Some of his early experimental work was performed in Saint Paul's Cathedral where he timed the descents of a large number of pellets of different characteristics from the ceiling to the floor. By dropping these pellets from different altitudes, he was able to show that the drag of the air increased as the square of the velocity of the pellets. By comparing the times of fall of pellets in air and in several liquids, he found that the drag exerted by fluids increased in proportion to their densities. He established that the drag of particles was proportional to the square of their diameters by comparing the times of fall of pellets of varying diameters.

He also constructed a theoretical argument to support these experimental results. The Newtonian theory of the resistance of fluids was too oversimplified to be regarded as adequate for the treatment of most recent problems in aerodynamics, but Professor T. H. von Kármán has remarked that Newton's argument leads to correct results for particles travelling at velocities many times the speed of sound.

John Bernoulli of Switzerland (1667-1748) obtained an exact analytical solution in finite terms for the general problem of the motion of a particle acted upon by a uniform gravitational acceleration and a Newtonian or "square law" drag in an atmosphere of constant density. Bernoulli's solution, which he found in 1711, was effected by a reduction to quadratures. The required quadratures could be performed by numerical methods as in Bernoulli's time or by use of a mechanical integraph such as that invented by Abdank-Abakanowicz about one hundred years later. The formulas available for numerical quadrature in Bernoulli's time were the Trapezoidal Rule, Simpson's Rule and the Newton-Cotes system of rules, all of which used equispaced ordinates of the function for which a quadrature was to be computed. The Trapezoidal Rule had been used in the method of exhaustion employed by the ancients for finding the area under a curve. Thomas Simpson of England (1710-1761), long professor of mathematics at Woolwich Arsenal, discovered a rule using three ordinates at each step of the quadrature. Simpson's Rule permitted a parabola with a vertical axis to be passed through the three points over which the quadrature was to be formed. Roger Cotes (1682-1716) collaborated with Newton in the formulation of a system of rules which permitted polynomials of higher order to be passed through a number of points in forming a quadrature. Other numerical methods of computing quadratures were discovered about this time — Euler's involved ordinates and derivatives of the function and Gregory's involved ordinates and differences of the function.

The most important early successor of Newton in ballistics was Leonhard Euler of Switzerland (1707-1783). Euler knew a method of measuring the muzzle velocity of cannon; he analyzed the results of experimental range firings in order to determine the drag of cannon balls. The letter e was first used to denote the base of the natural system of logarithms in one of Euler's papers on ballistics. He was the first major writer on theoretical dynamics whose work was cast in analytical rather than geometrical form; he gave the general equations of motion of a rigid body of which the first on linear momentum and the first on angular momentum are usually written as

$$\frac{d}{dt}(mu) - mv\theta_3 + mw\theta_2 = X \text{ and } \frac{dh_1}{dt} - h_2\theta_3 + h_3\theta_2 = L.$$

He used a simple fourth-order system of differential equations to describe the acceleration of the center of mass of projectiles and discovered two famous approximate methods of solving these equations of motion in order to determine trajectories. The mean value short-arc method given by Euler for computing the trajectory of a projectile in air of constant density with drag proportional to the square of the velocity was used in the computation of Otto's Tables one hundred years later. Euler also developed the solutions of equations of planetary motion in power series in the independent variable by using the classical expansion theorem of Brook Taylor of England (1685-1731). The Taylor expansion short-arc method has long been used for treating the initial stages of motion of projectiles and it has been employed occasionally to determine complete trajectories. The Taylor expansion method, however, was found very early to be inconvenient in practice for solving the normal equations of motion of projectiles. The Gregory expansion method, using finite differences, has generally superseded the Taylor expansion method, using derivatives, in numerical computation of trajectories. The four best known formulas for polynomial interpolation using finite differences - Gregory's, Gauss', Stirling's and Bessel's - were all known to Newton and a quadrature formula using finite differences was also given by Gregory. Taylor, who was regarded by his contemporaries as the founder of the theory of finite differences, apparently employed his methods in some simple computations in ballistics.

Accurate drag functions for projectiles were first determined by experiments using the ballistic pendulum which was invented in 1740 by Benjamin Robins of England (1707-1751). The initial angular momentum of the ballistic pendulum was derived from a bullet of small mass, fired into a pendulum bob. The height of the pendulum swing was observed and the striking momentum of the bullet thereby computed. The effect of drag on striking velocity was determined by firing bullets at a pendulum bob at different distances from a gun of assumed fixed muzzle velocity. Thus, Robins reported in his New Principles of Gunnery of 1742 that he had obtained mean velocities of impact of musket bullets fired at the ballistic pendulum of 1670, 1550 and 1425 feet per second. These bullets were twelve gauge round shot, that is twelve of them weighed one pound: they were of 0.75 caliber, in current terminology. The average retardation, or deceleration due to drag, was roughly 2750 feet per second per second. This large retardation, more than one hundred times greater than the acceleration of gravity, established the importance of terms representing the drag in the equations of motion of projectiles.

The retardation coefficients were found to be greater for projectiles moving at speeds near to the speed of sound than for projectiles moving at speeds very much less than the speed of sound. The speed of sound had been measured in 1640 in an experiment performed by the French scientists Marin Mersenne and Pierre Gassendi. Physicists of the nineteenth century found that the retardation coefficient was a complicated function of the Mach number or ratio of the speed of the projectile to the speed of sound. Ballisticians of the eighteenth century, however, concluded merely that the retardation coefficients of projectiles varied with the speed of the projectile, increasing at high speeds. Before the invention of the ballistic pendulum, the retardation, or deceleration due to drag, had usually been expressed in the form kv^2 where k was the constant Newtonian retardation coefficient and v the velocity of the projectile. After the middle of the eighteenth century the deceleration due to drag was often represented in the form bv^n where b and n were constants made to fit the results of observations. When it was found that no constant values could be found for b and n which would fit the experimental measures over all values of the velocity, the deceleration due to drag was represented by a sequence of zone functions of the form $b_i v^{n_i}$ where b_i and n_i yielded an adequate representation of the deceleration due to drag over the selected intervals in velocity. A representation of the retardation by zone laws was frequently employed under the name of Mayevski laws in the United States Army Ordnance Department as late as the First World War.

Jean-le-Rond D'Alembert of France (1717-1783) reduced to quadratures the problem of motion of a projectile in air of constant density with retardation proportional to $j + bv^n$ where j, b and n were constants and v was the velocity of the projectile. D'Alembert's solution involved functions which were somewhat similar to those which had appeared in Bernoulli's reduction of the "square law" problem: these functions were tabulated during the nineteenth century. The constant j was never used extensively in ballistic computations since it had no physical interpretation in the theory of retardation. However j had, of course, no restriction on algebraic sign: the D'Alembert solution was used for the case $j + bv^2$ in simple computations made in the Ballistic Research Laboratory during the Second World War on the motion of rockets during the propulsive regime. These computations employed j as a negative quantity, an assumed constant propulsive acceleration during burning, and bv^2 as the opposed "square law" retardation of the rocket during the same regime.

Ballisticians of the latter part of the eighteenth century sometimes attempted to reduce their observations on retardation to values which would have resulted had the actual density of the air been equal to an assigned standard density at sea-level. The observed, or unreduced, retardation coefficient had been denoted by k', the standard, or reduced, coefficient by k, the observed air density by ρ', and the standard air density at sea-level by ρ_o. Then

$$k' = k\rho'/\rho_o.$$

The ratio of the observed air density, ρ', to the standard air density at sea-level, ρ_o, has been described as the observed relative density and denoted by the symbol H'. The observed relative density has long been assigned the form of a product, commonly written as

$$H' = H(1 + \Delta H/H).$$

The standard relative air density, H, has been treated as an assigned function of the altitude of the projectile above the earth's surface, y. The observed quantity $\Delta H/H$ has been described as the relative density excess. Various forms have been assigned to H(y) by ballisticians since the latter part of the eighteenth century: an early form of H was the function H_1 where

$$H_1 = 1/(1 + hy)$$

and h is a constant. Adrian M. Legendre of France (1752-1833) used the function H_1 in his <u>Dissertation sur la question balistique proposee par l'Academie royal des Sciences et Belles-Lettres de Prusse pour le prix de 1782</u>. In this paper, Legendre reduced to quadratures the problem of motion of a projectile acted upon by a constant gravitational acceleration and a retardation proportional to the product of the square of the velocity and the relative density function, H_1. In 1917, American ballisticians adopted a standard relative density function of the form

$$H = e^{-hy}$$

where h is constant. The exponential form of standard relative air density was first used by Laplace.

The smoothbore hand-guns and cannon of European armies of the eighteenth century were effective only at short distances because of their low muzzle velocities and large internal clearances. The low striking momentum and inaccuracy of projectiles fired from smoothbore guns led to the return of the artillery rocket and the appearance of the rifle in European warfare during the early part of the nineteenth century. Incendiary rockets had been extensively employed in Italy and Germany during the fourteenth century, but they were gradually abandoned in European land warfare after 1450, largely because of their tendency to explode during manufacture or upon firing. Incendiary and even explosive rockets were, however, continuously developed in the Middle East during the centuries in which the rocket was little used as a military weapon in western Europe. The armies of Haidar Ali and his son Tippoo Sultan, princes of Mysore in India during the latter part of the eighteenth century, included brigades of specialists in the use of the war rocket. These Indian rockets were iron tubes weighing six to twelve pounds and were mounted on stabilizing poles about ten feet long. The Indian rockets were somewhat inaccurate, but they alarmed the flanks and rear of British soldiery campaigning in India, especially during the battle of Seringapatam in 1799. The effectiveness of the Indian rockets interested William Congreve of England (1772-1828) who developed incendiary rockets which attained ranges as great as three thousand yards. Congreve's rockets were employed very effectively by British armies during the Napoleonic Wars and the War of 1812. Rocket battalions were presently organized in Austria, Prussia, France, Poland, Russia and the United States. The military rockets of the early nineteenth century were somewhat inaccurate and their long guiding sticks were inconvenient in field operations. The need for the guiding sticks was removed by the invention of the spin-stabilized rocket by the American, William Hale, in 1855. Hale achieved rotation by placing three curved metal vanes in the nozzles of his motors. Spinning rockets attained stability because of dynamical reasons similar to those which caused shell fired from rifled weapons to be stable. Hale's invention was undoubtedly suggested by the spin-stabilized shell fired from rifled ordnance which had been developed a few years earlier and which superseded rockets as artillery projectiles during the latter part of the nineteenth century.

Rifled gun barrels were first employed generally in small arms before being used in artillery. Thus, the rifled barrels proposed by Da Vinci in the latter part of the fifteenth century first appeared in the hand-guns of the forces of the Landgrave of Hesse in 1631, but were unknown in ordnance until 1661 and then only used in one experimental cannon made in Prussia. Breech-loading was old in principle in 1740 when Robins advanced a formidable argument for the advantages of breech-loading rifles of all calibers. The "needle gun," a breech-loading infantry rifle invented by Johann Dreyse before 1836, was issued to some Prussian regiments in 1841, four years before a serviceable breech-loading artillery rifle was first made by Major Cavalli of Sardinia. Robins knew that the muzzle velocity of a projectile could be increased by reducing the amount of gas which escaped past its base. An elongated bullet, with a base which expanded on firing to fill the bore, was employed in the Minié rifle issued to British troops in 1851, considerably before the successful development of a cylindro-ogival artillery shell with a rotating band made of copper or a soft metal alloy.

The early work on the theory of gun design and interior ballistics was associated in an interesting way with the development of the theory of rocket propulsion and the resistance of fluids to the motion of projectiles. A correct theory of the nature of rocket propulsion had been advanced by Desaguliers in the eighteenth century. In the nineteenth century Saint Robert of France undertook the problem of determining the motion of a rocket which would attain a velocity permitting it to depart indefinitely from the earth. He assumed that the rocket would be acted upon by a gravitational acceleration inversely proportional to the square of the distance from the center of the earth and an atmospheric retardation proportional to the square of the velocity and to the density of the air. He employed the exponential law enunciated by Laplace to describe the diminution of the density of the air with increasing altitude above the surface of the earth. Increased jet velocities for rockets and muzzle velocities for guns were partly a consequence of the invention of the ballistic pendulum which had made it possible to test the efficacy of various powders. Antoine L. Lavoisier (1743-1794), the founder of modern chemistry, became director of the French National Powder Factory just prior to the French Revolution and initiated many experiments on the properties of propellants. Joseph L. Lagrange (1736-1783) made several contributions to the theory of wave propagation, apparently discovering that surges of the powder gas in the bore of the gun resulted in the loss of gas momentum which would otherwise have increased the muzzle velocity of the projectile. Lagrange formulated the problem of motion of the shell and the powder gas inside the bore of the gun. Pierre S. Laplace (1749-1827) who became Examiner of the Royal Artillery in 1784, had collaborated with Lavoisier in founding the theory of specific heats. Laplace recognized that sudden compression increased the elasticity of the air and corrected Newton's inaccurate argument on the propagation of sound. Newton's theory had yielded a value of the speed of sound which was lower than that found from the experiments of Mersenne, Gassendi and others. Laplace gave an accurate expression for the speed of sound in terms of the pressure, density and ratio of specific heats of the air. The modern proportionality relation between the speed of sound and the absolute temperature was first given after the work of William Thomson, better known as Lord Kelvin (1824-1907), but some ballisticians of the early nineteenth century computed the speed of sound waves correctly by the formula given by Laplace. Laplace probably knew that the formation of sound waves by a projectile travelling at a sonic or supersonic speed increased the drag of the air beyond the resistance expected if the air was treated as incompressible. His work introduced compressibility considerations into fluid dynamics. Mach's aerodynamic number, the ratio of the speed of the projectile to the speed of sound in the surrounding medium, was clearly related to the compressibility of the air and has great significance in ballistics. The importance of this number was probably inferred by some scientists as early as Bernhard Riemann (1826-1866) but it was first carefully considered by Ernst Mach, a German physicist of the late nineteenth century. Mach demonstrated the presence of shock waves formed by a projectile moving at supersonic speeds.

Advances in ordnance engineering greatly increased the range and accuracy of gunnery during the nineteenth century. This development led to the theoretical treatment of some differential effects in artillery fire. The range effects of a constant superficial density excess and the range and deflection effects of a constant superficial wind were considered by ballisticians of the latter part of the eighteenth century. Correct formulas were known for computing these effects before the middle of the nineteenth century, although these relations were sometimes derived by complicated and even innacurate arguments. The results were probably obtained originally by accurate methods early in the nineteenth century and the complicated derivations given later to explain the effects. Formulas for differential effects which were correct for flat nearly horizontal fire were usually obtained before results accurate under more general conditions were known. The first perturbing, or "abnormal," accelerations accurately treated in ballistics were those due to the small Coriolis, or "compound centrifugal" forces resulting from the rotation of the earth. The early treatment of the effects of rotation of the earth in ballistics was a consequence of their interest to theoretical astronomers. Some speculation about these effects had begun during the sixteenth century with the controversy on the Copernican hypothesis. The opponents of the Copernican hypothesis argued that if the earth rotated, a body dropped from a tower would deviate to the west of the point of fall by the amount which the surface of the earth moved eastward while the body was falling. Galileo recognized that a body dropped vertically downward would be released with the eastward velocity of the top of the tower and would, therefore, fall slightly east of the base of the tower since the top of the tower would have a greater eastward velocity than the surface of the earth. Galileo's argument was not regarded as conclusive by most of his contemporaries, but Newton defended the idea of the easterly deviation in 1679. Newton wrote that

> "a falling body ought by reason of the earth's diurnal motion to advance eastward and not fall to the west as the vulgar opinion is."

Hooke performed experiments which established that the deviation of a body dropped from a tower was, in fact, eastward of the point vertically below the point of release. D'Alembert and Clairaut knew how to form particular differential equations which appear to govern the motion of a body acted upon by known forces relative to an inertial frame of reference and the apparent forces which result from an arbitrary acceleration of the observer's reference frame with respect to an inertial frame. Such equations include, as a special case, the equations of motion of a body with respect to a reference frame fixed with respect to the surface of the earth. Power series solutions in the time of flight were found during the latter part of the eighteenth century for the effects of rotation of the earth on a falling body. Laplace treated the effects of the Coriolis acceleration on the flight of a projectile in air in the fourth volume of the Mecanique Celeste. He first considered "the descent of a body from a great height," but he followed his solution for the easterly deviation by determining the westerly deviation of a projectile fired vertically and finally examined the problem of the effects of the earth's rotation on the elements of the trajectory of a body which had "a projectile motion in space." He assumed an atmosphere of constant density with drag proportional to the square of the velocity and deduced correctly the equations of variation of coordinates which would govern the effects. He obtained solutions of these equations by a series in powers of the arc length on the normal trajectory. In 1835, Gaspard G. de Coriolis described the complete significance of the accelerations which apparently acted upon bodies as a consequence of the rotation of the earth. The effects of rotation of the earth were not generally introduced into artillery firing tables until after the work of Forrest R. Moulton in the Ordnance Department of the U.S. Army in 1917-1918. However, such effects were sometimes computed earlier by European ballisticians for interesting cases such as fire from the German long-range guns employed against Paris during the First World War.

The conception of the drag as a function of aerodynamic numbers derived from the properties of the air and the shape and velocity of the projectile was gradually evolved during the eighteenth and nineteenth centuries. Newton had recognized that the drag of an elongated body in a yawing position would be augmented by a quantity proportional to the drag of the body at zero yaw and to a function of the angle of yaw. A yaw drag proportional to the square of the sine of the yaw has sometimes been deduced from Newton's argument. The next aerodynamic numbers discovered after the yaw were the shape ratios of the projectile. The shape numbers were also anticipated by Newton who deduced the form of the head of a projectile of minimum resistance using a variational principle and his conception of drag. A roughness number was apparently anticipated by Robins. The aerodynamic numbers were dimensionless power products of the physical quantities appearing in the problem of fluid flow about the projectile. An incomplete set of the partial differential equations governing fluid motion was given by Euler, although more complete equations were first derived by Navier and Stokes in the nineteenth century. D'Alembert and Lagrange found some general solutions of simplified forms of the incompressible fluid equations in terms of arbitrary functions, which yield some information: they also obtained a few important special solutions. Although practical special solutions of the Navier-Stokes equations have never been obtained for an artillery projectile of the shape actually employed, much information has been obtained about the character of fluid flow about bodies of simpler shape. The theory of dimensional analysis which has been used to derive the aerodynamic numbers since the nineteenth century was initiated with the arguments of Joseph Fourier (1768-1830). Fourier was greatly influenced by the work of the French republican commission which founded the metric system of weights and measures. Aerodynamic numbers have been derived by Fourier's methods since the middle of the nineteenth century. An aerodynamic number of considerable significance for low-speed aircraft was discovered by Osborne Reynolds in his study of the flow of fluids through pipes. Viscosity was the principal property of a gas which appeared in the Reynolds number. Maxwell and others used the kinetic theory of gases to derive relations between the coefficient of viscosity and other measureable characteristics of a gas. However, aerodynamic theory remained incapable of providing a method for computing the drag of actual projectiles accurately.

The importance of accurate experimental methods for determining the drag of projectiles increased steadily during the nineteenth century because of developments in interior ballistics and ordnance engineering which increased both the magnitude and the consistency of the muzzle velocities of projectiles. Ballisticians of many countries sought improved methods of measuring drag after about 1850. A method sometimes employed in France involved determination of the retardation coefficient, k, by inversion of the Piton-Bressant formula. The Piton-Bressant formula is a special case, for flat horizontal fire, of the expression for the drop as a function of the slant distance along the line of departure as given in Chapter V of this book. The French had used the Piton-Bressant formula to determine mean values of k from measurements of range, muzzle velocity and drop from flat nearly horizontal fire. The computed values of k were corrected to those which would have been obtained if the density of the air at the time of firing had been an assigned standard for sea-level conditions. The values of the retardation coefficient so obtained were used to compute the striking velocity which, averaged with the muzzle velocity, yielded a mean speed assumed to correspond to the computed value of k. This somewhat inaccurate method was sometimes used to determine the retardation function of a projectile for comparison with results obtained by experiments with the ballistic pendulum.

Retardation functions of small arms projectiles were accurately determined during the latter part of the nineteenth century by a method which was developed in Germany. This procedure involved shooting a bullet through both sides of paper bands surrounding two or more widely separated spinning drums. The distance travelled by the paper around the drum between the nearly diametrically opposed entrance and exit of the bullet was measured first. Since the rate of revolution and the radius of the drum were known, the velocity of the projectile at each drum could be computed. The loss of velocity between drums was found and, since the distance between the drums was known, the retardation of the projectile could be inferred. The intersected drum method yielded improved retardation functions for bullets of small caliber but was inapplicable to large projectiles because sufficiently large drums could not be employed.

An accurate method of determining the drag of artillery projectiles was developed in England about the middle of the nineteenth century by Francis Bashforth. Bashforth utilized a chronograph which recorded time signals derived from an electric circuit which was partly based on principles developed by Wheatstone, the inventor of the Wheatstone bridge. Signals on Bashforth's chronograph were initiated by the breaking of a screen of fine copper wire with the passage of the projectile. Bashforth's method of reducing his observations may have been suggested originally by John Couch Adams, the celebrated British theoretical astronomer who had predicted the position of the planet Neptune prior to its discovery. This reduction employed finite differences of the measured times at which the projectile passed screens separated by equal distances in horizontal range. The time effects of range wind and air density excess were correctly determined in the reduction. Retardation data were obtained in several European countries during the latter part of the nineteenth century by ballisticians who used methods generally similar to those employed by Bashforth. These data were combined between 1880 and 1900 to determine the retardation function known as the Gâvre from the location of the French commission which considered the problem of projectile resistance. The Gâvre function was purported to correspond to a definite standard projectile with a known set of shape factors, although precisely similar projectiles were not employed in the original experiments. The Gâvre data were graduated so as to yield a function of the speed at the time of firing. Experimental retardation coefficients should have been combined at equal values of the Mach number and not at equal values of the speed of the projectile irrespective of the speed of sound at the time of the experiment. The Gavre retardation coefficient has a smaller slope near the speed of sound than that which would have been obtained by eliminating the statistical dispersion of the speed of sound at the time of measurement. The Gâvre retardation coefficient was also graduated and extrapolated somewhat differently by ballisticians of different nations. The Gâvre coefficient used by American ballisticians after the First World War had a remarkable curvature below the abscissa of lowest experimental velocity whereas Italian ballisticians apparently assumed, more rationally, a constant retardation coefficient for low velocities. The Mayevski zone laws employed a constant retardation coefficient for velocities below 790 feet per second.

The conception of a standard projectile having a caliber of one inch, a mass of one pound and a definite set of shape factors appeared in the work of Bashforth. He also used an assigned standard for the density of the air and his computed retardations correspond to those of the standard projectile moving in air of standard density, but at the observed velocity irrespective of the temperature of the air at the time of firing. The relation between the general retardation coefficient, k, the unit retardation coefficient, B, and the ballistic coefficient, C, appeared about the time of Bashforth. Thus,

$$k = B/C.$$

The definition of the ballistic coefficient in terms of mass, diameter and coefficient of form also was used during the latter part of the nineteenth century.

R. H. Kent and H. P. Hitchcock, among others, recognized that the unit retardation coefficients of projectiles actually employed during the First World War were unlikely to have numerical values equal to those of the Gâvre coefficient which was usually used in ballistic computations at the time. Kent developed the solenoid chronograph for measuring the retardation coefficients of projectiles from the time intervals occupied in passage between a sequence of solenoid coils. This method obviated the tipping and resistance effects of the screens of copper wire which had been generally employed in earlier procedures. Kent introduced the dimensionless drag coefficient, K_D, into American ballistic computations during 1937. He and Hitchcock determined accurate drag coefficients for six types of modern artillery shell. These coefficients have generally replaced the Gâvre coefficient in most ballistic computations in the United States. Kent's projectile type 2 was a narrow boat-tailed projectile with a long ogive. The drag coefficient of projectile type 2 was smaller and also had a steeper rise near the speed of sound than the Gâvre coefficient corresponding to the same values of the Mach number. An interesting verification of the shape of Kent's drag coefficient for the projectile of type 2 was presented by the theoretical form of the drag coefficient of a needle-shaped projectile which was determined by T. H. von Karman in 1934. Although the yaw drag had been known to Newton and was somewhat considered in the work of R. H. Fowler and his associates during the First World War, Kent and Hitchcock were apparently the first to determine a yaw drag coefficient from experimental range firings. The drag coefficients deduced by Kent and Hitchcock were reduced to values which would have been obtained at zero yaw.

The normal equations of motion which have been used by American ballisticians in recent years assume a constant uniformly directed field of gravitational acceleration and a retardation equal to the product of the square of the projectile's speed, a constant reciprocal ballistic coefficient, an exponential relative density function and a normal retardation coefficient. The normal retardation function has been regarded since the First World War as a function of the Mach number, thus requiring assignment of a normal speed of sound as a function of the altitude above sea-level. The exponential function of the altitude now used for the normal relative temperature was chosen more recently. Most of the methods used before the First World War for integrating the equations of motion could not have been applied to the solution of the normal equations in their present form because the older methods were usually dependent upon an assumption that the retardation of a projectile was equal to the product of a constant and a function of the speed only. Almost all methods of computing trajectories for more than one hundred years after Euler's time depended upon modifying or, as A. A. Bennett said, "wrenching," the differential equations of motion into forms which could be reduced to quadratures. The usual assumptions were that the trajectory could be regarded as nearly flat and nearly horizontal in the treatment of the terms in the equations of motion which express the components of the retardation. These assumptions were basic to the original form of the important method devised about 1880 by F. Siacci of Italy. Siacci's method, described in Chapter V of this book, constituted a reduction to quadratures in terms of the "pseudo-velocity," a vertical projection of the velocity along the slant line of departure. The elements of the trajectory were given in the method of Siacci through simple quadratures involving only products of the unit retardation function of the pseudo-velocity and powers of the pseudo-velocity. These quadratures, the Siacci functions, were tabulated in convenient form for use in numerical computation.

The ballistic coefficient as used in the method of Siacci was regarded as a rough average of the tabulated Gâvre retardation divided by the actual retardation of the projectile. In practice, the ballistic coefficient was assumed to vary with the quadrant elevation of the gun. The ballistic coefficients obtained from reduction of range firings were allowed to absorb some of the effects of the simplifying assumptions used in reducing the equations of motion to quadratures, in addition to the effects of the departures of actual retardation from tabulated superficial retardation. The diminution of the density of the air with altitude above sea-level was recognized and its reduction of the drag at high altitudes absorbed in the ballistic coefficient by rough mean values according to various formulas some of which had an empirical basis while others had been found by theoretical arguments. Maximum ordinates greater than three miles were unusual prior to the First World War. Under these conditions, the approximations of the older methods were somewhat satisfactorily absorbed in a slowly varying ballistic coefficient inferred from the tabulated Siacci space function and the observed range and muzzle velocity. The Paris long-range gun was reportedly invented as a consequence of the employment of the method of Siacci in circumstances in which its use was completely inadequate for accuracy in computation. Prediction of the probable value of the coefficient of form of a projectile from its shape alone was fraught with uncertainty: accurate inference of the ballistic coefficient always required actual range firing. Apparently the German Admiralty sought, during the early part of the First World War, to design a powerful naval gun with a range about three times that attainable with typical field artillery of the time. The first experimental shell produced for this gun was fired at a quadrant elevation of approximately forty-five degrees and travelled much of their trajectories in air of density approximately half that at sea-level. These shell attained ranges nearly twice as large as those which had been predicted. The ballisticians who had made the predictions found their error in the customary assumptions of the Siacci method as used at the time. These men, therefore, undertook the design of a gun which could shoot shell to very great ranges. They proposed to obtain these ranges by using shell which would travel most of their trajectories at altitudes such that the drag of the earth's atmosphere would be less than one tenth the drag which would have been developed in flat fire near sea-level. The principal problem involved was that of obtaining a powder which would yield a muzzle velocity of five thousand feet per second for a shell weighing about one hundred pounds. The required propellant was developed by Fritz Haber, the famous German chemist. The Paris long-range gun, of approximately nine-inch caliber, was designed to fire shell at a fixed elevation of about fifty degrees. Some of the first shell fired from the Paris gun attained ranges greater than ninety miles.

After the First World War, H. P. Hitchcock tabulated Siacci functions for the various new retardation coefficients which had been determined for modern projectiles by the experiments of R. H. Kent. Kent and Hitchcock discovered that the assumption of the Siacci method on flat nearly horizontal fire could be modified in such a way as to secure equations reducible to quadratures for flat fire at any angle of elevation. Kent and Hitchcock used the pseudo-velocity as dependent variable and the range as independent variable: their method was the source of the process described in Chapter V of this book, where the slant distance along the line of departure has been used as independent variable. Kent and Hitchcock's method employed the Siacci space function in the fundamental equation, but all the other quadratures used in their equations were different from those employed by Siacci. Kent and Hitchcock's method permitted approximate retention in the retardation of the diminution of the density of the air with altitude. This method was extensively used during the Second World War in computing elements of the trajectories of projectiles fired by small arms against aircraft and by guns aboard aircraft.

Artillery such as the German Paris long-range gun and the French railway cannon required accurate firing tables. Such firing tables could only be obtained from trajectories computed by methods which did not employ Siacci's assumption of flat horizontal fire. The methods developed for accurate computation of trajectories at large elevations were influenced by the methods used in celestial mechanics from the time of the Swiss, Leonhard Euler, in the eighteenth century, to that of the American, Forrest R. Moulton, during the First World War. These methods have been described, since about 1880, as short-arc procedures because they depended upon use of a step-wise advance in computing trajectories. Short-arc methods have always been intrinsically susceptible of indefinitely great accuracy in producing solutions of somewhat arbitrarily assigned forms of the differential equations of motion of projectiles, assuming that a sufficient number of figures were employed in each step of the computation and that the steps were divided at sufficiently narrow intervals in the chosen independent variable. The short-arc procedures have been subclassified into two groups described as mean-value and expansion methods. Mean-value methods, employing appropriate averages of functions of coordinates on successive arcs of the trajectory, were first used in the cosine-average method of Euler for solving the equations of motion of projectiles. Expansion methods, employing various types of developments about the values of coordinates at the beginning of successive arcs, were related to the Taylor series method suggested by Euler for computing orbits in celestial mechanics.

Recent methods of computing trajectories by using the theorem of mean value over short arcs began with the work of F. Gossot in France about 1880. Later French ballisticians developed many methods similar to the procedure originally suggested by Gossot:

this work was summarized in 1921 in the Traité de Ballistique Extérieure of P. Charbonnier, Engineer General of French Artillery. Methods of this type were employed in the computation of the French Commission Artillerie Lourde sur Voie Ferrée, or A. L. V. F., ballistic tables for long-range artillery published about 1923. The accuracy of the mean-value short-arc methods depended largely upon the width of the interval employed, assuming that a sufficiently large number of significant figures could be used at each step of the computation. However, the comparative convenience in computation of the various mean-value methods was greatly influenced by the dependent and independent variables chosen for the procedure. Greater lengths of interval for the same accuracy and convenience were obtained by advantageous choices among a wide variety of dependent and independent variables. An improved mean-value method has sometimes been found by considering functions of independent variables which vary slowly along the trajectory: slowly varying functions were readily estimated by extrapolation in advance of computation at each step of the trajectory. Mean-value short-arc devices have also been used with slowly varying dependent variables in order to facilitate expansion short-arc methods. The device of choosing an appropriate dependent variable from a property of slow variation in an independent variable was utilized in A. A. Bennett's Tangent-Reciprocal Method. Bennett's procedure was described in 1919 by Dunham Jackson in The Method of Numerical Integration in Exterior Ballistics. The Tangent-Reciprocal Method was arranged for rapid and convenient computation by the finite difference expansion procedure first employed in numerical integration of the equations of motion of projectiles by F. R. Moulton. Two devices derived from the mean-value principle were used in the primarily expansion method of Charles B. Morrey. Morrey used his method during the Second World War for rapid and convenient computation of trajectories, employing wide intervals in the range, the independent variable.

Finite difference developments generally replaced power series in expansion methods for numerical integration of differential equations in physics during the latter part of the nineteenth century. Laplace gave the finite difference formulas which have been generally used in the United States since the First World War for step-wise numerical integration of the normal equations of motion of projectiles. Although Bashforth and Adams employed many techniques of the calculus of finite differences during the nineteenth century, the general introduction of finite differences into computation developed during the First World War with the work of Karl Pearson in England and F. R. Moulton in the United States. Pearson introduced finite difference methods in numerical interpolation and graduation in the preparation of firing tables for British ordnance. Moulton developed the finite difference method of numerical integration of differential equations which has since been employed for computing trajectories in the United States. Moulton's method, as used in ballistics, provided that the numerical solution satisfy the differential equations at each stage of the computation. Moulton's method differed in this respect from the older finite difference procedure for solving differential equations which had been given by J. C. Adams. Adams' method employed only expansions in extrapolation and checked only by Gregory's or Laplace's quadrature rules expressed in finite differences.

Although the finite difference method of numerical integration of differential equations was susceptible of all accuracy required for purposes of exterior ballistics, it was somewhat laborious for hand computation even when performed with the aid of an electric computing machine. A rapid method of computing trajectories, including observed departures from standard conditions, was initiated with the application of the differential analyzer to the solution of the equations of motion of projectiles by L. S. Dederick beginning about 1926. All trajectory computations for artillery fire and many for aircraft and ground gunfire from small arms made between 1926 and 1948 by the Ordnance Department of the United States Army have been performed by the differential analyzer.

It has some times appeared desirable, at least in France, Great Britain, and the United States, to construct general ballistic tables similar to the A. L. V. F. tables published by the French Heavy Railway Artillery Commission and the Exterior Ballistic Tables Computed by Numerical Integration published by the United States Army Ordnance Department. The differential effects on trajectory elements resulting from perturbing conditions could not always be determined by varying the initial conditions and ballistic coefficient which form the arguments of ballistic tables. A method became necessary for computing these small effects directly rather than by using the laborious process of separately integrating differential equations including terms due to each distinct type of departure from standard conditions. The first direct and general procedure for determining differential effects was the method of variation of coordinates given by Moulton in his New Methods in Exterior Ballistics of 1926. Moulton, using the time as independent variable and assuming that the temperature of the air was constant on the trajectory, derived a sixth-order set of differential equations to govern the variation of coordinates. The second-order system in the deflection was separable from the fourth-order system governing the variations in range and altitude. Moulton obtained a solution of the deflection equations by a reduction to quadratures. He also applied the theory of the fundamental set of solutions of a group of linear differential equations in arranging a procedure for numerical solution of his fourth-order system of equations of variation in range and altitude. A very convenient and direct method of computing differential effects was developed about 1919 by G. A. Bliss in his study of the adjoint system of equations of variation. Bliss, assuming a constant temperature of the air and using time as the independent variable found a fourth-order system of linear differential equations with a matrix of coefficients adjoint to that of the equations of variation of coordinates given by Moulton. Bliss' dependent variables were the variations of the trajectory element induced at the end of the trajectory divided by variations in the four coordinates and components of velocity, assumed at successive instants in time along the trajectory. All differential effects of general interest in the ballistics of a particle were readily computed after a solution of the adjoint equations had been obtained by numerical integration. Bliss found one integral of the adjoint system, thus reducing the equations which had to be solved numerically to a system of the third order. A second integral was found by T. H. Gronwall in 1919, thus reducing the procedure for determining the differential effects on an assigned element of the trajectory to the numerical solution of a single differential equation of the second order. About 1930, Dederick, using range as independent variable, derived a fourth-order system of adjoint equations, of which he also found two integrals. During the latter part of the Second World War, a system of normal equations of motion became generally used which assumed an exponential law for the diminution of the relative temperature of the air with altitude above sea-level. The adjoint equations given in Chapter VIII of this book were derived for variations from normal trajectories computed with this assigned law of temperature. These adjoint equations, with time as the independent variable, were not so readily solved as the set derived by Bliss, who had used the assumption of a constant temperature of the air in the basic normal equations of motion. However, by using slope as independent variable and the properties of the fundamental set of solutions of a system of linear differential equations, the problem has been reduced to the numerical solution of a simple second-order system of differential equations followed by two quadratures.

Unless a cannon ball spun extremely rapidly, the only appreciable aerodynamic force acting upon it was an excessive drag which was regarded as that of a particle. The first need for treating the projectile as a body subject to aerodynamic forces other than the drag arose with the employment of rifled artillery in the latter part of the nineteenth century. Some information on the character of the general aerodynamic force system acting on a projectile had long been available — physicists from the time of da Vinci had recognized the aerodynamic lift and overturning moment on elongated yawing bodies moving in fluids. The dependence of the lift and overturning moment on the square of the velocity and on the density of the air was known in the latter part of the eighteenth century. The Magnus force acting to swerve a spinning cannon ball from the path of a particle acted upon only by drag and gravity was recognized early in the nineteenth century. However, aerodynamic forces other than the drag were first considered in ballistics with the advent of spinning shell in gunnery.

Two observed phenomena of the flight of spinning shell were of great interest to the artillerists of the late nineteenth and early twentieth centuries. The first of these phenomena was the tendency of projectiles of some designs with some twists of rifling and some muzzle velocities to tumble in flight. Tumbling shell developed huge drags which greatly shortened their ranges. Some other shell "trailed" on their trajectories in a satisfactory manner and attained consistent ranges. Still other shell tended to maintain their axes of figure parallel to the initial direction of flight. Artillerists began to describe projectiles which tumbled as unstable, those which trailed satisfactorily as stable and those which maintained their axes parallel to the line of departure as superstable. The principal reason for this variation in the behavior of spinning shell was found in the relation now described as the first stability condition. The first stability condition involved the aerodynamic overturning moment, the rate of spin, the air speed and the moments of inertia of projectiles. This relation was initially found by considerations analogous to those employed in the theory of tops: it was probably known to George Greenhill in England and Karl Cranz in Germany early in the twentieth century. The stability of shell was largely assured by experimental firings under varying conditions. About 1920 R. H. Kent began to employ a visual demonstration which illustrated instability, stability and superstability of projectiles under different conditions.

The second phenomenon which puzzled the artillerists of the late nineteenth century was the right deflection of a shell fired from a gun with a right-hand twist of rifling. One term in the deflection due to the rotation of the earth was known to be a right deviation from the line of departure for firing in the northern hemisphere. However the observed right deflection of a shell with right-hand spin was usually too large to have been caused by the deviation resulting from the right-hand term in the Coriolis acceleration. Artillerists also discovered that the deflection of a shell fired with a left-hand spin was to the left of the line of fire. Thus a deflection of spinning shell was shown to be caused by an aerodynamic effect of spin. This effect was found to be independent of other known perturbing, or "abnormal," conditions, such as the cross wind and Coriolis acceleration, which also affected deflection. The deflection effect dependent upon the spin was described as the drift. The side-jump was measured systematically from about this time and it was presently shown that the drift was not caused by side-jump. The observed sign of the drift was found to be opposite to that which would result from the Magnus force on a spinning shell. The drift was known to depend upon the curvature of the trajectory and its cause was known to some ballisticians before the First World War. The drift has usually been inferred from observed deflections reduced for the effects of cant, side-jump, cross wind and rotation of the earth. The drift was first treated theoretically by Fowler, Gallop, Lock and Richmond about 1919. If all the aerodynamic coefficients of the shell were known, the drift could be predicted in advance of range firings by methods given in Chapter XII of this book.

The character of the general aerodynamic force system which acts on the projectile first became recognized with the advent of aircraft. Wind tunnel measurements of the drag, lift and overturning moment acting on aircraft were intensively examined from the beginning of the twentieth century. These forces and torques were considered by the English astronomer, G. H. Bryan, in his early work on the aerodynamics of aircraft. The drag, lift, overturning moment, damping moment and spin-retarding moment appeared in the equations of motion given by Fowler, Gallop, Lock and Richmond in their important paper "The Aerodynamics of a Spinning Shell" published in 1920. Fowler and his associates found solutions of their equations which hold for short distances along the trajectory. Fowler explained the right-hand drift of a right-hand spinning projectile as a consequence of the aerodynamic lift resulting from the average right-hand pointing of the projectile during flight. He analyzed experimental firings from which he made some predictions on the magnitude of various aerodynamic forces and torques acting on the projectile during flight. Solutions of Fowler's equations applicable to many special cases of projectile motion were obtained by Kent and Hitchcock in the period between the First and Second World Wars. Kent introduced dimensionless aerodynamic coefficients denoted by the symbol K with subscripts indicating the particular forces and torques involved. Kent's coefficients were similar in form to those employed by Prandtl and others in the treatment of aerodynamics of aircraft.

Effective use of wind tunnel measurements of aerodynamic forces and torques was made in planning the design of bombs somewhat earlier than in considerations on the design of shell. American bombs of the First

World War were frequently unstable, partly because the leading edges of the fins were placed far forward on the body of the bomb. E. J. Loring and H. L. Dryden advanced, as an indication of static stability in bomb design, the criterion that the center of pressure of a bomb in a position of yaw in a steady air stream must lie behind the center of mass. Dryden made many measurements of the drag, lift and restoring moments acting on bombs of various shapes. He used considerations based upon his experimental work to determine the shape and mechanical characteristics which were employed in the design of American bombs during the Second World War.

During the period immediately following the First World War, Loring noted that stable bombs dropped at low altitudes exceeded the ranges which would have been expected in vacuo. Loring advanced the tentative explanation that this curious effect was in some fashion a consequence of the aerodynamic lift acting on the projectile. He suggested that the lift acted in such a manner as to decrease the actual drop of the bomb from that which would be expected for a normal time of flight. About 1938, it was learned from the experiments of Colonel H. H. Zornig and others that the measured times of flight of stable bombs dropped from low altitudes were, in fact, longer than would be inferred from the action of aerodynamic drag alone. R. H. Kent explained the "kiting effect" on the bomb as a consequence of the aerodynamic lift due to the fact that the average position of the axis of the bomb during flight was above the tangent to the trajectory. H. P. Hitchcock and others found some particular solutions of the equations governing this effect and established some of its properties. A more exact explanation and a convenient method of computing this deviation of the bomb from normal motion were devised in 1942 and are given in Chapter XI of this book. This method used the concepts of steady lift and oscillatory swerve. The steady lift due to mean yaw resulting from the curvature of the trajectory was separated from the oscillatory swerve due to sinusoidal yaw. This theory of the planar yaw of bombs was extended in 1943 to treat the effects of the general yaw of spinning projectiles. The extension led to a convenient method of computing the horizontal drift of an artillery shell. The theory has also predicted a vertical drift for a spinning shell. The existence of a vertical drift was apparently not suspected prior to 1943, but it has since received some experimental confirmation.

During 1942, John L. Synge introduced into ballistics a complete aerodynamic force system for projectiles with small angles of yaw and small angular velocities. The angular changes were computed with respect to a system of axes fixed in the projectile which may be regarded for a short time as parallel to axes fixed in space. Synge's work began with the recognition that an aerodynamic force which had not been noted by earlier ballisticians would necessarily act on the projectile if the damping torque, denoted by Fowler as H, existed. The required force, which has been described as the pitching force, was subsequently measured experimentally. A. C. Charters and one of the writers determined the pitching force on several bomb models mounted on an oscillating device in the wind tunnel at Wright Field. These measurements were later extended by G. B. Shubauer, who used the wind tunnel of the United States Bureau of Standards. The spin-retarding torque earlier remarked by Fowler was also accurately measured first by experiments performed in the United States during the Second World War. Thomas D. Carr, about 1942, found an accurate method of determining the loss of spin of a projectile in flight. Carr's measurements of loss of spin were used by Kent to compute the spin-retarding torque coefficients of projectiles. Kent and Charters derived a relation between the spin-retarding torque and the part of the drag due to skin-friction. Since the head drag of the projectile could be computed directly and the total drag determined accurately from data secured from the aerodynamic spark range developed by Charters, Kent was able to deduce the base drag of the projectile by subtracting the computed head drag and inferred skin drag from the measured total drag. The reduction of spark range data, described in Chapter XIII of this book, has made it possible to infer all the significant aerodynamic forces and torques acting on the projectile from measurements of experimental firings. Some of the aerodynamic coefficients computed from spark range experiments have been compared with the values determined in the supersonic wind tunnel of the Ballistic Research Laboratory. The stability and flight characteristics of projectiles can be predicted by these two methods from model experiments made in advance of range firings.

The motion of the rocket has been examined by experimental methods which were somewhat different from those used in measurements on the flight of the bomb or the shell. Experimental and theoretical work on rocketry in the Ballistic Research Laboratory during the Second World War was actively directed by the astronomer, Edwin Hubble. Hubble and his assistants, notably Dirk Reuyl and Marvin Cobb, developed very accurate methods of measuring position and velocity of rockets by high-speed photography. Radar measurements of the position of several rockets throughout complete trajectories were secured by L. A. Delsasso. Velocities were also obtained for some rockets by the sky screen method originally discovered in Canada. Mathematicians working under Hubble's direction, especially A. P. Morse, J. W. Green, A. S. Peters, P. A. White, H. L. Meyer, J. V. Lewis, C. John and the writers devised procedures for reducing these measurements and computing firing tables for various new types of rockets as they were developed. Most of the important military rockets of the first part of the Second World War were propelled by gas jets derived from the combustion of solid fuels. Although successful liquid-fuelled jet-propelled aircraft were first made by Frank Whittle in England, the first effective liquid-fuelled military rockets were built in Germany. The British jet-propelled airplanes and the German very long-range missiles of 1944 and 1945 were products of researches which began early in the twentieth century.

A very long-range rocket must arrive at a trajectory inclination of approximately forty-five degrees with a high velocity after it has passed through that part of the earth's atmosphere which has any appreciable density. Thus, the axis of the Vergeltungswaffen 2, the V-2, was gyroscopically controlled during the burning time in order to provide for a fuel cut-off when the trajectory arrived at an angle of inclination of forty-five degrees, the elevation required for the attainment of the maximum range in vacuo. The V-2 rockets fired by the Germans were powered by continuously operating jets with approximately sixty seconds burning time and attained maximum velocities of about five thousand feet per second. These velocities were made possible by researches on jet-propulsion with liquid fuels. Robert H. Goddard, an American physicist, had performed the first important experiments with rockets propelled by the liquid fuels earlier suggested by the Russian engineer, Ziolkovsky, who had been encouraged by the chemist Mendeleyeev. Some of Goddard's researches were described in 1919 in a Smithsonian report entitled A Method of Reaching Extreme Altitudes. Goddard's work became known to European physicists shortly after the First World War. Some of these men noted that the rate of burning of a liquid fuel could be more readily controlled and varied in an engine than the burning of a solid propellant. Experiments with liquid fuel and an oxidizer projected from tanks into a firing chamber resulted in low pressures and small heat developed by combustion. These experiments showed that the walls of the combustion chamber and its auxiliary tanks could be made thinner if liquid fuels were employed in rockets than if solid fuels were used. This information indicated the possibility of using steadily proceeding controlled burning in a light casing with low combustion heating.

Some development of Goddard's ideas appeared in the work of Herman Oberth, a professor of mathematics in Transylvania who wrote an important book on rocket motion which was published in 1923. This book, which was entitled Die Rakete zu Planeten Raumen, was a treatise on the theory of rocketry as applied to interplanetary travel. Oberth considered various navigational problems of travel in free space, but his most important contributions to rocket development were theoretical arguments on the proper direction for experimental work in order to attain high velocities. The basis of these arguments was a relation between the initial mass of the rocket, hereafter denoted by m_o, the mass of the rocket after all the fuel has been burned, m_b, the exhaust velocity of the jet relative to the rocket, q, and the velocity of the rocket after all the fuel has been burned, V_b. The rocket was assumed to begin its motion from rest and to move thereafter in a space free of matter, that is, without being subject to aerodynamic or gravitational forces. It was not difficult to show that

$$V_b = q \log_e (m_o/m_b).$$

This relation showed that a rocket which was to have a high velocity after burning must have large values of the exhaust velocity of the jet, q, and the mass ratio (m_o/m_b). The best liquid fuels are theoretically susceptible of a higher exhaust velocity than most solid fuels. Oberth considered various types of propellants for use in rockets and suggested alcohol and liquid hydrogen as fuels and liquid oxygen as an oxidizing agent. These suggestions contributed to the development of the V-2: the fuel actually employed by the V-2 was a modification of that originally proposed by Oberth. Willy Ley, about 1931, suggested a fuel composed of ethyl alcohol and water in the proportion of three parts of alcohol to one part of water. An important advantage of the watered alcohol and liquid oxygen mixture was that the exhaust gases contained molecules of water vapor which have a low molecular mass. The highest attainable exhaust velocity would result from a mixture of atomic hydrogen and liquid oxygen. This fact was responsible for the interest of American rocket engineers in recent experiments with jets derived from the combustion of atomic hydrogen. Oberth's velocity relation also showed the importance of increasing the mass ratio, (m_o/m_b), if the same exhaust velocity, q, could be obtained with a given rocket. The mass ratio could be increased for a fixed payload if the internal pressure and combustion heating could be decreased. The motor of the V-2 was force-fed by centrifugal pumps. Its internal pressure was very low. The watered alcohol employed in the V-2 served as a coolant before it was used as a fuel. The fuel was injected into the combustion chamber through five holes in the sheet-steel walls of the motor. Direct introduction of the mixture from the cooling jackets protected the walls of the combustion chamber from excessive local heating. These features of the design made it possible to build the V-2 with a mass ratio of three, the initial mass being about twelve tons. Oberth's relation showed that, in the absence of drag and gravity, the final velocity of the V-2 should be slightly greater than the exhaust velocity. Post-war American experiments with nearly vertical fire have yielded maximum ordinates of nearly one hundred and twenty miles. These altitudes would correspond to ranges of about two hundred and forty miles for fire at an elevation of forty-five degrees. The jet velocity, q, might possibly be doubled for the V-2, but this would yield a maximum range only about four times that attainable with the existing rocket. The nature of the materials from which the V-2 was constructed makes it seem unlikely that a similar rocket could be built with a mass ratio greater than about six, thus an ultimate range of less than four thousand miles may be expected with a V-2 type of rocket having optimum characteristics for very long-range fire.

The simplest principle available for achievement of intercontinental ranges is the multi-stage, or step, procedure. This employs a primary rocket with a high mass ratio as the initial driving engine for a smaller rocket. The reaction of the motor of the smaller rocket is initiated after the case of the primary has been cast off following exhaustion of its fuel. The step principle has been utilized in a two-stage combination of the V-2 and the Wac Corporal which attained an altitude of approximately two hundred and fifty miles when fired nearly vertically at White Sands Proving Ground on February 24, 1949. This combination would have achieved a range of about five hundred miles had the inclination of the trajectory of the second rocket been approximately forty-five degrees at the time when its fuel was exhausted. An intercontinental rocket will be obtained if a two-stage rocket, each stage having a mass ratio of about six and an exhaust velocity of eight to ten thousand feet per second, can be constructed. Such a two-stage rocket would be a practically possible development even at the present time.

The theory of motion of an intercontinental rocket will require treatment as if its flight were that of a small satellite of the earth. The number of stages of a step rocket is, however, limited only by the mechanical feasibility of constructing sufficiently large motors. A three-stage rocket of the same general characteristics as those indicated for the foregoing two-stage rocket could readily attain the velocity required to circle the earth indefinitely as an artificial satellite. In order to achieve a rocket which would depart indefinitely from the earth, it would be necessary that the rocket attain the so-called escape velocity which has a value of about thirty-five thousand feet per second. The escape velocity from the earth could be achieved by a four-stage rocket having the exhaust velocity and successive mass ratio indicated above. It seems probable that exterior ballisticians will presently be concerned with problems which were formerly considered only by astronomers.

TABLES

S- and T-functions based on Gâvre drag function

$$S = \int_U^{3600} (1/G_1)\, dU \qquad T = \int_U^{3600} (1/UG_1)\, dU$$

U	S	T	U	S	T
f/s	ft	sec	f/s	ft	sec
3600	0.00	.000	3350	687.97	.189
3590	26.72	.007	3340	716.37	.208
3580	53.50	.015	3330	744.85	.217
3570	80.34	.023	3320	773.39	.226
3560	107.26	.031	3310	802.01	.235
3550	134.24	.039	3300	830.69	.244
3540	161.28	.047	3290	859.45	.252
3530	188.40	.055	3280	888.27	.261
3520	215.58	.063	3270	917.17	.270
3510	242.82	.071	3260	946.13	.279
3500	270.14	.079	3250	975.16	.288
3490	297.52	.087	3240	1004.26	.297
3480	324.96	.095	3230	1033.42	.306
3470	352.48	.103	3220	1062.66	.315
3460	380.06	.111	3210	1091.97	.324
3450	407.71	.119	3200	1121.35	.333
3440	435.43	.127	3190	1150.81	.342
3430	463.21	.135	3180	1180.33	.351
3420	491.07	.143	3170	1209.93	.360
3410	518.99	.151	3160	1239.59	.369
3400	546.99	.159	3150	1269.33	.378
3390	575.05	.167	3140	1299.13	.388
3380	603.17	.175	3130	1329.01	.398
3370	631.37	.183	3120	1358.95	.408
3360	659.63	.191	3110	1388.97	.418
3350	687.97	.189	3100	1419.05	.428

S- and T-functions based on Gâvre drag function

$$S = \int_U^{3600} (1/G_1)\, dU \qquad T = \int_U^{3600} (1/UG_1)\, dU$$

U	S	T	U	S	T
f/s	ft	sec	f/s	ft	sec
3100	1419.05	.428	2850	2194.56	.688
3090	1449.21	.438	2840	2226.53	.699
3080	1479.43	.448	2830	2258.56	.710
3070	1509.73	.458	2820	2290.67	.721
3060	1540.09	.468	2810	2322.85	.733
3050	1570.53	.478	2800	2355.11	.745
3040	1601.03	.488	2790	2387.44	.756
3030	1631.61	.498	2780	2419.85	.768
3020	1662.27	.508	2770	2452.32	.780
3010	1693.01	.518	2760	2484.87	.792
3000	1723.81	.528	2750	2517.49	.804
2990	1754.69	.538	2740	2550.19	.816
2980	1785.63	.548	2730	2582.97	.828
2970	1816.65	.558	2720	2615.82	.840
2960	1847.75	.568	2710	2648.75	.852
2950	1878.92	.578	2700	2681.74	.864
2940	1910.16	.589	2690	2714.81	.876
2930	1941.47	.600	2680	2747.96	.888
2920	1972.85	.611	2670	2781.18	.900
2910	2004.30	.622	2660	2814.48	.912
2900	2035.83	.633	2650	2847.86	.925
2890	2067.44	.644	2640	2881.31	.938
2880	2099.11	.655	2630	2914.83	.951
2870	2130.86	.666	2620	2948.43	.964
2860	2162.67	.677	2610	2982.11	.977
2850	2194.56	.688	2600	3015.86	.990

S- and T-functions based on Gâvre drag function

$$S = \int_U^{3600} (1/G_1)\, dU \qquad T = \int_U^{3600} (1/UG_1)\, dU$$

U f/s	S ft	T sec	U f/s	S ft	T sec
2600	3015.86	.990	2350	3884.91	1.347
2590	3049.69	1.003	2340	3920.72	1.362
2580	3083.59	1.016	2330	3956.61	1.377
2570	3117.57	1.029	2320	3992.59	1.392
2560	3151.63	1.042	2310	4028.66	1.408
2550	3185.76	1.055	2300	4064.82	1.424
2540	3219.97	1.068	2290	4101.06	1.440
2530	3254.26	1.082	2280	4137.39	1.456
2520	3288.63	1.096	2270	4173.80	1.472
2510	3323.07	1.110	2260	4210.30	1.488
2500	3357.59	1.124	2250	4246.89	1.504
2490	3392.19	1.138	2240	4283.56	1.520
2480	3426.86	1.152	2230	4320.33	1.536
2470	3461.61	1.167	2220	4357.18	1.553
2460	3496.44	1.182	2210	4394.13	1.570
2450	3531.35	1.197	2200	4431.16	1.587
2440	3566.34	1.212	2190	4468.29	1.604
2430	3601.41	1.227	2180	4505.51	1.621
2420	3636.56	1.242	2170	4542.82	1.638
2410	3671.79	1.257	2160	4580.23	1.655
2400	3707.11	1.272	2150	4617.74	1.672
2390	3742.50	1.287	2140	4655.34	1.690
2380	3777.98	1.302	2130	4693.04	1.708
2370	3813.54	1.317	2120	4730.84	1.726
2360	3849.18	1.332	2110	4768.74	1.744
2350	3884.91	1.347	2100	4806.76	1.762

S- and T-functions based on Gâvre drag function

$$S = \int_{U}^{3600} (1/G_1)\, dU \qquad T = \int_{U}^{3600} (1/UG_1)\, dU$$

U	S	T	U	S	T
f/s	ft	sec	f/s	ft	sec
2100	4806.76	1.762	1850	5794.62	2.263
2090	4844.88	1.780	1840	5835.85	2.285
2080	4883.10	1.798	1830	5877.23	2.308
2070	4921.42	1.817	1820	5918.76	2.331
2060	4959.86	1.836	1810	5960.45	2.354
2050	4998.40	1.855	1800	6002.31	2.377
2040	5037.06	1.874	1790	6044.33	2.401
2030	5075.82	1.893	1780	6086.51	2.425
2020	5114.70	1.912	1770	6128.87	2.449
2010	5153.68	1.931	1760	6171.40	2.473
2000	5192.78	1.950	1750	6214.11	2.497
1990	5232.01	1.970	1740	6257.01	2.522
1980	5271.36	1.990	1730	6300.10	2.547
1970	5310.83	2.010	1720	6343.37	2.572
1960	5350.42	2.030	1710	6386.84	2.597
1950	5390.13	2.050	1700	6430.52	2.622
1940	5429.98	2.070	1690	6474.39	2.648
1930	5469.95	2.091	1680	6518.48	2.674
1920	5510.06	2.112	1670	6562.78	2.701
1910	5550.30	2.133	1660	6607.30	2.728
1900	5590.67	2.154	1650	6652.06	2.755
1890	5631.18	2.175	1640	6697.04	2.782
1880	5671.83	2.197	1630	6742.26	2.810
1870	5712.62	2.219	1620	6787.73	2.838
1860	5753.54	2.241	1610	6833.45	2.866
1850	5794.62	2.263	1600	6879.41	2.894

S- and T-functions based on Gâvre drag function

$$S = \int_U^{3600} (1/G_1)\, dU \qquad T = \int_U^{3600} (1/UG_1)\, dU$$

U f/s	S ft	T sec	U f/s	S ft	T sec
1600	6879.41	2.894	1350	8141.56	3.754
1590	6925.65	2.923	1340	8198.32	3.796
1580	6972.15	2.952	1330	8255.77	3.839
1570	7018.93	2.982	1320	8313.95	3.883
1560	7066.01	3.012	1310	8372.88	3.928
1550	7113.39	3.042	1300	8432.61	3.974
1540	7161.08	3.073	1290	8493.20	4.021
1530	7209.08	3.104	1280	8554.69	4.069
1520	7257.39	3.136	1270	8617.14	4.118
1510	7306.03	3.168	1260	8680.62	4.168
1500	7355.03	3.201	1250	8745.18	4.219
1490	7404.39	3.235	1240	8810.9	4.272
1480	7454.13	3.269	1230	8877.9	4.326
1470	7504.25	3.303	1220	8946.2	4.382
1460	7554.77	3.338	1210	9015.9	4.439
1450	7605.69	3.373	1200	9087.2	4.499
1440	7657.03	3.408	1190	9160.2	4.560
1430	7708.82	3.444	1180	9235.0	4.623
1420	7761.07	3.480	1170	9311.7	4.688
1410	7813.80	3.517	1160	9390.6	4.756
1400	7867.04	3.555	1150	9471.7	4.826
1390	7920.81	3.594	1140	9555.3	4.899
1380	7975.12	3.633	1130	9641.6	4.975
1370	8029.99	3.673	1120	9730.9	5.054
1360	8085.46	3.713	1110	9823.2	5.137
1350	8141.56	3.754	1100	9919.0	5.224

S- and T-functions based on Gâvre drag function

$$S = \int_{U}^{3600} (1/G_1)\ dU \qquad T = \int_{U}^{3600} (1/UG_1)\ dU$$

U	S	T	U	S	T
f/s	ft	sec	f/s	ft	sec
1100	9919.0	5.224	850	14044.8	9.563
1090	10018.4	5.315	840	14292.8	9.856
1080	10121.8	5.410	830	14547.3	10.161
1070	10229.3	5.510	820	14808.4	10.478
1060	10341.3	5.615	810	15076.0	10.806
1050	10458.0	5.726	800	15350.2	11.147
1040	10579.6	5.842	790	15630.8	11.500
1030	10706.4	5.965	780	15917.8	11.865
1020	10838.7	6.094	770	16211.1	12.244
1010	10976.6	6.230	760	16510.7	12.635
1000	11120.3	6.373	750	16816.7	13.041
990	11269.9	6.523	740	17129.0	13.460
980	11425.8	6.681	730	17447.3	13.893
970	11587.8	6.847	720	17771.9	14.341
960	11756.2	7.022	710	18102.8	14.804
950	11931.1	7.205	700	18439.8	15.282
940	12112.6	7.397	690	18782.9	15.775
930	12300.6	7.598	680	19132.0	16.285
920	12495.2	7.809	670	19487.4	16.812
910	12696.5	8.029	660	19849.0	17.356
900	12904.6	8.259	650	20216.7	17.917
890	13119.3	8.498	640	20590.7	18.497
880	13340.6	8.749	630	20970.9	19.096
870	13568.7	9.009	620	21357.5	19.714
860	13803.5	9.281	610	21750.2	20.353
850	14044.8	9.563	600	22149.3	21.013

S- and T-functions based on Gâvre drag function

$$S = \int_U^{3600} (1/G_1)\, dU \qquad T = \int_U^{3600} (1/UG_1)\, dU$$

U	S	T
f/s	ft	sec
600	22149.3	21.013
590	22554.8	21.694
580	22966.9	22.398
570	23385.6	23.127
560	23811.0	23.880
550	24243.4	24.659
540	24682.7	25.465
530	25129.3	26.300
520	25583.0	27.164
510	26044.2	28.060
500	26513.3	28.989

ϕ_2-function for approximating drop

r	$\phi_2(r)$	Δ'	$\frac{1}{2}g\phi_2(r)$	Δ'
0.75	0.91086		14.643	
.74	.90702	-384	14.581	-62
.73	.90316	-386	14.519	-62
.72	.89928	-388	14.457	-62
.71	.89537	-391	14.394	-63
.70	.89144	-393	14.331	-63
.69	.88748	-396	14.267	-64
.68	.88350	-398	14.203	-64
.67	.87949	-401	14.139	-64
.66	.87545	-404	14.074	-65
.65	.87138	-407	14.008	-66
.64	.86729	-409	13.943	-65
.63	.86317	-412	13.876	-67
.62	.85902	-415	13.810	-66
.61	.85484	-418	13.742	-68
.60	.85064	-420	13.675	-67
.59	.84641	-423	13.607	-68
.58	.84214	-427	13.538	-69
.57	.83785	-429	13.469	-69
.56	.83352	-433	13.400	-69
.55	.82916	-436	13.330	-70
.54	.82477	-439	13.259	-71
.53	.82034	-443	13.188	-71
.52	.81588	-446	13.116	-72
.51	.81138	-450	13.044	-72
0.50	0.80685	-453	12.971	-73

ϕ_2-function for approximating drop

r	$\phi_2(r)$	Δ'	$\frac{1}{2}g\phi_2(r)$	Δ'
1.00	1.00000		16.076	
0.99	0.99666	-334	16.022	-54
.98	.99330	-336	15.968	-54
.97	.98992	-338	15.914	-54
.96	.98653	-339	15.860	-54
.95	.98312	-341	15.805	-55
.94	.97969	-343	15.750	-55
.93	.97624	-345	15.694	-56
.92	.97278	-346	15.638	-56
.91	.96930	-348	15.583	-55
.90	.96580	-350	15.526	-57
.89	.96228	-352	15.470	-56
.88	.95874	-354	15.413	-57
.87	.95518	-356	15.356	-57
.86	.95160	-358	15.298	-58
.85	.94800	-360	15.240	-58
.84	.94438	-362	15.182	-58
.83	.94074	-364	15.123	-59
.82	.93708	-366	15.065	-58
.81	.93340	-368	15.005	-60
.80	.92970	-370	14.946	-59
.79	.92598	-372	14.886	-60
.78	.92223	-375	14.826	-60
.77	.91846	-377	14.765	-61
.76	.91467	-379	14.704	-61
0.75	0.91086	-381	14.643	-61

G-function based on Gâvre drag function

v²/100	0	1	2	3	4	5	6	7	8	9
0	0.00000	0174	0244	0297	0342	0381	0416	0447	0477	0504
10	.00530	0555	0578	0600	0621	0642	0661	0680	0699	0716
20	.00734	0750	0767	0783	0798	0813	0828	0842	0856	0870
30	.00883	0897	0910	0922	0935	0947	0959	0971	0983	0994
40	.01006	1017	1028	1039	1049	1060	1070	1080	1090	1100
50	.01110	1120	1130	1139	1148	1158	1167	1176	1185	1193
60	.01202	1211	1220	1228	1236	1245	1253	1261	1269	1277
70	.01285	1293	1301	1308	1316	1323	1331	1338	1346	1353
80	.01360	1367	1375	1382	1389	1396	1402	1409	1416	1423
90	.01430	1436	1443	1449	1456	1463	1468	1475	1482	1488
100	.01495	1501	1507	1513	1519	1525	1531	1537	1543	1549
110	.01555	1561	1567	1573	1578	1584	1590	1596	1601	1607
120	.01612	1618	1623	1629	1634	1639	1645	1650	1655	1661
130	.01666	1671	1676	1682	1687	1692	1697	1702	1707	1712
140	.01717	1722	1727	1732	1737	1742	1747	1752	1757	1762
150	.01766	1771	1776	1781	1785	1790	1795	1800	1804	1809
160	.01814	1818	1823	1827	1832	1836	1841	1845	1850	1855
170	.01859	1863	1868	1872	1876	1881	1885	1890	1894	1898
180	.01903	1907	1911	1916	1920	1924	1928	1933	1937	1942
190	.01946	1950	1954	1958	1962	1967	1971	1975	1979	1983
200	.01987	1991	1995	1999	2003	2007	2011	2015	2019	2024
210	.02028	2032	2036	2040	2044	2047	2051	2055	2059	2063
220	.02067	2071	2075	2079	2083	2086	2090	2094	2098	2102
230	.02106	2110	2113	2117	2121	2125	2129	2132	2136	2140
240	.02144	2148	2151	2155	2159	2163	2167	2171	2174	2178
250	.02182	2186	2189	2193	2197	2200	2204	2208	2211	2215
260	.02219	2222	2226	2230	2233	2237	2240	2244	2248	2251
270	.02255	2258	2262	2266	2269	2273	2277	2280	2284	2288
280	.02291	2295	2298	2302	2306	2309	2313	2316	2320	2324
290	.02327	2331	2334	2338	2342	2345	2349	2352	2356	2360
300	.02363	2367	2370	2374	2377	2381	2385	2388	2392	2396
310	.02399	2403	2406	2410	2413	2417	2420	2424	2427	2431
320	.02434	2438	2442	2445	2449	2452	2456	2460	2463	2467
330	.02470	2474	2477	2481	2484	2488	2492	2495	2499	2502
340	.02506	2509	2513	2516	2520	2524	2527	2531	2535	2538
350	.02542	2545	2549	2553	2556	2560	2563	2567	2571	2574
360	.02578	2582	2585	2589	2593	2596	2600	2604	2607	2611
370	.02615	2618	2622	2625	2629	2633	2636	2640	2644	2647
380	.02651	2655	2658	2662	2666	2669	2673	2677	2681	2684
390	.02688	2692	2695	2699	2703	2706	2710	2714	2718	2721
400	.02725	2729	2733	2737	2741	2745	2748	2752	2756	2760
410	.02764	2768	2771	2775	2779	2783	2786	2790	2794	2798
420	.02802	2805	2809	2813	2817	2821	2825	2829	2833	2837
430	.02841	2844	2848	2852	2856	2860	2864	2868	2872	2876
440	.02880	2884	2888	2892	2896	2900	2904	2908	2912	2916
450	.02920	2924	2928	2932	2937	2941	2945	2950	2954	2958

G-function based on Gâvre drag function

$v^2/100$	0	1	2	3	4	5	6	7	8	9
450	0.02920	2924	2928	2932	2937	2941	2945	2950	2954	2958
460	.02962	2966	2970	2974	2979	2983	2987	2991	2995	2999
470	.03003	3007	3012	3016	3020	3024	3028	3032	3037	3041
480	.03045	3049	3053	3058	3062	3067	3071	3076	3080	3084
490	.03089	3093	3097	3102	3106	3111	3115	3120	3124	3128
500	.03133	3137	3142	3146	3151	3155	3160	3164	3169	3173
510	.03178	3183	3187	3192	3196	3201	3206	3210	3215	3220
520	.03224	3229	3234	3238	3243	3248	3252	3257	3261	3266
530	.03271	3276	3281	3286	3291	3295	3300	3305	3310	3315
540	.03320	3325	3330	3335	3340	3344	3349	3354	3359	3364
550	.03369	3374	3379	3384	3389	3394	3399	3405	3410	3415
560	.03420	3425	3430	3435	3440	3446	3451	3456	3461	3466
570	.03471	3477	3482	3487	3493	3498	3504	3509	3515	3520
580	.03525	3531	3536	3542	3547	3553	3558	3564	3569	3575
590	.03580	3586	3591	3597	3602	3608	3614	3620	3625	3631
600	.03637	3643	3648	3654	3660	3666	3671	3677	3683	3689
610	.03695	3701	3707	3713	3719	3725	3730	3736	3742	3748
620	.03754	3760	3766	3772	3778	3785	3791	3797	3804	3810
630	.03816	3822	3828	3834	3841	3847	3854	3860	3866	3873
640	.03879	3885	3892	3896	3905	3911	3918	3924	3931	3937
650	.03944	3950	3957	3963	3970	3976	3983	3990	3997	4003
660	.04010	4017	4023	4030	4037	4044	4050	4057	4064	4071
670	.04078	4086	4093	4100	4107	4114	4121	4129	4136	4143
680	.04150	4157	4164	4172	4179	4186	4193	4200	4207	4215
690	.04222	4230	4237	4244	4252	4259	4267	4275	4282	4290
700	.04298	4305	4313	4320	4328	4336	4343	4351	4359	4367
710	.04374	4382	4390	4398	4406	4414	4422	4430	4438	4446
720	.04455	4463	4471	4479	4487	4496	4504	4512	4521	4529
730	.04537	4546	4554	4562	4571	4579	4588	4596	4605	4613
740	.04622	4630	4639	4648	4656	4665	4674	4683	4691	4700
750	.04709	4717	4726	4736	4745	4754	4763	4772	4781	4790
760	.04799	4808	4818	4827	4836	4845	4855	4864	4874	4883
770	.04893	4902	4912	4922	4931	4941	4950	4960	4969	4979
780	.04988	4998	5008	5018	5028	5038	5048	5058	5068	5078
790	.05088	5098	5108	5118	5128	5139	5149	5159	5169	5180
800	.05190	5200	5210	5220	5231	5241	5252	5263	5273	5284
810	.05295	5306	5316	5327	5338	5349	5360	5372	5383	5394
820	.05405	5416	5427	5438	5449	5460	5471	5483	5494	5505
830	.05516	5527	5539	5550	5562	5573	5585	5596	5608	5619
840	.05631	5643	5655	5666	5678	5690	5702	5714	5726	5738
850	.05750	5762	5774	5786	5798	5811	5823	5835	5848	5860
860	.05872	5885	5897	5910	5922	5935	5948	5960	5973	5986
870	.05999	6012	6025	6038	6051	6064	6077	6090	6103	6116
880	.06129	6142	6155	6168	6182	6195	6208	6222	6235	6249
890	.06262	6276	6290	6304	6317	6331	6345	6358	6372	6386
900	.06399	6413	6427	6441	6455	6468	6482	6496	6510	6525

G-function based on Gâvre drag function

V²/100	0	1	2	3	4	5	6	7	8	9
900	0.06399	6413	6427	6441	6455	6468	6482	6496	6510	6525
910	.06539	6554	6568	6583	6597	6612	6627	6642	6656	6671
920	.06686	6700	6715	6729	6744	6759	6773	6788	6803	6818
930	.06833	6848	6863	6878	6893	6909	6924	6939	6955	6970
940	.06986	7001	7016	7032	7047	7063	7079	7095	7111	7127
950	.07142	7158	7173	7189	7205	7221	7236	7252	7268	7284
960	.07300	7316	7332	7348	7364	7380	7396	7413	7429	7445
970	.07461	7478	7494	7511	7527	7544	7560	7577	7593	7610
980	.07626	7643	7660	7677	7694	7711	7728	7746	7763	7780
990	.07797	7814	7831	7848	7865	7883	7900	7917	7934	7952
1000	.07969	7986	8003	8020	8037	8055	8072	8089	8107	8124
1010	.08142	8159	8177	8195	8212	8230	8248	8265	8283	8300
1020	.08318	8336	8354	8371	8389	8407	8425	8443	8461	8480
1030	.08498	8516	8534	8552	8570	8588	8607	8625	8643	8662
1040	.08680	8698	8716	8735	8753	8771	8790	8808	8827	8845
1050	.08863	8882	8900	8919	8937	8956	8974	8993	9012	9030
1060	.09049	9067	9086	9104	9123	9141	9160	9178	9197	9215
1070	.09234	9253	9271	9290	9309	9327	9346	9365	9384	9402
1080	.09421	9440	9458	9477	9495	9514	9533	9552	9570	9589
1090	.09608	9627	9645	9664	9683	9701	9720	9739	9758	9776
1100	.09795	9814	9833	9852	9871	9890	9909	9928	9947	9966
1110	.09985	0004*	0022	0041	0060	0079	0098	0116	0135	0154
1120	.10173	0192	0210	0229	0248	0267	0286	0304	0323	0342
1130	.10360	0379	0398	0417	0435	0454	0473	0492	0510	0529
1140	.10548	0566	0585	0604	0623	0641	0660	0679	0698	0716
1150	.10735	0753	0772	0790	0809	0827	0846	0864	0883	0901
1160	.10919	0938	0956	0974	0993	1011	1029	1047	1066	1084
1170	.11102	1120	1138	1157	1175	1193	1211	1230	1248	1266
1180	.11284	1302	1320	1338	1357	1375	1393	1411	1429	1447
1190	.11465	1483	1501	1519	1537	1555	1573	1591	1608	1626
1200	.11644	1661	1679	1697	1715	1732	1750	1768	1786	1803
1210	.11821	1839	1856	1874	1892	1909	1927	1945	1962	1980
1220	.11997	2014	2032	2049	2066	2083	2100	2117	2134	2151
1230	.12168	2185	2202	2219	2236	2253	2270	2287	2304	2321
1240	.12338	2355	2372	2389	2406	2423	2440	2456	2473	2490
1250	.12507	2523	2540	2557	2573	2590	2606	2622	2639	2655
1260	.12671	2688	2704	2720	2736	2753	2769	2785	2801	2817
1270	.12833	2849	2865	2881	2897	2913	2929	2945	2960	2976
1280	.12992	3008	3024	3039	3055	3071	3087	3102	3118	3134
1290	.13149	3165	3180	3196	3211	3227	3242	3258	3273	3289
1300	.13304	3319	3335	3350	3365	3380	3396	3411	3426	3441
1310	.13456	3471	3486	3501	3516	3531	3546	3561	3575	3590
1320	.13605	3620	3634	3649	3664	3679	3694	3708	3723	3738
1330	.13753	3767	3782	3797	3811	3826	3841	3855	3870	3884
1340	.13899	3913	3928	3942	3957	3971	3985	4000	4014	4028
1350	.14042	4056	4070	4084	4098	4112	4126	4140	4154	4167

G-function based on Gâvre drag function

/100	0	1	2	3	4	5	6	7	8	9
150	0.14042	4056	4070	4084	4098	4112	4126	4140	4154	4167
160	.14181	4195	4209	4222	4236	4250	4264	4277	4291	4305
170	.14318	4332	4346	4359	4373	4386	4400	4413	4426	4440
180	.14453	4466	4480	4493	4506	4520	4533	4546	4559	4573
190	.14586	4599	4612	4625	4638	4651	4664	4677	4690	4703
100	.14716	4729	4741	4754	4767	4780	4792	4805	4818	4830
110	.14843	4856	4868	4881	4893	4906	4918	4931	4943	4956
120	.14968	4981	4993	5006	5018	5031	5043	5055	5068	5080
130	.15092	5105	5117	5129	5141	5153	5166	5178	5190	5202
140	.15214	5226	5238	5250	5262	5274	5286	5298	5310	5322
150	.15334	5346	5358	5370	5381	5393	5405	5417	5428	5440
160	.15452	5464	5475	5487	5499	5510	5522	5533	5545	5556
170	.15567	5578	5590	5601	5612	5623	5634	5645	5657	5668
180	.15679	5690	5701	5713	5724	5735	5746	5757	5768	5779
190	.15790	5801	5812	5823	5834	5845	5856	5867	5878	5889
500	.15900	5911	5922	5933	5944	5955	5966	5977	5987	5998
510	.16009	6020	6030	6041	6052	6063	6073	6084	6094	6105
520	.16115	6126	6136	6147	6157	6167	6178	6188	6198	6209
530	.16219	6229	6240	6250	6260	6271	6281	6291	6302	6312
540	.16322	6332	6343	6353	6363	6373	6384	6394	6404	6414
550	.16425	6435	6445	6455	6465	6476	6486	6496	6506	6516
560	.16526	6536	6546	6556	6565	6575	6585	6595	6605	6614
570	.16624	6634	6643	6653	6663	6672	6682	6692	6701	6711
580	.16720	6730	6739	6749	6758	6768	6777	6787	6796	6806
590	.16815	6825	6834	6844	6853	6863	6872	6882	6891	6901
500	.16910	6920	6929	6938	6948	6957	6966	6976	6985	6994
510	.17003	7012	7021	7030	7039	7048	7057	7066	7075	7084
520	.17093	7102	7111	7120	7129	7138	7147	7156	7164	7173
530	.17182	7191	7200	7208	7217	7226	7235	7244	7252	7261
540	.17270	7279	7288	7296	7305	7314	7323	7332	7340	7349
550	.17358	7367	7376	7384	7393	7402	7411	7419	7428	7436
560	.17445	7454	7462	7471	7480	7488	7497	7506	7514	7523
570	.17531	7540	7548	7557	7565	7574	7582	7590	7599	7607
580	.17615	7623	7632	7640	7648	7656	7664	7672	7681	7689
590	.17697	7705	7713	7722	7730	7738	7746	7754	7763	7771
700	.17779	7787	7795	7804	7812	7820	7828	7836	7845	7853
710	.17861	7869	7877	7885	7893	7902	7910	7918	7926	7934
720	.17942	7950	7958	7966	7974	7982	7990	7998	8006	8014
730	.18022	8030	8037	8045	8053	8061	8068	8076	8084	8091
740	.18099	8107	8114	8122	8130	8137	8145	8153	8160	8168
750	.18175	8183	8190	8198	8206	8213	8221	8228	8236	8243
760	.18251	8259	8266	8274	8281	8289	8296	8304	8311	8319
770	.18326	8333	8341	8348	8355	8363	8370	8377	8385	8392
780	.18399	8406	8414	8421	8428	8435	8442	8450	8457	8464
790	.18471	8478	8486	8493	8500	8507	8514	8522	8529	8536
300	.18543	8550	8558	8565	8572	8579	8586	8594	8601	8608

G-function based on Gâvre drag function

$v^2/100$	0	1	2	3	4	5	6	7	8	9
1800	0.18543	8550	8558	8565	8572	8579	8586	8594	8601	8608
1810	.18615	8622	8629	8636	8643	8650	8657	8664	8671	8678
1820	.18685	8692	8699	8706	8713	8720	8727	8734	8740	8747
1830	.18754	8761	8768	8775	8782	8789	8796	8802	8809	8816
1840	.18823	8830	8837	8844	8851	8858	8865	8872	8879	8886
1850	.18893	8900	8907	8914	8921	8928	8934	8941	8948	8955
1860	.18962	8968	8975	8982	8989	8996	9002	9009	9016	9023
1870	.19029	9036	9042	9049	9055	9062	9068	9075	9082	9088
1880	.19095	9101	9108	9114	9121	9127	9134	9140	9147	9153
1890	.19160	9166	9173	9179	9186	9192	9199	9205	9211	9218
1900	.19224	9231	9237	9243	9250	9256	9263	9269	9275	9282
1910	.19288	9294	9301	9307	9313	9320	9326	9332	9339	9345
1920	.19351	9357	9364	9370	9376	9382	9388	9395	9401	9407
1930	.19413	9420	9426	9432	9438	9445	9451	9457	9463	9470
1940	.19476	9482	9488	9494	9501	9507	9513	9519	9525	9532
1950	.19538	9544	9550	9556	9562	9568	9574	9580	9586	9592
1960	.19598	9604	9610	9615	9621	9627	9633	9639	9644	9650
1970	.19656	9662	9668	9674	9680	9685	9691	9697	9703	9709
1980	.19715	9721	9727	9733	9738	9744	9750	9756	9762	9768
1990	.19774	9780	9785	9791	9797	9803	9809	9815	9820	9826
2000	.19832	9838	9844	9850	9855	9861	9867	9873	9879	9884
2010	.19890	9896	9902	9907	9913	9919	9924	9930	9936	9941
2020	.19947	9953	9958	9964	9970	9975	9981	9986	9992	9997
2030	.20003	0009	0014	0020	0025	0031	0037	0042	0048	0053
2040	.20059	0065	0070	0076	0081	0087	0092	0098	0103	0109
2050	.20114	0119	0125	0130	0136	0141	0147	0152	0157	0163
2060	.20168	0173	0179	0185	0191	0196	0201	0207	0212	0217
2070	.20222	0228	0233	0239	0244	0250	0255	0261	0266	0272
2080	.20277	0283	0288	0294	0299	0305	0310	0316	0321	0327
2090	.20332	0337	0343	0348	0354	0359	0365	0370	0375	0381
2100	.20386	0391	0397	0402	0408	0413	0418	0424	0429	0434
2110	.20440	0445	0450	0456	0461	0466	0472	0477	0482	0488
2120	.20493	0498	0503	0508	0514	0519	0524	0529	0534	0539
2130	.20544	0549	0554	0559	0564	0569	0574	0579	0584	0589
2140	.20594	0599	0604	0609	0614	0619	0624	0629	0634	0639
2150	.20644	0649	0654	0659	0664	0669	0674	0678	0683	0688
2160	.20693	0698	0703	0708	0713	0718	0722	0727	0732	0737
2170	.20742	0747	0752	0757	0762	0766	0771	0776	0781	0786
2180	.20791	0796	0801	0806	0811	0816	0820	0825	0830	0835
2190	.20840	0845	0850	0854	0859	0864	0869	0874	0879	0883
2200	.20888	0893	0898	0902	0907	0912	0917	0922	0926	0931
2210	.20936	0941	0945	0950	0955	0960	0964	0969	0974	0978
2220	.20983	0988	0992	0997	1002	1007	1011	1016	1021	1025
2230	.21030	1035	1039	1044	1049	1053	1058	1063	1068	1072
2240	.21077	1082	1086	1091	1096	1101	1105	1110	1115	1119
2250	.21124	1129	1134	1138	1143	1148	1153	1157	1162	1167

G-function based on Gâvre drag function

$v^2/100$	0	1	2	3	4	5	6	7	8	9
2250	0.21124	1129	1134	1138	1143	1148	1153	1157	1162	1167
2260	.21171	1176	1181	1185	1190	1195	1200	1204	1209	1214
2270	.21218	1223	1228	1232	1237	1242	1247	1251	1256	1261
2280	.21265	1270	1275	1279	1284	1289	1293	1298	1302	1307
2290	.21311	1316	1320	1325	1329	1334	1338	1342	1347	1351
2300	.21355	1360	1364	1368	1373	1377	1381	1386	1390	1395
2310	.21399	1403	1408	1412	1417	1421	1425	1430	1434	1439
2320	.21443	1447	1452	1456	1460	1465	1469	1473	1478	1482
2330	.21486	1491	1495	1499	1504	1508	1512	1517	1521	1525
2340	.21530	1534	1538	1543	1547	1551	1556	1560	1564	1569
2350	.21573	1577	1582	1586	1590	1595	1599	1603	1608	1612
2360	.21616	1620	1625	1629	1633	1637	1642	1646	1650	1654
2370	.21658	1662	1667	1671	1675	1679	1683	1687	1692	1696
2380	.21700	1704	1708	1712	1717	1721	1725	1729	1733	1737
2390	.21741	1745	1749	1753	1758	1762	1766	1770	1774	1778
2400	.21782	1786	1790	1794	1798	1802	1806	1810	1814	1818
2410	.21822	1826	1830	1834	1838	1843	1847	1851	1855	1859
2420	.21863	1867	1871	1875	1879	1883	1887	1891	1895	1899
2430	.21903	1907	1911	1915	1919	1923	1927	1931	1935	1939
2440	.21943	1947	1951	1955	1959	1963	1967	1971	1975	1979
2450	.21983	1987	1991	1995	1999	2003	2007	2011	2015	2019
2460	.22023	2027	2031	2035	2038	2042	2046	2050	2054	2058
2470	.22062	2066	2070	2074	2078	2081	2085	2089	2093	2097
2480	.22101	2105	2108	2112	2116	2120	2124	2128	2131	2135
2490	.22139	2143	2147	2150	2154	2158	2162	2166	2170	2173
2500	.22177	2181	2185	2188	2192	2196	2200	2204	2207	2211
2510	.22215	2219	2223	2226	2230	2234	2238	2242	2245	2249
2520	.22253	2257	2261	2264	2268	2272	2276	2280	2283	2287
2530	.22291	2295	2298	2302	2306	2310	2313	2317	2321	2325
2540	.22328	2332	2336	2340	2343	2347	2351	2354	2358	2362
2550	.22365	2369	2373	2376	2380	2384	2387	2391	2395	2398
2560	.22402	2406	2409	2413	2417	2420	2424	2428	2431	2435
2570	.22439	2443	2446	2450	2454	2457	2461	2465	2468	2472
2580	.22476	2479	2483	2487	2490	2494	2497	2501	2505	2508
2590	.22512	2516	2519	2523	2526	2530	2534	2537	2541	2544
2600	.22548	2552	2555	2559	2562	2566	2570	2573	2577	2580
2610	.22584	2588	2591	2595	2598	2602	2606	2609	2613	2616
2620	.22620	2623	2627	2631	2634	2638	2641	2645	2648	2652
2630	.22655	2659	2662	2666	2669	2673	2676	2680	2683	2687
2640	.22690	2694	2697	2701	2704	2708	2711	2715	2718	2722
2650	.22725	2729	2732	2736	2739	2743	2746	2750	2753	2757
2660	.22760	2763	2767	2770	2774	2777	2781	2784	2787	2791
2670	.22794	2797	2801	2304	2808	2811	2815	2818	2821	2825
2680	.22828	2831	2835	2838	2842	2845	2848	2852	2855	2859
2690	.22862	2865	2869	2872	2876	2879	2882	2886	2889	2893
2700	.22896	2899	2903	2906	2910	2913	2916	2920	2923	2927

G-function based on Gâvre drag function

$v^2/100$	0	1	2	3	4	5	6	7	8	9
2700	0.22896	2899	2903	2906	2910	2913	2916	2920	2923	292[illegible]
2710	.22930	2933	2937	2940	2943	2947	2950	2953	2957	296[illegible]
2720	.22963	2967	2970	2973	2977	2980	2983	2987	2990	299[illegible]
2730	.22996	3000	3003	3006	3010	3013	3016	3019	3023	302[illegible]
2740	.23029	3032	3036	3039	3042	3045	3049	3052	3055	305[illegible]
2750	.23062	3065	3068	3071	3075	3078	3081	3084	3088	309[illegible]
2760	.23094	3097	3101	3104	3107	3110	3113	3117	3120	3123
2770	.23126	3129	3133	3136	3139	3142	3145	3148	3152	315[illegible]
2780	.23158	3161	3164	3168	3171	3174	3177	3180	3184	31[illegible]
2790	.23190	3193	3196	3200	3203	3206	3209	3212	3216	321[illegible]
2800	.23222	3225	3228	3231	3234	3238	3241	3244	3247	325[illegible]
2810	.23253	3256	3260	3263	3266	3269	3272	3275	3278	3281
2820	.23284	3287	3290	3293	3296	3300	3303	3306	3309	3312
2830	.23315	3318	3321	3324	3327	3330	3333	3336	3340	3343
2840	.23346	3349	3352	3355	3358	3361	3364	3367	3371	337[illegible]
2850	.23377	3380	3383	3386	3389	3392	3395	3398	3402	3[illegible]
2860	.23408	3411	3414	3417	3420	3423	3426	3429	3432	34[illegible]
2870	.23438	3441	3444	3447	3450	3453	3456	3459	3462	3465
2880	.23468	3471	3474	3477	3480	3483	3486	3489	3492	3495
2890	.23498	3501	3504	3507	3510	3513	3516	3519	3522	3525
2900	.23528	3531	3534	3537	3540	3543	3546	3549	3552	3555
2910	.23558	3561	3564	3567	3570	3573	3576	3579	3582	3585
2920	.23588	3591	3594	3597	3600	3603	3606	3608	3611	3614
2930	.23617	3620	3623	3626	3629	3632	3634	3637	3640	3643
2940	.23646	3649	3652	3655	3658	3660	3663	3666	3669	3672
2950	.23675	3678	3681	3684	3687	3690	3692	3695	3698	3701
2960	.23704	3707	3710	3713	3716	3719	3722	3724	3727	3730
2970	.23733	3736	3739	3742	3745	3748	3750	3753	3756	3759
2980	.23762	3765	3768	3770	3773	3776	3779	3782	3784	3787
2990	.23790	3793	3796	3798	3801	3804	3807	3810	3812	3815
3000	.23818	3821	3824	3826	3829	3832	3835	3837	3840	3843

G-function based on Gâvre drag function

$v^2/100$	0	10	20	30	40	50	60	70	80	90
3000	0.23818	3846	3874	3902	3930	3957	3984	4011	4038	4065
3100	.24092	4119	4145	4171	4197	4223	4249	4275	4300	4325
3200	.24350	4375	4400	4425	4450	4475	4500	4525	4549	4574
3300	.24598	4623	4647	4671	4695	4719	4743	4767	4790	4814
3400	.24837	4861	4884	4907	4930	4953	4976	4999	5022	5045
3500	.25067	5090	5112	5134	5156	5178	5200	5222	5244	5266
3600	.25287	5309	5330	5352	5373	5394	5415	5436	5457	5478
3700	.25498	5519	5539	5560	5580	5601	5621	5641	5662	5682
3800	.25702	5722	5742	5762	5782	5802	5822	5841	5861	5881
3900	.25900	5920	5939	5959	5978	5997	6016	6035	6054	6073
4000	.26092	6111	6130	6149	6167	6186	6205	6223	6242	6260
4100	.26279	6297	6316	6334	6353	6371	6389	6408	6426	6444
4200	.26462	6480	6498	6516	6534	6551	6569	6586	6604	6621
4300	.26638	6655	6673	6690	6707	6724	6741	6758	6775	6792
4400	.26808	6825	6841	6858	6874	6891	6908	6924	6941	6957
4500	.26974	6990	7007	7023	7039	7055	7072	7088	7104	7120
4600	.27136	7152	7168	7184	7200	7215	7231	7247	7262	7278
4700	.27294	7309	7325	7341	7356	7372	7388	7403	7419	7434
4800	.27450	7466	7481	7497	7512	7528	7543	7559	7574	7590
4900	.27605	7621	7636	7651	7667	7682	7697	7713	7728	7743
5000	.27759	7774	7790	7805	7820	7836	7851	7866	7882	7897
5100	.27912	7927	7942	7957	7972	7987	8002	8017	8032	8047
5200	.28062	8077	8092	8106	8121	8136	8151	8165	8180	8195
5300	.28210	8224	8239	8254	8268	8283	8298	8312	8327	8341
5400	.28356	8370	8385	8399	8414	8429	8443	8458	8472	8487
5500	.28501	8516	8530	8544	8559	8573	8587	8602	8616	8630
5600	.28644	8658	8673	8687	8701	8715	8729	8744	8758	8772
5700	.28786	8800	8815	8829	8843	8857	8871	8885	8899	8913
5800	.28927	8941	8955	8969	8983	8997	9011	9025	9039	9053
5900	.29067	9081	9095	9109	9123	9137	9151	9165	9179	9193
6000	.29207	9221	9235	9249	9263	9277	9291	9305	9318	9332
6100	.29346	9360	9374	9388	9401	9415	9429	9443	9457	9470
6200	.29484	9498	9512	9526	9539	0553	9567	9580	9594	9608
6300	.29621	9635	9649	9662	9676	9690	9703	9717	9731	9744
6400	.29758	9772	9785	9799	9813	9826	9840	9854	9867	9881
6500	.29895	9909	9922	9936	9950	9964	9978	9991	0005*	0019
6600	.30033	0047	0061	0074	0088	0102	0116	0130	0143	0157
6700	.30171	0185	0198	0212	0226	0240	0253	0267	0281	0294
6800	.30308	0322	0335	0349	0363	0376	0390	0404	0417	0431
6900	.30444	0458	0472	0485	0499	0512	0526	0540	0553	0567
7000	.30580	0594	0608	0621	0635	0649	0663	0676	0690	0704
7100	.30717	0731	0744	0758	0772	0785	0799	0812	0826	0839
7200	.30853	0867	0880	0894	0907	0921	0934	0948	0961	0975
7300	.30988	1002	1015	1029	1043	1056	1070	1083	1097	1110
7400	.31124	1138	1151	1165	1178	1192	1205	1219	1233	1246
7500	.31260	1274	1287	1301	1314	1328	1342	1355	1369	1382

G-function based on Gâvre drag function

$V^2/100$	0	10	20	30	40	50	60	70	80	90
7500	0.31260	1274	1287	1301	1314	1328	1342	1355	1369	1382
7600	.31396	1410	1423	1437	1450	1464	1477	1491	1504	1518
7700	.31531	1545	1558	1572	1585	1599	1612	1626	1639	1653
7800	.31666	1680	1693	1707	1721	1734	1748	1761	1775	1788
7900	.31802	1816	1829	1843	1857	1870	1884	1897	1911	1924
8000	.31938	1952	1965	1979	1992	2006	2020	2033	2047	2060
8100	.32074	2088	2101	2115	2128	2142	2156	2169	2183	2196
8200	.32210	2224	2237	2251	2264	2278	2292	2305	2319	2332
8300	.32346	3260	2374	2387	2401	2415	2429	2443	2456	2470
8400	.32484	2498	2511	2525	2539	2553	2566	2580	2594	2607
8500	.32621	2635	2648	2662	2675	2689	2703	2716	2730	2743
8600	.32757	2771	2784	2798	2811	2825	2839	2852	2866	2879
8700	.32893	2907	2920	2934	2947	2961	2975	2988	3002	3015
8800	.33029	3043	3056	3070	3084	3097	3111	3125	3138	3152
8900	.33166	3180	3193	3207	3220	3234	3248	3261	3275	3288
9000	.33302	3315	3329	3342	3355	3369	3382	3396	3409	3422
9100	.33436	3449	3463	3476	3490	3503	3517	3530	3544	3557
9200	.33571	3584	3598	3611	3625	3638	3652	3666	3679	3693
9300	.33706	3720	3733	3747	3761	3774	3788	3801	3815	3829
9400	.33842	3856	3869	3883	3897	3910	3924	3937	3951	3965
9500	.33978	3992	4005	4019	4033	4046	4060	4073	4087	4101
9600	.34114	4128	4141	4155	4169	4182	4196	4210	4223	4237
9700	.34251	4265	4278	4292	4306	4319	4333	4347	4360	4374
9800	.34388	4402	4415	4429	4443	4456	4470	4484	4497	4511
9900	.34525	4539	4552	4566	4580	4594	4608	4621	4635	4649
10000	.34663	4677	4690	4704	4718	4732	4746	4759	4773	4787
10100	.34801	4815	4828	4842	4856	4870	4883	4897	4911	4925
10200	.34938	4952	4966	4980	4993	5007	5021	5035	5049	5062
10300	.35076	5090	5104	5117	5131	5145	5159	5173	5186	5200
10400	.35214	5228	5242	5255	5269	5283	5297	5311	5324	5338
10500	.35352	5366	5380	5393	5407	5421	5435	5449	5463	5476
10600	.35490	5504	5518	5532	5545	5559	5573	5587	5601	5614
10700	.35628	5642	5656	5670	5683	5697	5711	5725	5739	5752
10800	.35766	5780	5794	5808	5821	5835	5849	5863	5877	5890
10900	.35904	5918	5932	5946	5959	5973	5987	6001	6015	6028
11000	.36042	6056	6070	6084	6097	6111	6125	6139	6153	6166
11100	.36180	6194	6208	6222	6235	6249	6263	6277	6291	6304
11200	.36318	6332	6346	6360	6373	6387	6401	6415	6429	6442
11300	.36456	6470	6484	6498	6511	6525	6539	6553	6566	6580
11400	.36594	6608	6622	6635	6649	6663	6677	6691	6704	6718
11500	.36732	6746	6760	6773	6787	6801	6815	6828	6842	6856
11600	.36870	6884	6897	6911	6925	6939	6953	6966	6980	6994
11700	.37008	7022	7035	7049	7063	7077	7090	7104	7118	7132
11800	.37145	7159	7173	7187	7200	7214	7228	7242	7255	7269
11900	.37283	7297	7310	7324	7338	7352	7365	7379	7393	7407
12000	.37420	7434	7448	7462	7475	7489	7503	7517	7530	7544

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